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Jakob Steiner's Construction of Conics Revisited

Pavel Pech[1], **Gunter Weiss**[2,3]

[1] *University of South Bohemia, České Budějovice*
Jeronýmova 10, ČZ 371 15 České Budějovice, Czech Republic
email: pech@pf.jcu.cz

[2] *University of Technology Dresden, Germany*

[3] *Vienna University of Technology, Austria*
email: weissgunter@gmx.at

Abstract. We aim at presenting material on conics, which can be used to formulate, e.g., GeoGebra problems for high-school and freshmen maths courses at universities. In a (real) projective plane two pencils of lines, which are projectively related, generate, in general, a conic. This fact due to Jakob Steiner [4] allows to construct points of a conic given by, e.g., 5 points. Hereby the problem of transfering a given cross-ratio of four lines of the first pencil to the corresponding ones in the second pencil occurs. To solve this problem in a graphically simple and uniform way we propose a method, which uses the well-known fact that a projective mapping from one line (or pencil) to another always can be decomposed into a product of perspectivities. By extending the presented graphical methods, we also construct tangents and osculating circles at points of a conic. The calculation following the graphic treatment delivers a parametrisation of conic arcs applicable also for so-called 2$^{\text{nd}}$ order biarcs. Even so the topic and its theoretical background is a matter of the 19$^{\text{th}}$ century, it is not at all well-known nowadays, as also is stated in [3]. Some of the presented constructions might also be new.

Key Words: conic, real projective plane, projectivity, Steiner's generation of a conic

MSC 2010: 51M15, 51N15

1. A simple linear procedure delivering a conic

In the plane of visual perception π, let two line-elements (A, p), (B, q) and a point C be given (see Figure 1). An arbitrary line x through C intersects p at P and q at Q. The lines AQ and BP intersect in a point X, which runs through a conic c, if x runs through the pencil at C. Obviously, c passes through the intersection point S of p and q and touches AC resp. BC at A resp. B. Let us call this the *PP-construction* of c.

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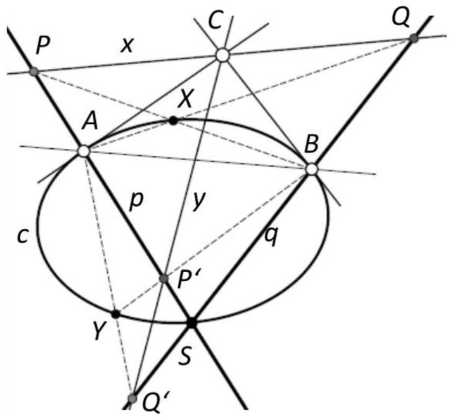

Figure 1: If x runs through the pencil at C, X runs along the conic c

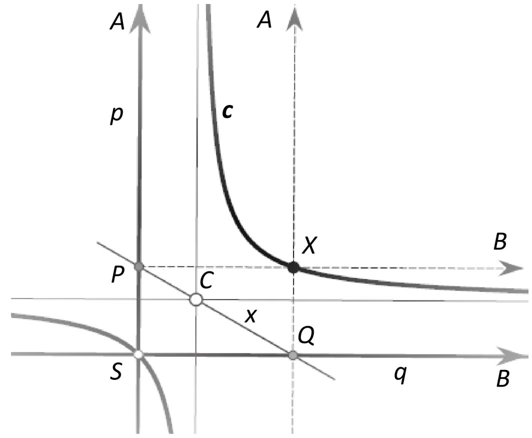

Figure 2: Euclidean interpretation of the projective coordinate frame

We see that only connections of points and intersections of lines are needed. Thus, for the place of action only a projective plane π is needed, which, because of the occurrence of a conic, must be coordinized by a *commutative* field $\mathcal{F}$ with char $\mathcal{F} \neq 2$. As we point to high-school applications, we restrict ourselves to the field $\mathcal{F} = \mathbb{R}$.

For the calculations, we use projective coordinates based on the frame

$$S = (1,0,0)\mathbb{R},\ A = (0,1,0)\mathbb{R},\ B = (0,0,1)\mathbb{R},\ C = (1,1,1)\mathbb{R}. \tag{1}$$

Line x intersects p at $P = (1,t,0)\mathbb{R}$ and q at $Q = (t-1,0,t)\mathbb{R}$, such that AQ and BP get equations $tx_0 + (1-t)x_2 = 0$ and $tx_0 - x_1 = 0$. For the intersection point X it follows

$$X = (1-t, t(t-1), t)\mathbb{R}. \tag{2}$$

A Euclidean interpretation of the projective coordinate frame and the described conic construction is shown in Figure 2.

