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Geometric Feature of a Structure Composed of Concave and Convex Parabolic Mirrors

Yusuke Tsuji, Hirotaka Suzuki

Department of Architecture, Kobe University
1-1, Rokkodai, Nada, Kobe, Japan
email: 151t702t@stu.kobe-u.ac.jp

Abstract. The convex side of a parabolic mirror realizes to convert focusing light into parallel one. Using this feature, we propose a structure which is composed of concave and convex parabolic mirrors. This new-type structure converts parallel light into highly-dense parallel light. The structure has several interesting geometric features. This structure realizes a conversion before passing a focal point. The structure offers the possibility of applications for industrial use. As one of the industrial applications, we suggest a daylighting system using this structure and we estimate its performance.

Key Words: reflection, paraboloids, daylighting system *MSC 2010:* 51N05, 51N20

1. Introduction

The convex side of a parabolic mirror converts focusing light into parallel one, in the same way as the concave side. This is a well-known feature of parabolic mirrors (e.g., [1]).

Figure 1 shows both sides of a parabolic mirror that convert focusing light into light parallel to the paraboloid's axis. Especially, the convex side realizes to convert focusing light before passing through the focal point. This feature has the extremely big advantage to make parabolic mirrors usable for daylighting systems. Daylighting systems are conduction systems of natural sunlight into the interior of buildings (e.g., [3]). For the study of daylighting systems, it is important to control the density of parallel light. However, in general, this control is not an easy task.

In this paper, we suggest a structure composed of concave and convex parabolic mirrors, and we prove the geometric feature of this structure. As one of its applications, we propose this structure to be used for a daylighting system, and we estimate performance of this system.

This paper is organised as follows: We explain the basic theory of a parabolic mirror in Section 2. Then, we confirm the features of a parabolic structure in Section 3, we consider an application for a daylighting system in Section 4, and finally summarize and conclude in Section 5.

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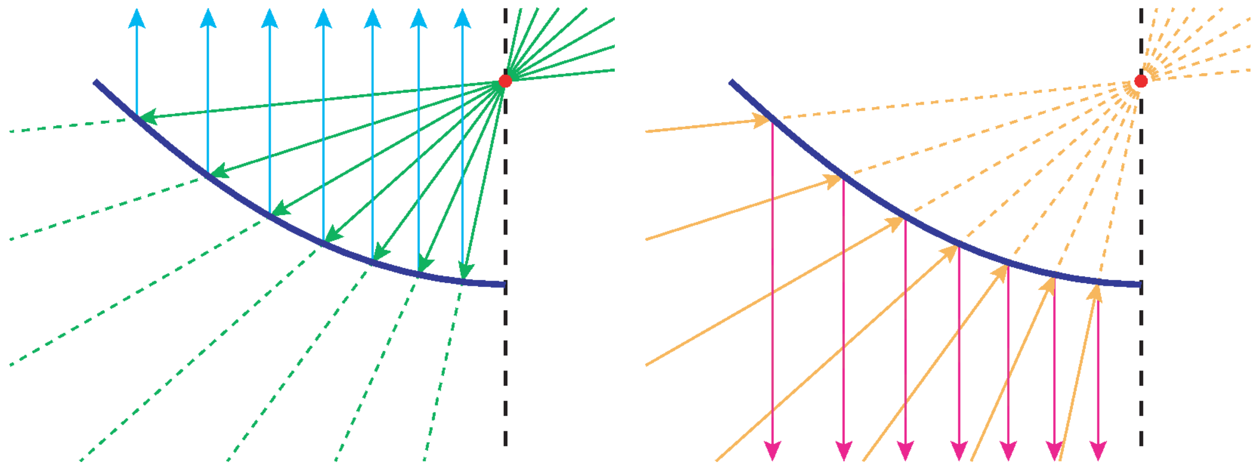

Figure 1: Two types of a parabolic mirror. As depicted on the left-hand side, the concave side of a parabolic mirror converts focusing light into parallel one after passing the focal point. The right-hand side shows that the convex side of a parabolic mirror convert focusing light into parallel one before passing the focal point.

