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The Cube: Its Billiards, Geodesics, and Quasi-Geodesics

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Abstract. The cube and its higher dimensional counterparts ("n-cubes") are well-known basic polytopes with a well-studied symmetry group, and from them one easily can derive other interesting polyhedrons and polytopes by a chamfering or adding process (cf. [3, 4, 9]).
The cube's geodesics and (inner) billiards, especially the closed ones, are already treated in [5, 2]. Hereby, a ray's incoming angle must equal its outcoming angle. There are many practical applications of reflections in a cube's corner, as, e.g., the cat's eye and retroreflectors or reflectors guiding ships through bridges. Geodesics on a cube can be interpreted as billiards in the circumscribed rhombidodecahedron. This gives a hint, how to treat geodesics on arbitrary polyhedrons. When generalising reflections to refractions, one has to apply Snellius' refraction law saying that the sine-ratio of incoming and outcoming angles is constant. Application of this law (or a convenient modification) to geodesics on a polyhedron will result in trace polygons, which might be called "quasi-geodesics". The concept "pseudo-geodesic", coined for curves c on smooth surfaces Φ, is defined by the property of c that its osculating planes enclose a constant angle with the normals n of Φ. Again, this concept can be modified for polyhedrons, too. We look for these three types of traces of rays in and on a 3-cube and a 4-cube.

Key Words: Polyhedron, cube, geodesic polygon, billiard polygon, Snellius' refraction law.

MSC 2010: 51M20, 52B10, 51N05

1. Introduction: The cube and related polyhedra

The properties of a cube or "hexahedron", as it is called by the ancient Greeks, are well-known. It is "the" space filling polyhedron in Euclidean space, and it also occurs, virtually or as a common set of vertices, at the other Platonic solids. Therefore, and roughly speaking,

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one can say that its symmetry group "dominates" that of the other Platonic solids and even those of Archimedean solids. Most of the latter are derived from the former by chamfering edges and vertices and even the snub cube and snub dodecahedron, too, can be related to the cube. The cube "generalizes" the square (which generalizes the segment) with respect to the dimension of the Euclidean spaces. The hypercubes in the 4-space resp. in an n-space are further generalizations in that sense.