For a synthetic proof for $c = \{X\}$ being a conic we use the concepts “perspectivity" (symbol $\overline{\overline{\wedge}}$) and “projectivity" (symbol $\overline{\wedge}$) as the finite product of perspectivities (c.f., e.g., [1]). We note that

$$A(AQ,\dots)\,\overline{\overline{\wedge}}\,q(Q,\dots)\,\overline{\overline{\wedge}}\,C(x,\dots)\,\overline{\overline{\wedge}}\,p(P,\dots)\,\overline{\overline{\wedge}}\,B(BP,\dots)\ \Rightarrow A(AQ,\dots)\,\overline{\wedge}\,B(BP,\dots). \tag{3}$$

Therefore, according to Jakob Steiner, the two projectively related pencils $A(AQ,\dots)$ and $B(BP,\dots)$ generate, in general, a conic. If there would be a self-corresponding element in the pencils, the projectivity would be a perspectivity and the result the axis of perspectivity instead of a conic. So we have to exclude $C \in p$ or q or AB to receive a conic c.

One might ask, how to position C such that the resulting conic c becomes a parabola in a projectively enclosed affine plane with ideal line u. Then, in the two projective pencils at A and B, there must occur a pair of parallel corresponding lines $AQ' \parallel BP'$ (see Figure 3). We start with pencils at A and B and define a projectivity from the first to the second by the parallel relation. These pencils are in perspective position with the ideal line u as perspectivity axis. Corresponding parallel lines intersect p and q at P_U and Q_U. Therefore, the pointsets $p(P_U)$ and $q(Q_U)$ are projective related.

Applying the dual version of Steiner's generation of a conic — we shall meet this construction in Section 7 — we find that the lines P_UQ_U envelop a conic h. It is obvious that

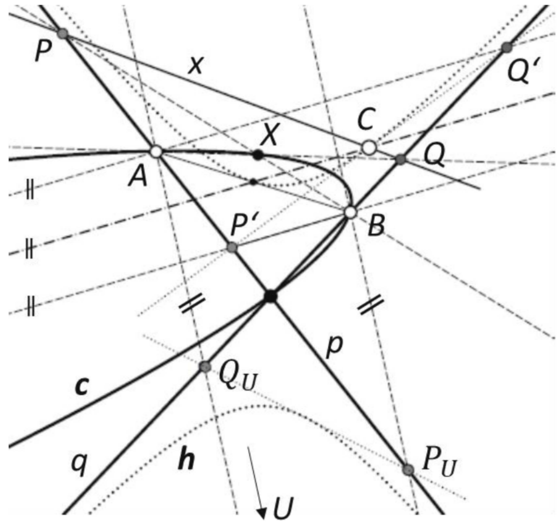

Figure 3: If C lies on the hyperbola h then the resulting conic c becomes a parabola

this conic is a hyperbola touching AB in the midpoint of segment A, B and has p and q as asymptotes. Choosing $C \in h$ the PP-construction delivers a parabola c. If C is an inner point of h, the resulting conic c is an ellipse, and if C is an outer point of h, then c is a hyperbola.

2. A first application: Construction of a conic through 5 given points

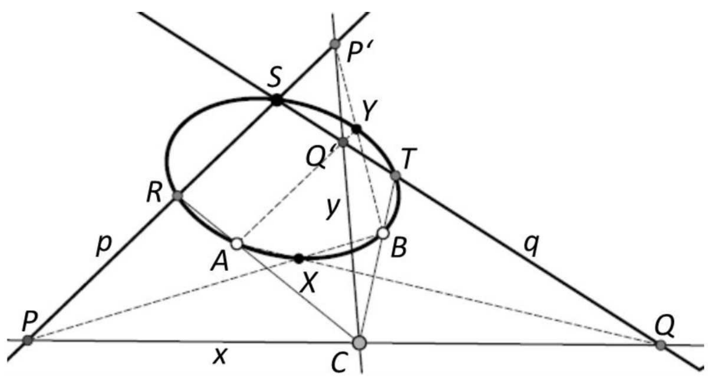

Figure 4: Construction of a conic through 5 given points

We choose two of the given points, say A and B, as vertices of pencils of lines, the line RS as p and ST as q. The lines AR and BT intersect in C, which will be used as vertex of a pencil $\{x\}$ (see Figure 4). Obviously, by the same arguments as (2), the set $\{X\}$ with $X = AQ \cap BP$ is a conic c passing through the given five points A, B, R, S, T.