2. Basic theory

It has already been known for a long time that the convex side of a parabolic mirror converts focusing light into parallel one (e.g., E. Hecht [1]). Suzuki et al. [5] explained this feature by solving an ordinary differential equations (hereafter, ODE) using numerical methods. Tsuji and Suzuki [7] found out the analytical solution of the ODE and proved the feature of a parabolic mirror.

According to [5], the ODE that defines the shape of a mirror to convert focusing light into parallel one reads:

$$x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0. \tag{1}$$

By [5], one of the solution is a parabolic equation. By [7], it can be rearranged as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{y/x - \sqrt{1+(y/x)^2}}, \tag{2}$$

which is a well-known form of a homogeneous ODE (see, e.g., [11]). According to the theory of homogeneous ODEs, we rearrange eq. (2) as

$$\frac{1}{\sqrt{1+r^2}}\,\mathrm{d}r = \frac{1}{x}\,\mathrm{d}x\,, \tag{3}$$

where $r = y/x$. Equation (3) is called a separable ODE. Integrating and rearranging this equation, we get

$$\sqrt{r^2+1} = K|x| - r. \tag{4}$$

Here, K is a positive constant. Solving (4) for y, we get finally

$$y = \begin{cases} \dfrac{1}{4p}x^2 - p & \text{for } x \geq 0, \\ -\dfrac{1}{4p}x^2 + p & \text{for } x < 0. \end{cases} \tag{5}$$

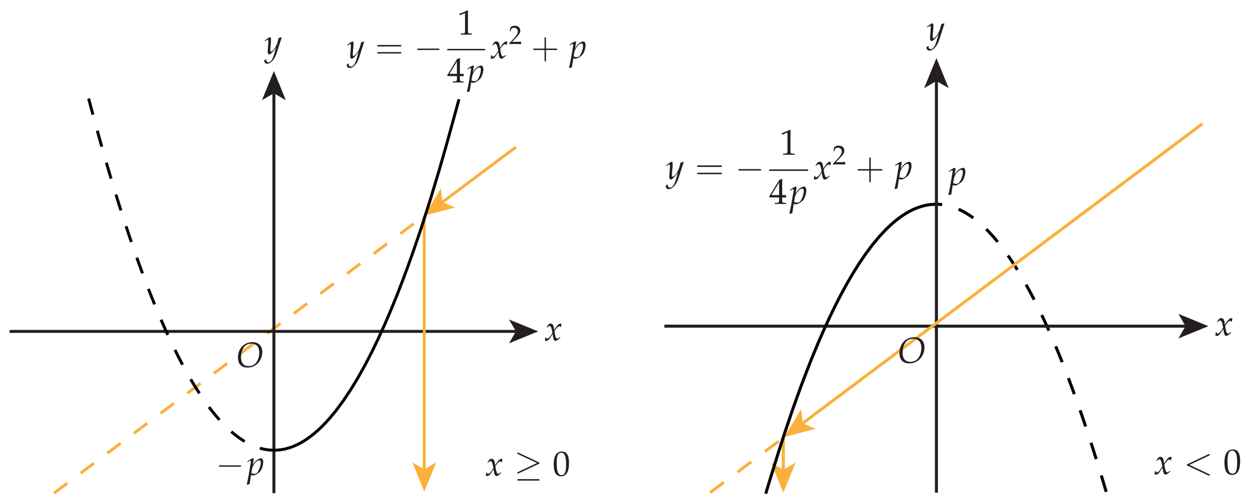

Figure 2: The result of eq. (5). The left figure shows the case $x \geq 0$, the right figure the case $x < 0$. In both cases, the depicted parabola converts focusing light into light parallel to the parabola's axis. In the case shown on the left-hand side, the focusing light is converted before passing the focal point. The figure on the right-hand side shows a conversion of focusing light after passing a focal point.

Here, $p = 1/2K$.

Figure 2 shows the solution of eq. (5). In the case $x \geq 0$ (left-hand figure), the convex side of a parabolic mirror converts focusing light before passing the focal point. In the case $x < 0$ (right-hand figure), the concave side of a parabolic mirror converts focusing light after passing the focal point.