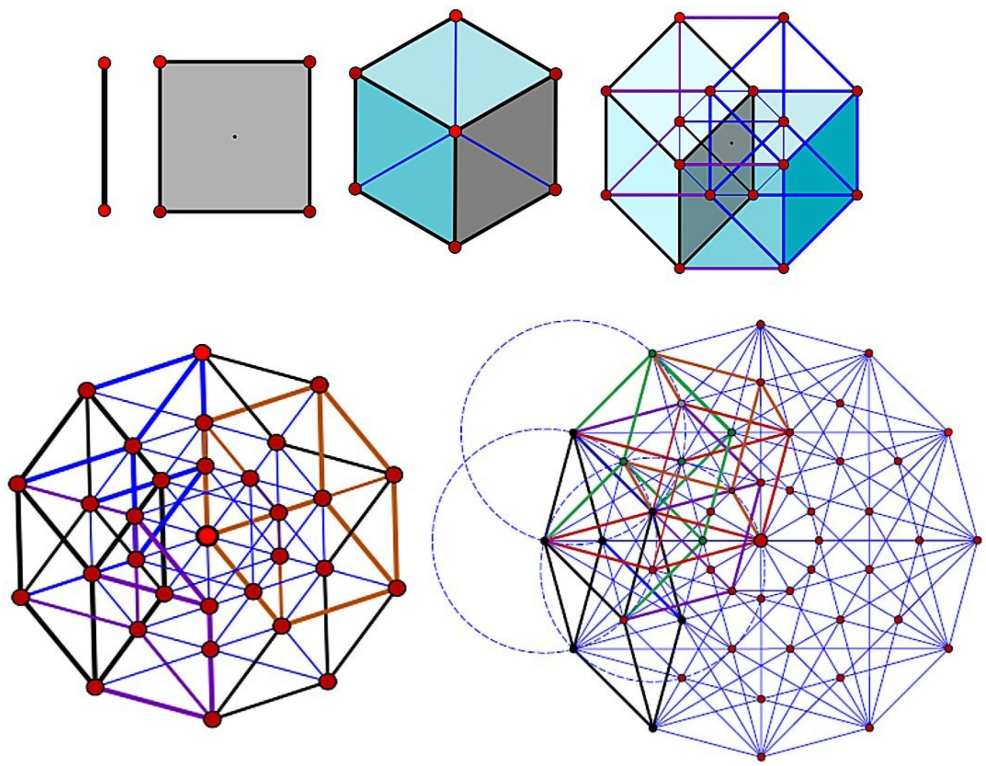

Figure 1: Series of "diagonal-projections" of n-cubes ($n = 1, \ldots, 6$). Some characteristic face-cubes are marked with edges or faces in different colors.

By the way, the normal projection of an n-cube in direction of a "diagonal-$(n-2)$-plane" onto a 2-plane results in $n-2$ concentric regular $2n$-gons. The eventually remaining $m = 2^n - 2n(n-2)$ vertices map to the common center of the $2n$-gons (see Figure 1). Similar projections onto a 3-space deliver space-filling polyhedrons and are treated, e.g., in [7].

2. Reflections and shortest paths

The reflection law states that "the ray's incoming angle equals its outcoming angle". We imagine that, at the point P, where the ray meets the reflecting (hyper-) surface Φ, we replace Φ by a planar mirror tangent to Φ and its normal n. A first and fundamental consequence of this consideration is that P is a regular point of Φ.

If Φ is the boundary of two media with different refractivity, one applies *Snell's law of refraction* stating that "the sine ratio of the angles of incoming and outcoming rays is constant", (see, e.g., [6]).

A light ray r starting from point A, passing through the "receiver point" B, and meeting Φ in between at P, must trace the fastest path from A over P to B. If the refractivity on both sides Φ is the same or r is reflected at Φ, this fastest path is also the shortest (with respect to Euclidean geometry). The shortest path problem connects the topic to the concept of geodesics, a concept of differential geometry. For a curve c on a (regular) surface Φ to be the shortest connection between two points $A, B \in \Phi$, the osculating planes of c contain the normals n of Φ along c. The heuristic imagination that, locally, c is reflected at the tangent plane of Φ at each point $P \in c$, suits very well to that orthogonality property of geodesics.

We aim at dealing with polyhedra and construct (closed) *geodesics* on them. E.g., a rubber band between two points of adjacent faces will cross the common edge such that the incoming angle equals the outcoming angle. This means that the trace of the rubber band can be interpreted as a reflection path at the (outer) symmetry plane of the two faces. Figure 2 shows such a closed geodesic on a cube passing each face of the cube only once. It is also an inner (closed) reflection path, a so-called *billiard*, in the subscribed rhombi-dodecahedron, meeting six of its twelve faces.

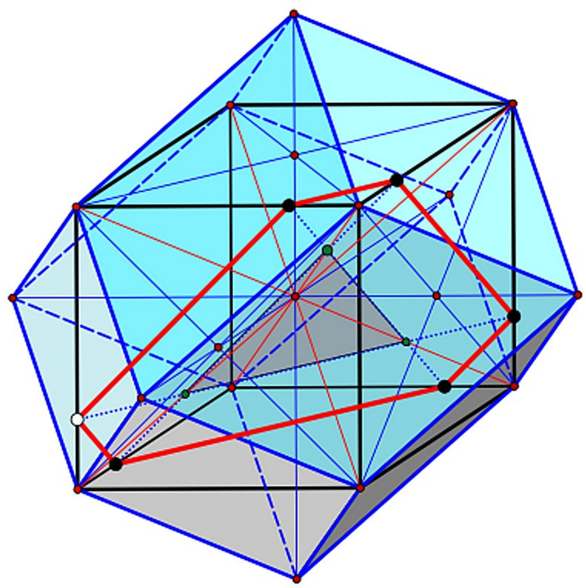

Figure 2: A closed geodesic (red) of a cube is a planar hexagon with sides parallel to face diagonals of the cube.