It is clear, how to proceed, if c is given by one line-element and three points or by two line-elements and one point.

3. Tangents of a conic

We start with a conic given by two line-elements and a point, as in Figure 5. We state:

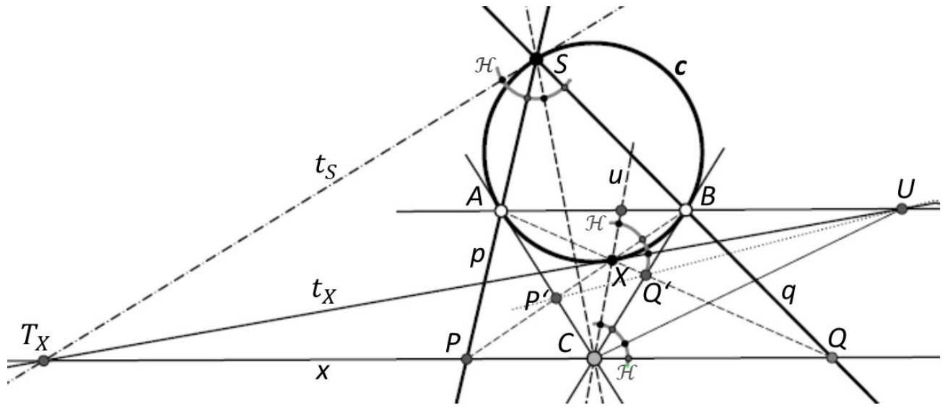

Figure 5: The tangent t_X to c at X is harmonic to XC with respect to AQ, BP

Corollary 1. *If $X \in c$ is derived as intersection of AQ with BP, then the tangent t_X to c at X is harmonic to XC with respect to AQ, BP.*

Proof. We take $XC = u$ as axis of a harmonic homology $\kappa\colon\ c \to c$. The centre of κ is the pole U of u with respect to c, and U is harmonic to u with respect to (A, B) (see Figure 5). As $X \in u$ the line $UX =: t_X$ must be tangent to c. Obviously, the pair (XA, XB) is harmonic to (u, t_X). □

To facilitate the construction of U, we note that $AC \cap BP =: P'$ and $BC \cap AQ =: Q'$ correspond in κ, such that we can find U as the intersection of $P'Q'$ with AB (see Figure 5).

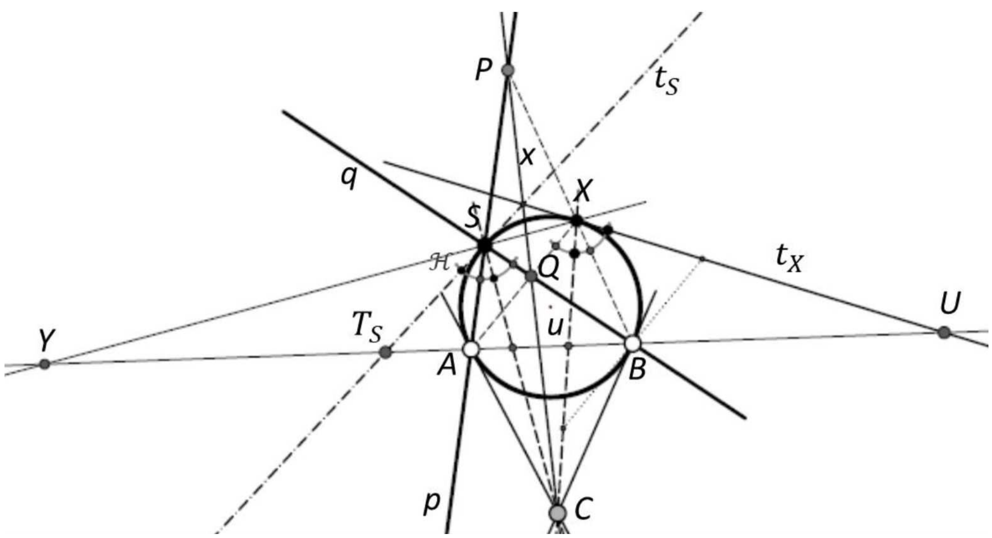

Figure 6: Illustration to Corollary 2

A further reduction of the construction lines is a consequence of

Corollary 2. *The harmonic point T_X to C with respect to (P, Q) is a point of t_X. Thereby $T_X \in t_S$, and t_S is harmonic to SC with respect to (p, q).*