Both structures can be used to convert focusing light into light parallel to the parabola's axis. Here, we aim to convert parallel light into highly dense parallel one. We can realize this

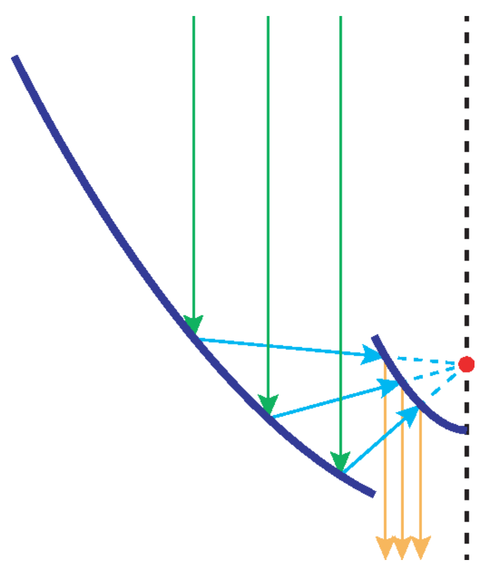

Figure 3: A structure composed of concave and convex parabolic mirrors. The concave mirror converts parallel light from above into focusing light. Then, the convex mirror converts it into highly-dense parallel light.

by using a combination of one concave side and one convex side of parabolic mirrors.

Figure 3 shows the structure (hereafter, the *CCCP system*) composed of a concave and a convex parabolic system. This system converts parallel light into highly-dense parallel one, and the light density can be controlled. The most remarkable feature of the CCCP system is that this structure does not have any focal points. We explain the advantages of this structure in Section 4.

3. Geometrical feature

3.1. Geometrical comprehension

As the result of (5), we get two types of parabolic mirrors for the conversion of focusing light into parallel one. Of course, this feature can also be proved in a geometric way, which is absolutely intuitive and well understandable.

Figure 4 shows the geometric background. The concave side corresponds to the case $x < 0$ in (5), the convex side to the case $x \geq 0$. The most important feature of Figure 4 is the convex side; it seems strange, but for the geometric point of view it is apparent. We can understand the correctness of this feature using well-known theories of geometry and physics (law of reflection and vertical angles).

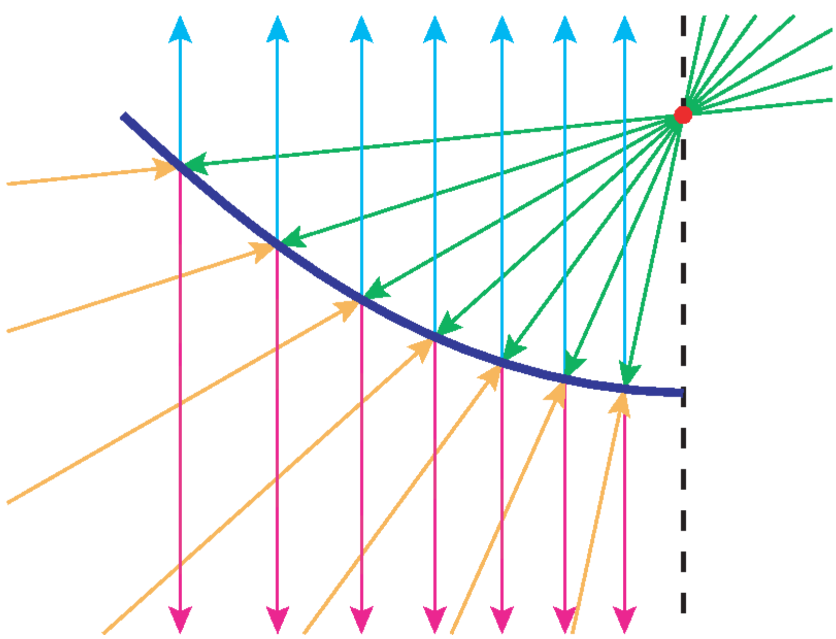

Figure 4: Focal property of a parabola: The concave side converts focusing light into parallel one after passing the parabola's focal point. The convex side converts it before passing the focal point.