The construction of a closed billiard path in a square (Figure 3) gives a hint, how to proceed in higher dimensions:

By suitable reflections of the square we get a rectification of the path. When we choose the start- and endpoint of this rectification on corresponding edges at the same position, we force the path to become closed. If these corresponding edges are parallel, the closure is independent of the starting point. A path meeting all sides of the square and being shortest must meet each side only once and therefore have sides parallel to the diagonals. In Figure 3, the exceptional billiard paths through vertices of the square are marked in red. Obviously, all shortest closed billiard traces have the same length, namely twice the length of the square's diagonal. A ray meeting a vertex of a face or a polyhedron will be excluded from consideration, even though such cases make sense as limits.

In the planar case, for the billiard we start from a (regular) point A on a side of a polygon. In the three-dimensional case, we start from an inner point of a face of a polyhedron, and, for a hypercube, the starting point can be arbitrarily chosen as inner point of its hyperface. The method to receive a closed billiard path is the same as for the square, using reflections of the

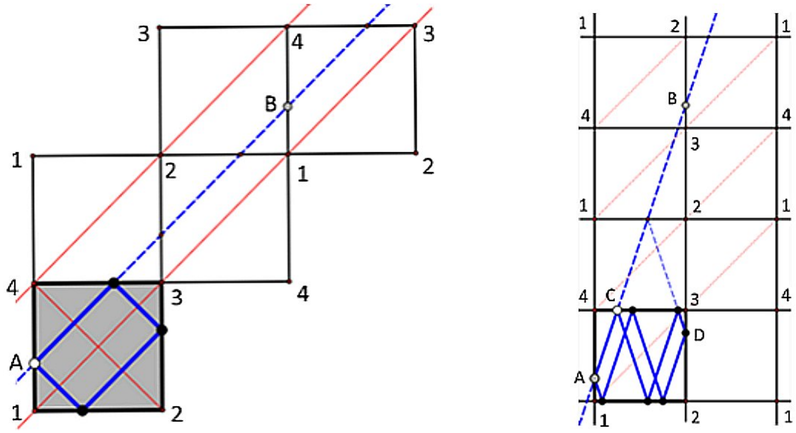

Figure 3: Closed billiards in a square. Suitable reflections of the square give a rectification of the billiard. It occurs as a closed billiard automatically, when choosing start- and endpoint on corresponding (parallel) edges at the same position.

(hyper-) cube at its faces. Figure 4 shows shortest closed billiard traces in a cube emanating from a point P of its faces.

If the trace meets all six face squares of the cube, the partial segments must be parallel to three diagonals of the cube (cf. [5]). Since three diagonals of the cube cannot be coplanar, the resulting trace hexagon cannot be planar. But it is symmetric with respect to the cube's center. As one of the cubes space diagonals is not parallel to the billiard's sides, one expects four such closed billiards starting from the same point P. If the starting point P belongs to a face-square, this number reduces to two (see Figure 4). It is easy to show that all simply closed billiards have the same length, which is twice the diagonal's length (see [5]). We summarize these statements in

Theorem 1. *Through an inner point P of a cube there pass four simply closed billiards meeting all six face squares once. These four billiard hexagons have a second point Q in common, which is symmetric to P with respect to the cube's center. Through an inner point P of a face square pass two closed billiards, through an inner point P of an edge the billiard hexagons degenerate to one parallelogram in the diagonal plane through this edge.*

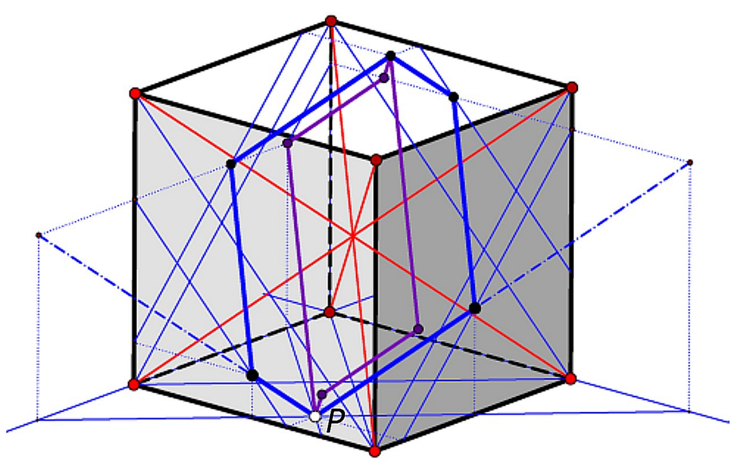

Figure 4: Closed billiards in a cube meeting each face exactly once. The sides of the hexagons are parallel to three diagonals of the cube.