Proof. The mentioned four points are X-perspective to the harmonic lines (t_X, XC, XA, AB) and finally S-perspective to the harmonic lines (t_S, SC, p, q) (see Figures 5 and 6). □

Remark 1. We will meet a well-known Euclidean variant of the PP-construction and Corollaries 1 and 2 in Section 5.

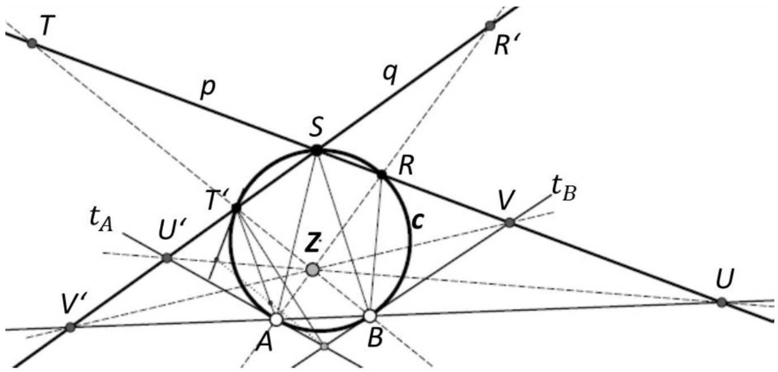

Figure 7: Construction of tangents t_A, t_B at A and B.

For a conic given by five points, we start with constructing the tangents t_A, t_B at A and B. We factorize the projectivity $\overline{\wedge}$: $A(AX,\dots) \to B(BX,\dots)$, which generates c, by perspectivities using $RS = p$ and $ST' = q$ as intermediate "axes". As S is self-corresponding, the induced mapping $p \to q$ is a perspectivity with centre $Z = AR \cap BT$. To $U := AB \cap p$ corresponds $U' \in q$ and therefore $t_A = AU'$. To $V' := AB \cap q$ corresponds $V \in p$ and therefore $t_B = BV$ (see Figure 7).

For further tangents we replace the given c by the two line-elements $(A, t_A), (B, t_B)$ and a further point and proceed as described above.

Remark 2. Besides these simple and maybe new tangent constructions there exist of course several other methods. For example, if one counts one of the five conic points twice, e.g. A, then the application of Pappus-Pascals theorem to these formally six points delivers the tangent t_A at A via the so-called Pascal-axis. This construction only makes use of incidences and its place of action is a projective plane.