3.2. The CCCP system

At a CCCP system, there are three types of conditions and four types of rays to distinguish.

Figure 5 shows the CCCP system and the expected rays. The rays of light in Region (A) miss, after their reflection in the first mirror, the second mirror. Parallel rays in the Region (B) are reflected by the first mirror into focusing light and, by the second mirror, again into parallel

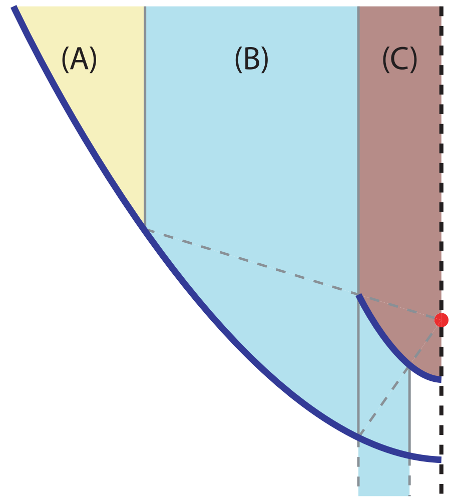

Figure 5: The CCCP system and the expected rays. There are three types of light rays: Region (A), Region (B), and Region (C). Only rays in Region (B) realize the requested conversion.

ray are generated. For rays in Region (C), the wrong side of the first mirror absorbs rays of light. Hereafter, we consider to rotate the first mirror about its focal point.

Figure 6 shows three types of conditions of the CCCP system:

• The left figure shows the slightly rotated case: When the first mirror rotates slightly, then we have three regions: Region (A), Region (B), and Region (C). In this case, we obtain completely the same as in the non-rotated case shown in Figure 5.

• The figure in the middle shows the rotated case: When the first mirror rotates, we can distinguish four regions: Region (A), Region (B), Region (C), and Region (X). Regions (A)–(C) are same as in the non-rotated case. Region (X) contains rays without any reflection; they are passing through between the two mirrors. In this region, the amount of incoming rays (i.e., rays in Region (B)) decreases when the rotation angle of the first mirror increases.

• The right figure shows the excessively rotated case: When the first mirror rotates excessively, then we have three cases: Region (A), Region (C), and Region (X). In this case, there are no rays to convert focusing light, i.e., no rays in Region (B).

In addition, we need to mention: Region (X) does not always show up. Basically, there are some special cases where the CCCP system does not have a Region (X).

4. Application to daylighting systems

4.1. Daylighting systems

Daylighting systems are the conduction systems of natural sunlight into the interior of buildings (e.g, [3]). There are two items for improving the performance: the light flow control and the density control. In general, it is difficult to realize both at the same time due to heat problems (e.g., [6]). Heat problems are must-be-avoid problems in architecture (e.g., [4]). In particular, when focusing on density control, it is not easy to solve the problems, since daylighting systems often have focal points. Almost all the studies on daylighting systems, that concentrated on density control, have parabolic, paraboloidal or approximated mirrors