3. Geodesics, pseudo- and quasi-geodesics

In Section 2, we started with the reflection law and Snell's refraction law and mentioned that, for a (regular) surface in Euclidean 3-space, its geodesics have at each point P an osculating plane containing the surface normal n at P. A generalization of the concept "geodesic" reads as follows (see [1] and Figure 5):

Definition 1. A curve on a surface in Euclidean 3-space is called a *pseudo-geodesic* if, at each of its points P, its osculating plane includes a fixed angle δ with the surface normal n at P. For $\delta = 0$, this curve is an ordinary geodesic, for $\delta = \pi/2$ it is an asymptotic curve.

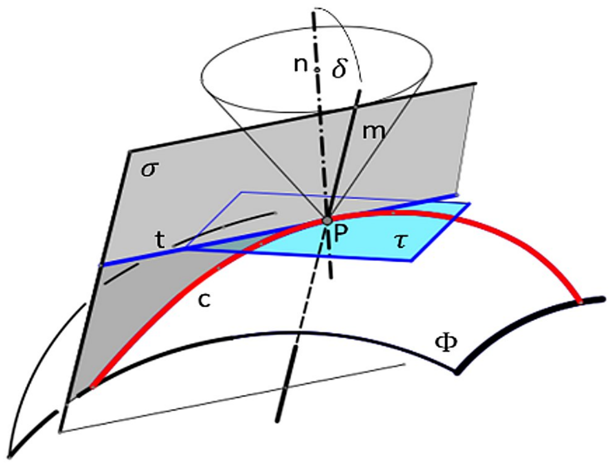

Figure 5: Sketch to Definition 1 of a pseudo-geodesic c on a surface Φ: σ = osculating plane at $P \in c$, τ = tangent plane at P to Φ, t = tangent of c, n = normal to Φ at P, m = principal normal of c, $\delta = \measuredangle nm$.

A natural extension of Definition 1 considers straight line segments also as pseudo-geodesic curves and we will meet them in faces of a polyhedron. There seem to be several possibilities, how to proceed at a point P of a common edge of two faces and we start with describing one by following Definition 1 (see Figure 6):

Let a convex polyhedron be given. We consider two adjacent faces and a point P on their common edge e. As replacement for the tangent plane in the regular case, we use the outer symmetry plane such that also the replacement of the normal n through P becomes well-defined. A ray g_1 in the first face ε_1 with endpoint P shall proceed in face ε_2 as g_2 such that the plane $\gamma := g_1 \vee g_2$ includes a given angle δ with the normal n. Figure 6 shows the situation in front and top projection, where e is a projecting line. All possible planes γ envelop a cone of revolution. Supposing that $d < \measuredangle\varepsilon_1\varepsilon_2$ there are two solutions g_2 to a given ray g_1. If we orient the half lines g_1, g_2, n emanating from P, the two solutions can be distinguished by the sign of the determinant $\det(g_1 g_2 n)$.

Intersecting the object depicted in Figure 6 with a plane parallel to the outer symmetry plane τ of $\varepsilon_1, \varepsilon_2$ allows to find the traces of the planes $\gamma = g_1 \vee g_2$ as tangents to the trace circle of a cone with half apex angle δ. Using the touching point X of the trace of γ resp. the angle $\xi := \measuredangle X'P'x'$ as parameter, the dependence of $\alpha_1(\varphi, \delta; \xi) := \measuredangle eg_1$ and $\alpha_2(\varphi, \delta; \xi) := \measuredangle eg_2$ can be described with $p := \tan\varphi$, $q := \tan\delta$ as follows:

$$G_i = \left(\frac{q}{\cos\xi} \pm p.\tan\xi,\ -p,\ -1\right),\ i = 1, 2, \tag{1}$$