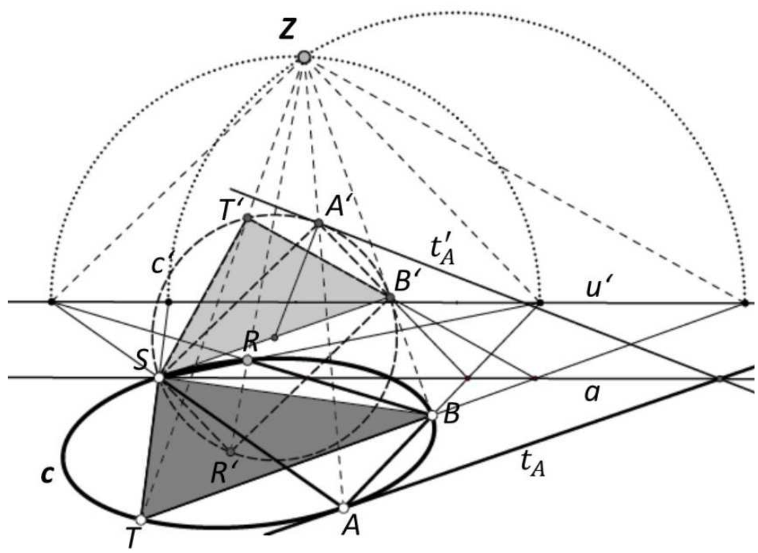

Figure 8: Conic c as the image of a circle k under a central projection

Another descriptive geometric way is to interpret c as the image of a circle k under a certain central projection. We need the projectively enclosed Euclidean plane π as place of action. Four of the given five points are interpreted as images of the vertices of a rectangle. As an implication we get the vanishing line of central projection and a pair of vanishing points of orthogonal directions. The fifth point, together with two diagonal endpoints of the quadrangle, defines a second pair of vanishing points to orthogonal directions due to Thales' theorem. Finally, the centre Z of the collineation $\kappa\colon\ c \to c'$ is a common point of the Thales-circles over the two pairs of vanishing points (see Figure 8). To complete the figure, one has to choose an axis $a \parallel u'$ of κ.

4. Osculating circles of a conic

In this section the place of action is the projectively enclosed Euclidean plane π realized as the classical plane of visual perception.

Let a conic c be given by a line-element (A, t_A) and three points R, S, T. We aim at the osculating circle k_A at A. This circle belongs to a parabolic pencil of circles, and it defines an elation $\varepsilon\colon\ c \to k_A$ with centre A and an axis a_A, which is unknown.

We choose an arbitrary circle k' of the parabolic pencil of circles and project R, S, T onto it with centre A. The triangle (RST) and the resulting triangle $(R'S'T')$ are in Desargues position and define a Desargues axis a', which acts as axis of a homology $\chi'\colon\ c \to k'$ with the center A. We make use of the following well-known fact:

Corollary 3. *The Desargues axes a, a', and a'' of three pairs of A-perspective triangles $((R'S'T'), (R''S''T''))$, $((RST), (R'S'T'))$, and $((RST), (R''S''T''))$ pass through a common point U.*

Proof. Let $\chi' :\ (RST) \mapsto (R'S'T')$ and $\chi'' :\ (RST) \mapsto (R''S''T'')$ be homologies with the common centre A and different axes a', a''. Then $a' \cap a'' =: U$ is a fixed point of $\chi = \chi'^{-1}\chi''\colon$ $(R'S'T') \mapsto (R''S''T'')$. Therefore, the axis a of χ must contain U. □

In the case depicted in Figure 9, χ is a homothety such that a is the ideal line of plane π and U must be an ideal point. As a result we get $a' \parallel a''$. Therefore, the axis $a_A = a''$ of the elation $\varepsilon : c \to k_A$ has to be chosen parallel to a' through A. This allows to reconstruct the triangle $(R''S''T'')$ similar to $(R'S'T')$ with its circumcircle as the demanded osculating circle k_A.

5. A generalization of the PP-construction

We started with a pair of lines p, q and a pair of points A, B and a suitably chosen point C to perform the PP-construction (Figures 1 and 3). We might think of p, q as a singular curve of 2^{nd} order and replace it by a regular conic d. Figure 10 shows the case, when $A, B \in d$. The result is a conic c, independent of which of the combinations AQ, BP or AP, BQ we choose. Again, the tangent t_X at X is harmonic to CX with respect to XA and XB.

Figure 11 shows the case, when $A, B \notin d$. The result is a curve of 4^{th} degree, and again both combinations AQ, BP or AP, BQ deliver points of the same curve c. Taking the line $x = t_d$ tangent to conic d, the points P and Q coincide with a point T of c. There the tangent t_T at T to c is harmonic to t_d with respect to TA and TB.

Another generalization replaces the pencil $\{x\}$ at C by two projectively related pencils $\{x\}$ and $\{y\}$ at C, and we keep the pair of lines p, q. Now we use $x \cap p = P$ and $y \cap q = Q$

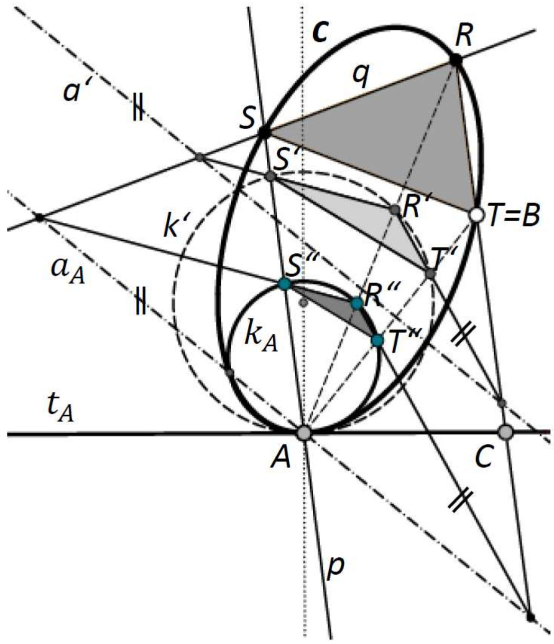

Figure 9: The circumcircle of the triangle $(R''S''T'')$ is the osculating circle k_A of c at A

