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Foldable and Self-Intersecting Polyhedral Cylinders Based on Triangles

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Dedicated to Prof. Kurt S. Anderson, Rensselaer Polytechnic Institute

Abstract. An infinitely long strip of paper is divided by a zigzagging line into congruent triangles with side lengths 1, a and b. On both rims of the strip the vertices V_k of the triangles are labeled from $-\infty$ to $+\infty$ with a shift n such that $(\mathrm{V}_0\mathrm{V}_1\mathrm{V}_n)$ is a representative triangle. Along the sides of the triangles folds with alternating fold angles are made. Under certain conditions on a, b and n and with appropriately chosen fold angles it is possible to bring every vertex V_k on the upper rim in coincidence with the vertex V_k of equal name on the lower rim. The resulting body is a polyhedral cylinder (PC). The vertices are distributed at equal intervals along a helix on the surface of a circular cylinder. For given lengths a and b up to $(n-2)$ PCs can be formed. There are foldable PCs and self-intersecting PCs. In the case $n=4$ self-intersecting PCs consist of a *core body* with congruent nonconvex pentagonal faces and of an infinite number of congruent tetrahedra, each tetrahedron in edge-to-edge contact with the core body along three edges.

Key Words: polyhedral cylinder, core body, foldability, flexible polyhedra, periodic framework

MSC 2010: 52C25, 53A17, 51M20

1. Problem statement

Figure 1 depicts an infinitely long strip of congruent triangles of side lengths 1, a and b. Every vertex V_k $(k=-\infty,\dots,0,\dots,+\infty)$ is represented twice, once on the lower rim of the strip and once on the upper rim with a shift n. The integer n of V_n in the representative triangle $(\mathrm{V}_0\mathrm{V}_1\mathrm{V}_n)$ is, in addition to a and b, the third parameter of the system. The coordinates x, y of this vertex are related to a, b:

$$a^2=x^2+y^2, \quad b^2=(x-1)^2+y^2, \quad x=(1+a^2-b^2)/2. \tag{1}$$

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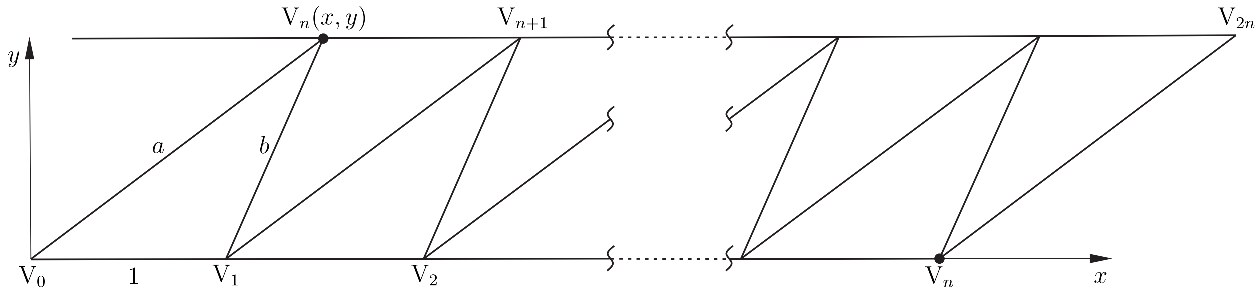

Figure 1: Strip of congruent triangles. Representative triangle ($V_0V_1V_n$). Coordinates (a, b) and (x, y) of V_n .

The sides of the triangles are interpreted as rods of a truss. The problem to be solved is stated as follows:

Given the parameters a, b and n, in how many ways can the truss be assembled such that vertices of equal name coalesce? Can the assembly be accomplished by folding the strip of triangles or do triangles intersect each other?

2. Solution

In the assembled state every vertex is center of a cluster of six triangles. Every triangle is part of three clusters. Figure 1 and the assembled structure are invariant with respect to

(a) replacing the label k by $k+m$ (m arbitrary),

(b) replacing the label k by $-k$.

With an appropriate value of m every cluster of six triangles is centered at V_0, and its diagonals are V_{-1}–V_1, $V_{-(n-1)}$–V_{n-1} and V_{-n}–V_n. The replacement (b) is a 180°-rotation about the midpoint of the line $V_{-(n-\ell)}$–$V_{n-\ell}$ with $\ell = n/2$ or $(n+1)/2$ depending on whether n is even or odd.