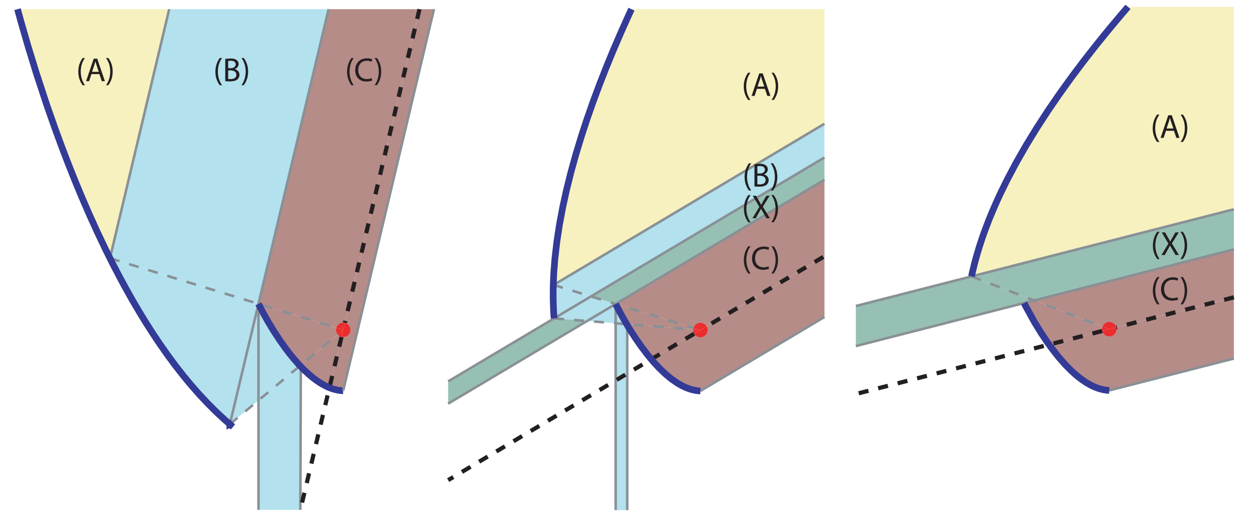

Figure 6: Three types of conditions for the CCCP system: Left: slightly rotated case, middle: rotated case, right: excessively rotated case.

in their systems. In order to solve heat problems, we suggest a new-type daylighting system with the CCCP system.

Figure 7 shows a new-type daylighting system with the CCCP system. This system has two degrees of freedom: a vertical rotation of the first mirror and a horizontal rotation of the system. The first mirror collects sunlight and reflects it to the second mirror while focusing.

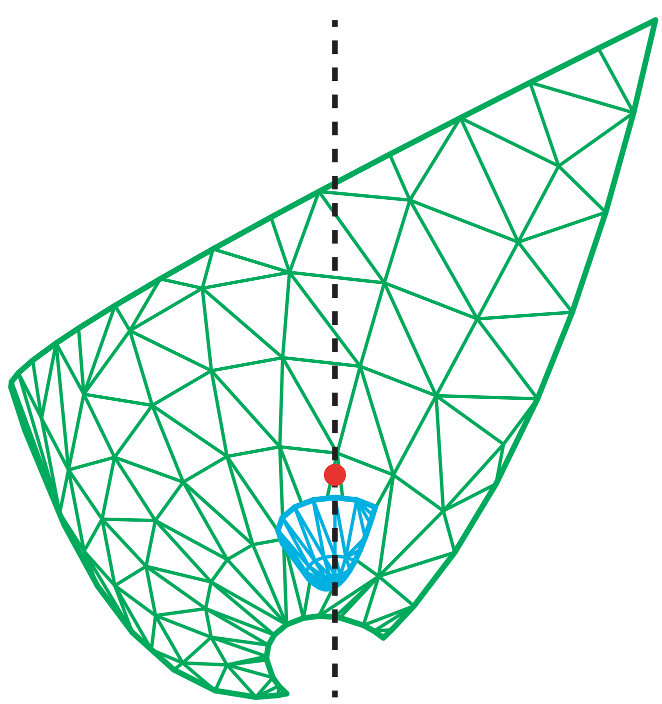

Figure 7: A new-type daylighting system with a CCCP system: The large concave paraboloidal mirror is the first mirror, the small convex one is the second mirror. The first mirror moves and tracks the sun. The second one is fixed. This system has two degrees of freedom.

The second mirror receives the light reflected from the first mirror and reflects it as parallel light before passing through a focal point.

Similar work has been done by I. Ullah et al. in [8, 9, 10]. They used a similar system composed of concave and convex parabolic mirrors (hereafter, *Ullah system*). However, there are two differences between the Ullah system and the CCCP system. These two differences are also two advantages of the CCCP system against the Ullah system:

• The first one is the degree of freedom. Ullah systems have three degrees of freedom. In contrast, the CCCP system has two degrees of freedom. It means that the CCCP system realizes the same result as the Ullah system with fewer moving parts. In fact, an Ullah system has only two moving parts: In order to reduce a moving part, the system applied optical fibres.