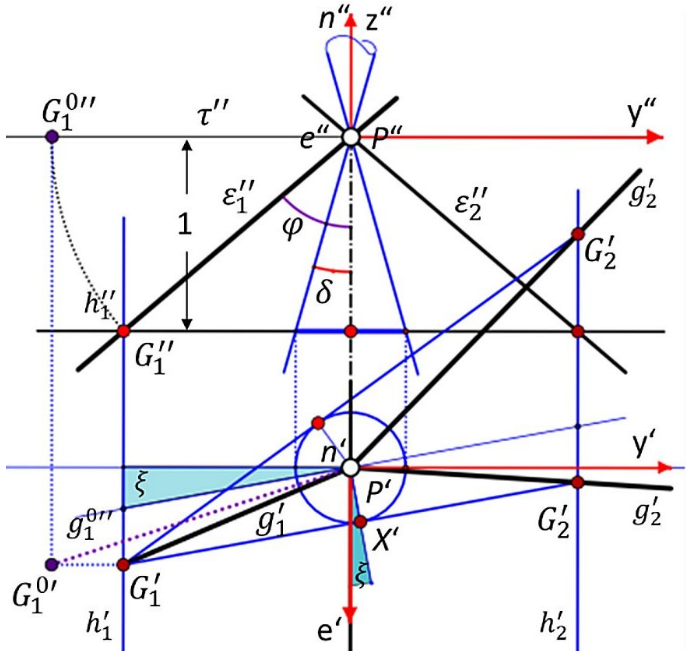

Figure 6: Front and top projection of two face planes $\varepsilon_1, \varepsilon_2$ with the two solutions $g_2 \subset \varepsilon_2$ to an incoming ray $g_1 \subset \varepsilon_1$.

$$\cos\alpha_i(\xi) = \left(\frac{q}{\cos\xi} \pm p.\tan\xi\right) / \sqrt{\left(\frac{q}{\cos\xi} \pm p.\tan\xi\right)^2 + p^2 + 1}. \tag{2}$$

For a given α_1, we can exploit equation (2) only numerically. Looking for a better practicable modification, we recall Snell's refraction law and the fact that the construction of a geodesic uses the net of a polyhedron.

Definition 2. An oriented polygon with vertices on edges of a polyhedron Φ and sides in the faces of Φ is called a quasi-geodesic, if, at each vertex P, which is an inner point of an edge e of Φ, the polygon's sides g_1, g_2 fulfil a "Snell's refraction condition" with respect to the edge e. If α_1, α_2 denote the angles $\measuredangle eg_1$, $\measuredangle eg_2$, the following Snell's refraction conditions SRC_i, $i = 1, 2, 3$, are convenient:

$$\begin{aligned}
&(SRC_1) \quad \sin\alpha_1 : \sin\alpha_2 = s_1 = \text{const.} \quad \text{(Snell's law)},\\
&(SRC_2) \quad \alpha_1 : \alpha_2 = s_2 = \text{const.},\\
&(SRC_3) \quad \tan\alpha_1 : \tan\alpha_2 = s_3 = \text{const.}.
\end{aligned}$$

Obviously, also other simple functions $\alpha_2 = f(\alpha_1)$ might be used. We will focus on (SRC_3), as it is easiest to handle. Figure 7 gives a sketch of the local situation in general within the net of a polyhedron, Figure 8 shows the front and top projection of two face planes of a polyhedron. In case of (SRC_3), for any ratio $\tan\alpha_1 : \tan\alpha_2 = s_3$, the enveloped cone of the planes $\gamma = g_1 \vee g_2$ degenerates into a line a normal to edge e, which can be considered as a proper replacement for the normal n.

For a cube's net, we find the trace of a quasi-geodesic polygon based on the condition (SRC_3) $\tan\alpha_1 : \tan\alpha_2 = 1/2$ (see Figure 9).