From these invariance properties it follows that in the assembled state all clusters of six triangles each are congruent. This congruence requires that the triangles are the faces of a polyhedral cylinder (PC). The vertices V_k are distributed at equal intervals along a helix on the surface of a straight circular cylinder of unknown radius r. In a (ξ, η, ζ)-system with its ζ-axis along the axis of the circular cylinder and with V_0 on the ξ-axis, the position vector $\vec{r}_k$ of V_k (k arbitrary) has the coordinates

$$[\, rc_k,\; rs_k,\; kz\,] \tag{2}$$

with only three unknowns φ, r, z ($0 \le z \le 1$). Here and in what follows the abbreviations $c_k = \cos k\varphi$ and $s_k = \sin k\varphi$ are used. For c_1 and s_1 also the simpler notations c and s, respectively, are used. The unknowns are determined by the side lengths of the representative triangle ($V_0V_1V_n$). Application of the formula $(\vec{r}_i - \vec{r}_j)^2 = 2r^2[1-\cos(i-j)\varphi] + (i-j)^2z^2$ results in the equations

$$2r^2(1-c) + z^2 = 1, \tag{3}$$

$$2r^2(1-c_{n-1}) + (n-1)^2z^2 = b^2, \quad 2r^2(1-c_n) + n^2z^2 = a^2. \tag{4}$$

Decoupling is based on the formulas

$$
\begin{aligned}
&1-c_k=(1-c)f_k(c), \qquad k^2-f_k=(1-c)F_k(c),\\
&f_k(c)=k+2\sum_{\ell=1}^{k-1}(k-\ell)c_\ell\,,\ F_2=2,\ F_k(c)=\frac{1}{3}\,k(k^2-1)+\frac{2}{3}\sum_{\ell=2}^{k-1}\ell(\ell^2-1)c_{k-\ell}\ (k\ge 3),\\
&(k+1)^2f_k-k^2f_{k+1}=-(1-c)[(k+1)^2F_k-k^2F_{k+1}],\\
&f_{k+1}-f_k=1+2\sum_{\ell=1}^{k}c_\ell\,, \qquad F_{k+1}-F_k=2\sum_{\ell=1}^{k}f_\ell\,.
\end{aligned}
$$

Elimination of $2r^2(1-c)$ from (3) and (4) leads to

$$(1-z^2)f_{n-1}+(n-1)^2z^2=b^2, \quad (1-z^2)f_n+n^2z^2=a^2. \tag{5}$$

Further decoupling results in the equations

$$(n^2-a^2)F_{n-1}-[(n-1)^2-b^2]F_n=0, \tag{6}$$

$$z^2=1-2r^2(1-c), \quad r^2=\frac{(n-1)^2-b^2}{2(1-c)^2F_{n-1}}\,. \tag{7}$$

Equation (6) is a polynomial equation of the order $n-2$ for c. From (1) it follows that lines $c=$const in Figure 1 are the circles

$$(x-x_0)^2+y^2=\varrho^2, \qquad x_0=n-\varrho, \qquad \varrho=n-1-\frac{F_{n-1}}{F_n-F_{n-1}}. \tag{8}$$

All circles are passing through the point $x=n$ (the vertex V_n) on the x-axis. Let $\mathrm{C_E}$ be the circle passing through the vertex V_n of the equilateral triangle $(\mathrm{V_0V_1V}_n)$. Its radius is

$$\varrho_{\mathrm{E}}=\frac{n^2-n+1}{2n-1}. \tag{9}$$

From $z^2\le 1$ it follows that $r^2>0$ if $z^2>0$. PCs with $z=0$ are flatfolded. On the line $z=0$ $a^2=f_n$, $b^2=f_{n-1}$. This line is the higher-order cycloid (Wunderlich [2])

$$x=1+\sum_{k=1}^{n-1}c_k, \qquad y=\Big|\sum_{k=1}^{n-1}s_k\Big| \qquad (0\le\varphi\le\pi). \tag{10}$$

The formula for y is proved by induction. In the interval $0\le\varphi\le\pi$ the cycloid is composed of a single branch connecting the points $(x=n\,,\ y=0)$ and $(x=y=0)$ and of $n-2$ branches connecting the points $(x=1\,,\ y=0)$ and $(x=y=0)$.

2.1. The case $n=3$

Equations (6), (7) and (8) are

$$c=\frac{a^2-4b^2+7}{2(b^2-4)}, \quad z^2=\frac{a^2-(b^2-1)^2}{15+a^2-6b^2}, \quad r^2=\frac{(4-b^2)^3}{(15+a^2-6b^2)^2}, \quad \varrho=2-\frac{1}{2c+3}\,. \tag{11}$$

A single PC exists for every point $\mathrm{V}_3(x,y)$ in the domain bounded by the ordinary cycloid $x=1+\cos\varphi+\cos 2\varphi$, $y=|\sin\varphi+\sin 2\varphi|$ $(0\le\varphi\le\pi)$ and by the circle $(x-2)^2+y^2=1$. In Figure 2 this domain is shaded. All PCs are foldable.