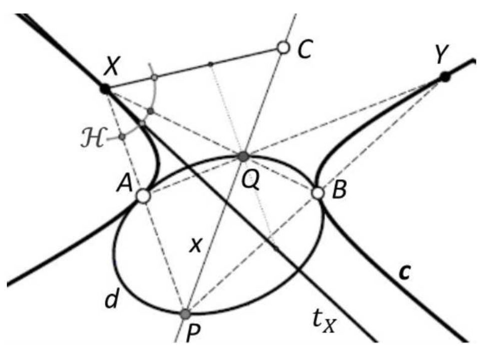

Figure 10: A generalization of the PP-construction, $A, B \in d$

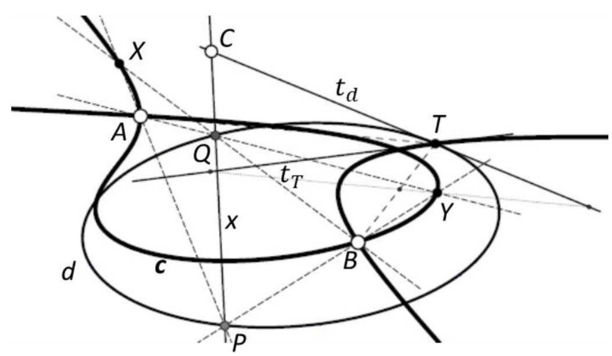

Figure 11: A generalization of the PP-construction, $A, B \notin d$

for defining $X := AQ \cap BP$. In Figure 12 we rotate x by 30° resp. 90° to get its projectively related line y. The results of the modified PP-construction are the conics labelled as c_{30}, c_{90} (see Figure 12).

6. The dual version of the PP-construction

The dual of line-elements are still line-elements. So we start with $(A, p), (B, q)$ as in Figure 1. The point C is replaced by a line d, the line x by a point X, which will run through x under the dual version of the PP-construction (see Figure 13).

We met this dual Steiner construction already in Figure 3. Projecting X from A resp. B to q resp. p we get points $Q \in q$, $P \in p$ and $x := PQ$ is an element of a dual conic c^*, i.e., a tangent of a conic c. Dualising the construction of tangents as described in Corollaries 1 and 2 results in the point of contact T of x with c as intersection of x with a line t harmonic to d with respect to XA, XB.

7. Steiner's conic construction and the algebraic point of view

When we speak of Steiner's construction of a conic we have the general case of two different projective pencils of lines $A(x,\dots), B(x',\dots)$ in mind, whereby the line connecting the pencils' centres A and B is not self-corresponding. If this line $AB =: u = u'$ is self-corresponding, the pencils are perspective and generate a line, the perspectivity axis v. For $u = u'$ one might understand the not defined intersection $x \cap x' = \{X\}$ as the whole line u, such that $u \cup v = c^{(2)}$ again becomes a (singular) curve of 2^{nd} degree. Similarly, if A and B coincide, there are, counted in algebraic sense, two real, imaginary or coinciding pairs of self-corresponding lines $u = u'$, $v = v'$, and again we can consider the singular curve $u \cup v = c^{(2)}$ as the result of Steiner's construction.

The dual point of view starts with two lines a, b and a linear mapping λ between the point sets $a(X\dots), b(X'\dots)$. If the intersection $a \cap b =: U \neq U'$, we have the general case of a projectivity λ, and XX' are tangents of a regular conic c. If $U = U'$, the lines XX' form a pencil with the common point V. The two pencils with vertices U and V can be considered as a singular dual curve $c^{*(2)}$ of 2^{nd} class. It is obvious, how to interpret the results, if $a = b$.