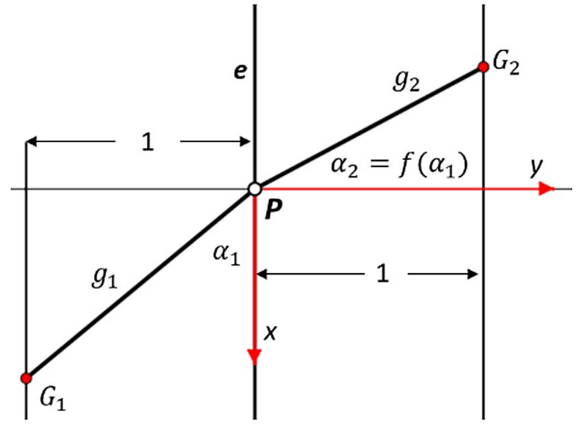

Figure 7: Sketch to Definition 2 of a quasi-geodesic on a polyhedron.

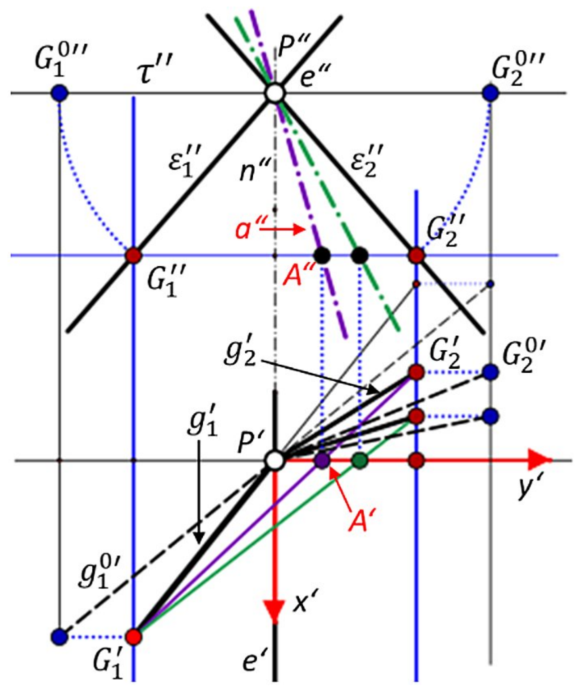

Figure 8: Front and top projection of two face planes $\varepsilon_1, \varepsilon_2$ with a projecting common edge e. In case of (SRC_3), the planes $\gamma = g_1 \vee g_2$ spanned by the in- and outcoming ray pass through a line a.

4. Square, cube, and hypercube: closed geodesics and billiards

In Section 2, we discussed the connection between billiards and geodesics. For example, in Figure 3, the (closed) billiard trace in the square is constructed in the same way as the closed geodesic on a cube, using the net of the cube. The shortest billiard path in the square has sides parallel to the square's diagonals, and so does the shortest geodesic on the cube. The billiard trace in the cube has segments parallel to 3 diagonals and we can expect that a closed geodesic on the hypercube in the Euclidean 4-space, which meets all 8 face cubes, consists of segments parallel to face-cubes' diagonals. To construct such a geodesic polygon, we use a "net" of the hypercube in the 3-space of one of its face cubes. Such a net was depicted by S. Dali in his famous painting *"Corpus Hypercubus"*. We proceed analogue to the cube's