• The second advantage is the centre of gravity. Ullah systems need to support two mirrors (i.e., one concave mirror and one convex mirror) from the bottom of the convex mirror. It means that the system needs a high moment of inertia, and a high load is applied to moving parts. In contrast, the CCCP system supports only the concave mirror and the load is reduced. In addition, it is easy to increase its size if needed.

We adopt paraboloidal mirrors instead of parabolic column mirrors. There are two advantages when using a paraboloid.

• The first one is its compactness. The system needs to follow the sunlight when used as a daylighting systems. Paraboloidal systems need smaller space than parabolic column systems, in particular with regard to the system width.

• The second advantage is the density. Paraboloids increase the density three-dimensionally and parabolic columns increase two-dimensionally.

4.2. Performance estimation

In the simulations, we used two-dimensional ray-tracing codes. Ray-tracing is a well-known calculation method in graphics. In this paper, we adopt a kind of ray-tracing called inverse ray-tracing. In order to calculate this simulations, we set two physical constants: the solar illuminance constant $E_\odot = 134000\,\mathrm{lx}$ and the transmittance $\tau = 0.65$.

Figure 8 shows ray-tracing simulations at the summer solstice. The left figure is the simulation at 10.00 o'clock. The rays received at the first mirror are separated into two types: Region (A) and Region (B). The right figure shows the simulation at 12.00 o'clock noon. All rays received at the first mirror are reflected and received by the second mirror, which reflects them as concentrated rays. Of course, we adjusted the parameters of the mirrors to realize this condition (i.e., at noon of summer solstice, the second mirror receives all rays reflected in the first mirror).

Figure 9 shows the CCCP system performance. The left figure displays the one-year survey. The one-year simulation is an effective and convenient estimation method for showing the performance of daylighting systems (cf. [2]). The illuminance rate r_E is defined as

$$r_E = \bar{E}_{\mathrm{out}} / \bar{E}_{\mathrm{in}}.$$

Here, $\bar{E}_{\mathrm{out}}$ is the outer mean illuminance (i.e., before being reflected in the first mirror), and $\bar{E}_{\mathrm{in}}$ is the mean inner illuminance (i.e., after reflection in the second mirror). When $r_E > 1$, then it means that the illuminance is increased by the daylighting systems. There are some concentric circles, and their centre at noon shows the highest illuminance rate. It corresponds

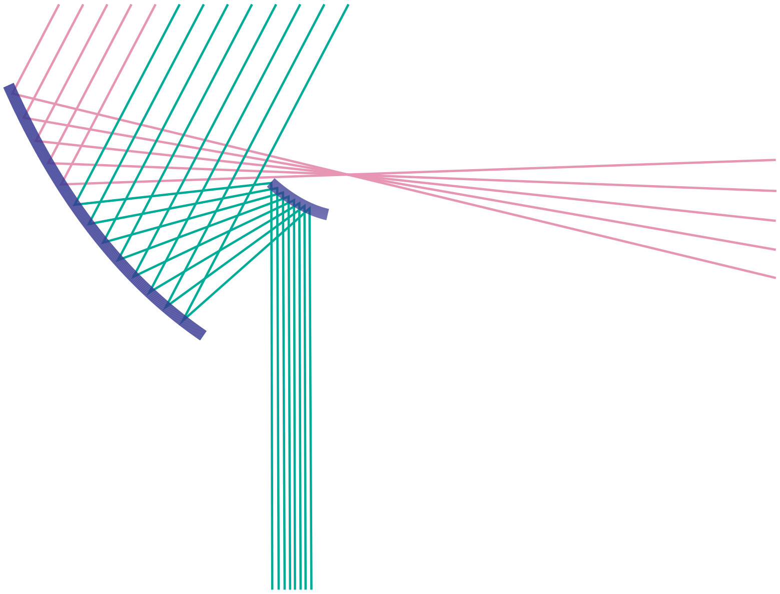

Figure 8: Ray-tracing simulations on 30° latitude at the summer solstice, at 10.00 o'clock (left) and 12.00 o'clock (right). At 10.00, there are two types of rays, converted and reflected rays.

to the summer solstice. The border of the outermost concentric circle at noon corresponds to the winter solstice. This figure of one-year survey shows that the CCCP system is available throughout the year.