We notice that the starting point of Steiner's construction, namely the two pencils $A(x\dots)$, $B(x'\dots)$ by themselves form a singular dual curve of type $c^{*(2)}$. It suggests itself to replace this curve by a regular conic c or a dual conic c^* and, after defining a projectivity $\lambda\colon\ c \to c$ resp. $\lambda^*\colon\ c^* \to c^*$, apply Steiner's construction for corresponding elements (see Figures 15 and 16). Also here we meet elementary geometric constructions of conics. As an analogue to a projectivity between lines a and b which is the product of perspectivities, we formulate:

Corollary 4. *A projectivity $\lambda\colon\ c \to c$ is defined by three pairs $R \mapsto R'$, $S \mapsto S'$, and $T \mapsto T'$. This projectivity λ is either the identity, an involutive projectivity (or "involution" in brief), or the product of two involutions.*

Proof. We define a first involution $\sigma_1\colon\ c \to c$ by the three pairs $R \mapsto R'' = S'$, $S \mapsto S'' = R'$, and $T \mapsto T''$ (see Figure 14). It has the "involution centre" $C_1 = RS' \cap SR'$. The second involution $\sigma_2\colon\ c \to c$ is defined by the pairs $R' \mapsto R''$, $S' \mapsto S''$, and $T' \mapsto T''$, and has the involution centre $C_2 = R'S' \cap T'T''$. Therewith, $\lambda = \sigma_1 \cdot \sigma_2$. $\square$

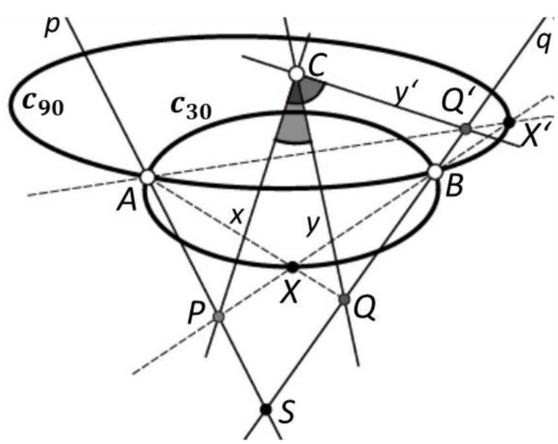

Figure 12: Another modified PP-construction

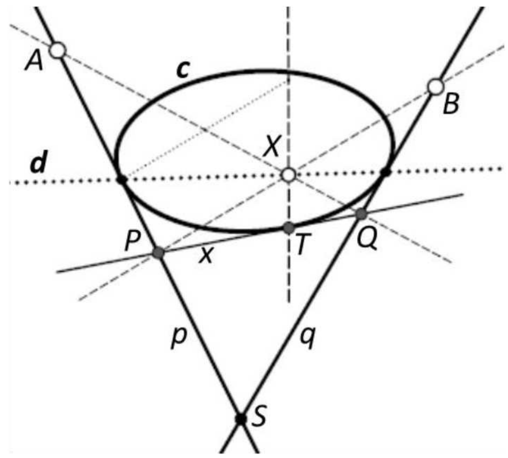

Figure 13: The dual version of the PP-construction

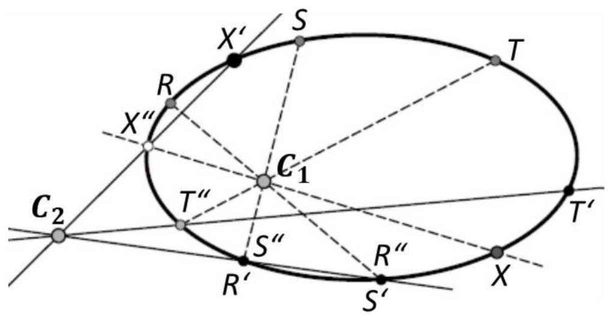

Figure 14: Involutions σ_1 and σ_2 with centres C_1 and C_2

As a modification of the PP-construction of Section 1, we can start with a conic c and two points C_1, C_2 as involution centres to define a projectivity $\lambda\colon\ c \to c$ by $\lambda = \sigma_1 \cdot \sigma_2$ (see Figure 15), which shows a well-known Euclidean construction of tangents of an ellipse.