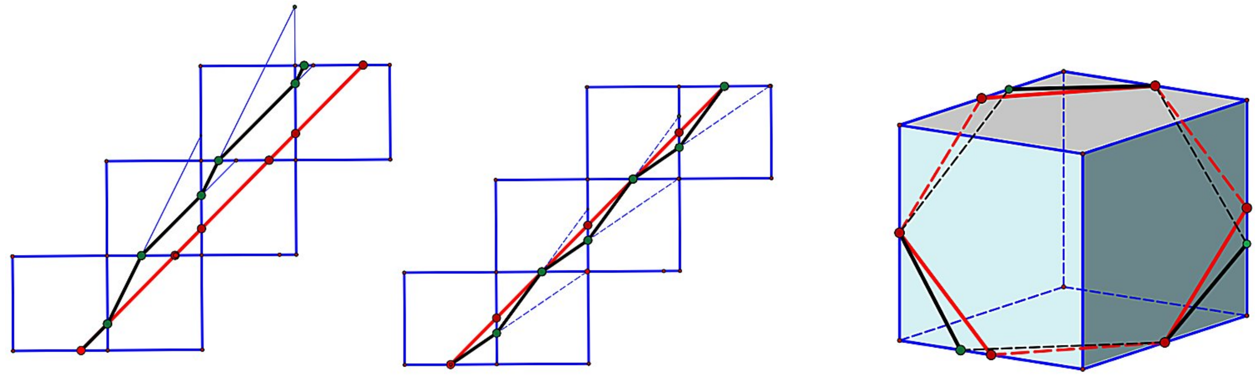

Figure 9: Net of a cube with developed geodesic (red) and quasi-geodesic (black). The axonometric image of the cube shows a closed geodesic and a closed quasi-geodesic.

case (Figure 9) and label the vertices in the net according to their position on the hypercube (Figure 10).

The red diagonal of the depicted cubes in Figure 10 symbolizes the rectified "critical" geodesic passing through vertices of the hypercube, while the black one represents the rectification of one of the general shortest closed geodesics. Figure 11 represents an axonometric view of the hypercube with a special case of such a closed geodesic polygon. It meets four edges of the hypercube and has therefore only 8 sides instead of the twelve sides in the general case. The sides are parallel to diagonals of face cubes, (marked as a dotted star shaped hexagon in Figure 11).

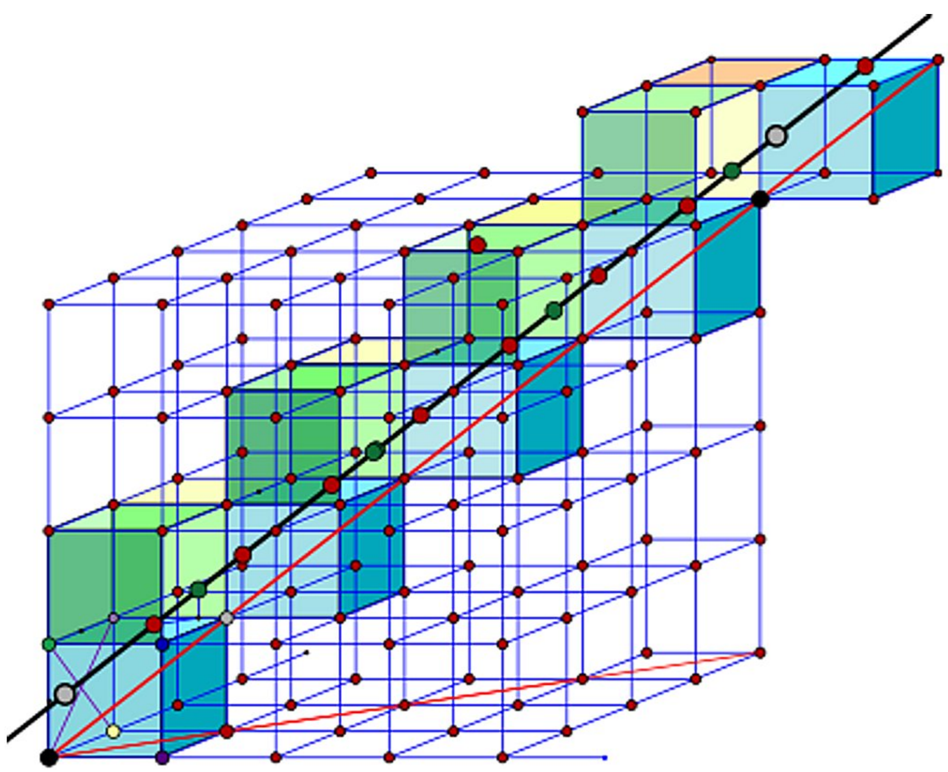

Figure 10: Net of a hypercube with developed shortest closed geodesic (black) together with the critical position of a geodesic through vertices of the hypercube (red).