The right figure shows three strategies estimation: the maximum effective time $T_{\max}$, the minimum effective time $T_{\min}$, and the annual lighting efficiency p_{yr}. The maximum effective time corresponds to the duration time at the summer solstice. The minimum effective time corresponds to the duration time at the winter solstice. The annual lighting efficiency cor-

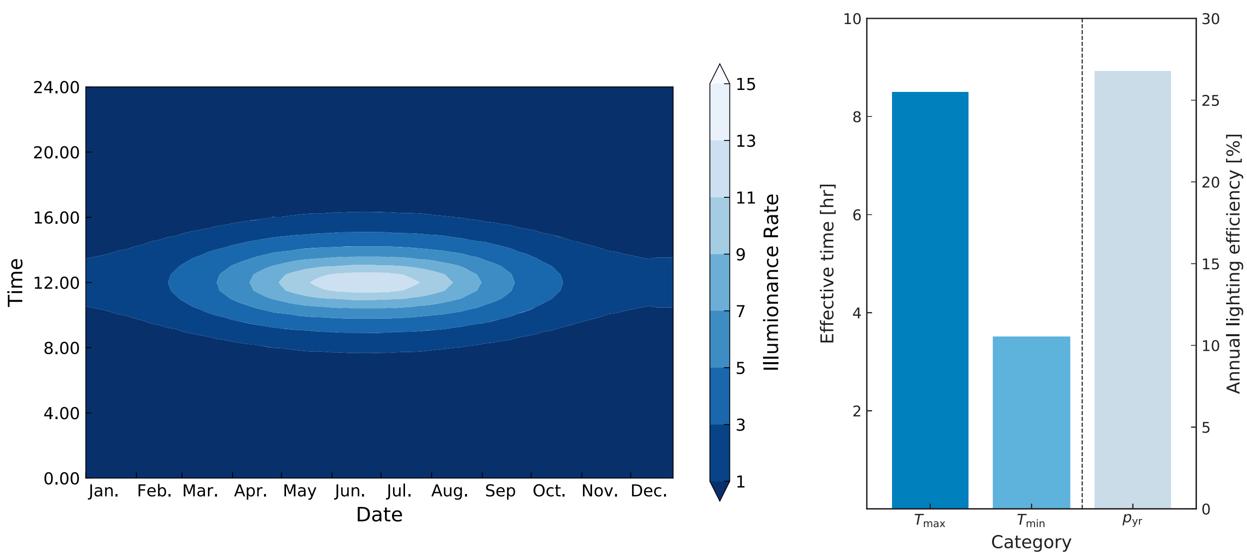

Figure 9: The CCCP system performance. The left figure shows a one-year survey of the system. The right figure is an estimation with three strategies: the maximum effective time, the minimum effective time, and the annual lighting efficiency.

responds to the percentage of a daylighting system available throughout the year. Basically, these three values show the one-year survey numerically. The one-year survey is a useful visual information when needed to know roughly the performance. The three strategies estimation helps to obtain the detailed performance or an obvious comparison.

5. Summary and conclusion

Aiming to propose the CCCP system, we investigate the geometric feature of the CCCP system, suggest an application as a daylighting system and analyse the performance of the CCCP daylighting system. The results are summarised below (point 1–3):

1. The well-known feature of a parabolic mirror is shown in eq. (1). This ODE can be solved in the form (5). These equations correspond to the convex side and the concave side of a parabolic mirror. In addition, this feature can be comprehended visually and geometrically, as shown in Figure 4.
2. The CCCP system can separate three type conditions: the slightly rotated, the rotated, and the excessively rotated. Rays of light in the CCCP system can be classified by four types: Region (A), Region (B), Region (C), and Region (X). The first mirror should be adjusted such that rays in Region (B) are not interrupted.
3. We propose an application as a daylighting system, simulate the CCCP daylighting system using ray-tracing codes, and estimate the performance.

For future works, we seek other useful applications and investigate the geometric feature of the CCCP system in more detail. Of course, we will propose an even more effective CCCP daylighting system.

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