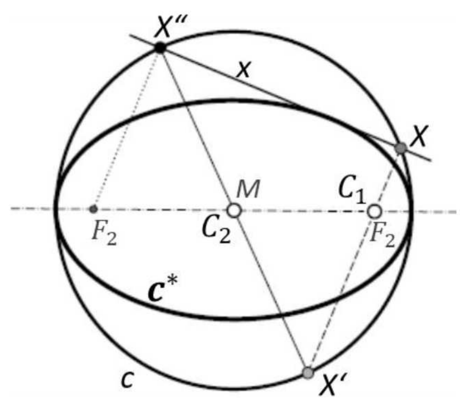

Figure 15: Well-known construction of tangents of an ellipse

The dual version of this extended PP-construction starts with a dual conic c^* and two lines c_1, c_2 to define a projectivity $\lambda^*\colon\ c^* \to c^*$ in the tangent set of c^* (see Figure 16), where we choose c_1 as the ideal line and c^* as a circle.

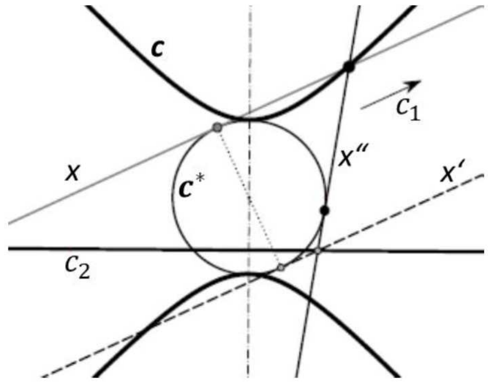

Figure 16: Dual version of the extended PP-construction

8. Poncelet's porism and Steiner's construction of conics

In Section 8, we considered two (different) pencils of lines as a singular curve of second degree. The extended PP-construction uses such a pair of pencils. It suggests itself to replace this pair of pencils by a regular dual conic c^* and use it to define a projectivity $\lambda : X \mapsto X'$ in the pointset of a regular conic d (see Figure 17). The lines XX' envelop a conic c as the result of this extension of Steiner's construction of a conic.

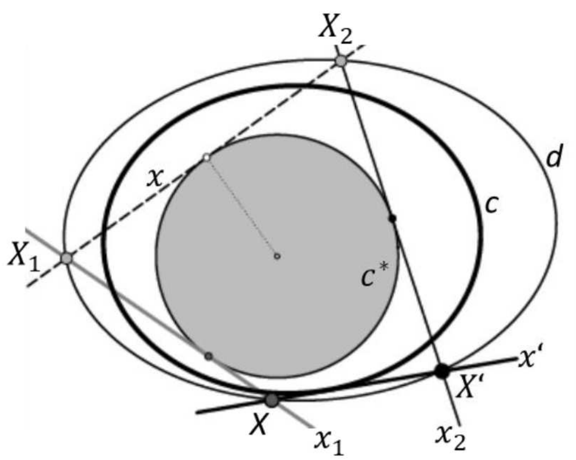

Figure 17: The lines XX' envelop a conic c

It is worth mentioning that, if $X = X'$, we get a fixed point of λ and the triangle (X_1, X_2, X) is as well inscribed to d and circumscribed to c^*. After relabelling this triangle, we receive each of its vertices as fixed points of λ such that, according to Corollary 4, the projectivity λ is the identity. This means that there exists a continuous one-parameter set of triangles inscribed to d and circumscribed to c^*, what represents an example of Poncelet's porism (see Figure 18).

9. Final remark

Dealing with constructive methods nowadays might look out of date. We justify the closer look to this material by both, the elegance of projective geometric synthetic reasoning in the

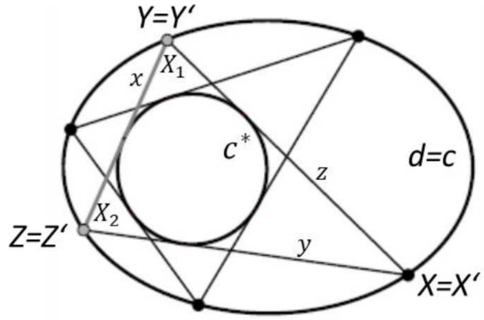

Figure 18: Poncelet's porism

sense of Jakob Steiner, and its applicability as training material for analytic geometry and computer aided problem proving via, e.g., GeoGebra.

Furthermore, even so it is possible to reconstruct the presented material from mostly German references, which are, to a big part, from 19th and early 20th century, we try to give a refreshed overview over a classical topic of constructive projective geometry.

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