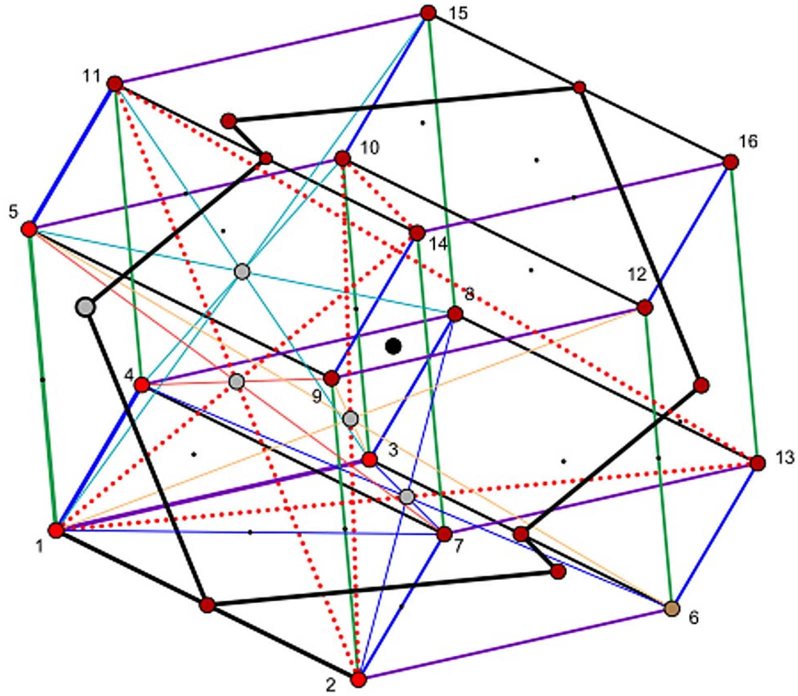

Figure 11: A special geodesic closed polygon on a hypercube. The sides of the polygon are parallel to diagonals of the face cubes of the hypercube.

5. Conclusion

The main aim of this article is to give a hint, how to modify the differential geometric concept "pseudo-geodesic", such that it becomes applicable for polyhedrons. Thereby a new concept "quasi-geodesic" is coined, which is based on generalizations of Snell's refraction law. The idea of interpreting geodesics as billiards in a lower dimensional case can be combined with the concept of quasi-geodesics. Here the reader should quicken his appetite to further and deeper going research. The article is based on a lecture given at the 18th ICGG Milan, August 2018 (see [8]).

A second aim is a didactical one: Even though there seems nothing new to be said about the cube itself, it is a surprisingly good basic object for generalizations and research, too, and one can teach many mathematical concepts based on it. There is a rich fundus of materials in references, see, e.g., Wikipedia, to satisfy one's curiosity.

References

[1] S. Dali: `https://en.wikipedia.org/wiki/Crucifixion_(Corpus_Hypercubus)`, accessed April 18, 2018.

[2] D. Fuchs, E. Fuchs: *Closed Geodesics on Regular Polyhedra.* Moscow Mathematical Journal **7**/(2), 265–279 (2007).

[3] J.F. Gabriel (ed.): *Beyond the Cube: The Architecture of Space Frames and Polyhedra.* Wiley, New York 1997.

[4] H. Martini: *Reguläre Polytope und Verallgemeinerungen.* In O. Giering, J. Hoschek (eds.): *Geometrie und ihre Anwendungen,* Carl Hanser Verlag, München Wien 1994, pp. 247–281.

[5] H. Schumann: *Raumgeometrische Entdeckungen durch Analogisieren am Beispiel "Würfelbillard"*. IBDG **35**/(2), 20–29 (2016).

[6] Snell's refraction law: `https://en.wikipedia.org/wiki/Snell%27s_law`, accessed April 20, 2019.

[7] L. Vörös: *N-Zonotopes and their Images: from Hypercube to Art in Geometry.* Proceedings 13th International Conference on Geometry and Graphics, Dresden/Germany 2008, pp. 253–254.

[8] G. Weiss: *The Cube: Its Relatives, Geodesics, Billiards, and Generalizations.* Proceedings 18th International Conference on Geometry and Graphics, Milan/Italy 2018, Springer-Verlag, AISC vol. 809, p. 471–483.

[9] G.M. Ziegler: *Lectures on Polytopes.* Graduate Texts in Mathematics Nr. 152, Springer Verlag, 1995.

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