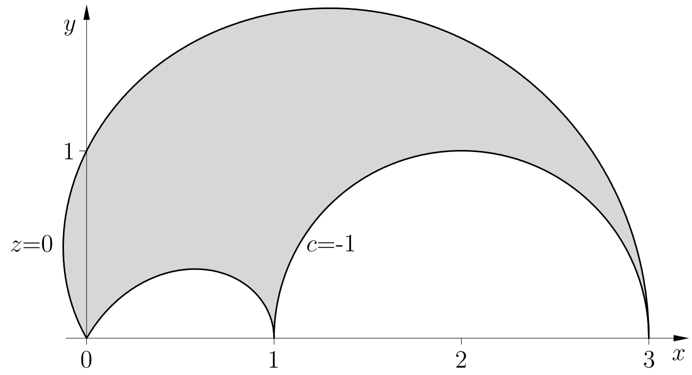

Figure 2: Boundaries $z = 0$ and $c = -1$ of the admissible domain for V_3.

2.2. The case $n = 4$

Equations (6), (7) and (8) are

$$2(9-b^2)c^2 + (20+a^2-4b^2)c + 2(2+a^2-2b^2) = 0, \tag{12}$$

$$z^2 = 1 - 2r^2(1-c), \quad r^2 = \frac{9-b^2}{8(1-c)^2(2+c)}, \quad \varrho = 3 - \frac{c+2}{2c^2+3c+2}. \tag{13}$$

The roots $c_{1_{1,2}}$ of (12) satisfy the equation $2(1+c_{1_1}+c_{1_2}) + c_{1_1}c_{1_2} = 0$ independent of a, b. In Figure 3 the radius ϱ of the circles $c = \text{const}$ is shown as function of c. The radius has its minimum $\varrho_{\min} = \frac{4}{7}(4-\sqrt{2})$ for $c = \sqrt{2}-2$. In Figure 4 the cycloid $z = 0$ is shown in thick lines. It is tangent to the circles $c = \text{const}$ with $\varrho = 2$ and $\varrho = \varrho_{\min}$ (with the latter one at the point $x = 2(23\sqrt{2}-32)$, $y = [8(746\sqrt{2}-1055)]^{1/2}$). According to Figure 3 the number of solutions $-1 \le c \le 0$ is two if V_4 is located in the domain between the circles $\varrho = \varrho_{\min}$ and $\varrho = 2$. The number of solutions $z^2 > 0$ changes by one every time V_4 crosses the line $z = 0$. The number is two for the equilateral triangle. Hence the conclusions:

Two PCs can be formed if V_4 *is located either in the vertically hatched domain or in the dark-shaded domain. A single PC can be formed if* V_4 *is located in the horizontally hatched domain. No PCs can be formed if* V_4 *is located elsewhere.*

In Figure 4 also the circle C_E with radius $\varrho_E = 13/7$ is shown. For all points V_4 on this circle Eq. (12) is $16c^2 + 17c + 2 = 0$. The roots are $c = (-17 \pm \sqrt{161})/32$. For the equilateral triangle Eqs. (13) yield the associated parameters z and r :

$$1.\quad c = (-17+\sqrt{161})/32, \quad \varphi \approx 97.743^\circ, \quad z \approx .2347, \quad r \approx .6453, \tag{14}$$

$$2.\quad c = (-17-\sqrt{161})/32, \quad \varphi \approx 158.090^\circ, \quad z \approx .1801, \quad r \approx .5010\,. \tag{15}$$

The PC with the parameters (14) is foldable. It is shown in Figure 6. The PC with the parameters (15) is not foldable. Not foldable means that triangles intersect. In Figure 5, a row of equilateral triangles is shown in the unfolded state. However, what follows is valid also for intersecting irregular triangles with side lengths 1, a and b. In the assembled state two triangles sharing a single vertex intersect. As example, the triangles $(V_0V_1V_4)$ and $(V_1V_2V_5)$ are considered. The line segment V_1–A_1 on the triangle $(V_0V_1V_4)$ coalesces with the line segment V_1–V_1 on the triangle $(V_1V_2V_5)$ where A_1 is a point on the edge V_0–V_4 and S_1 a point inside the triangle $(V_1V_2V_5)$. A_1 and S_1 have the same distance ℓ_1 from V_1. From the invariance properties it follows that all pairs of triangles sharing a single vertex intersect in like

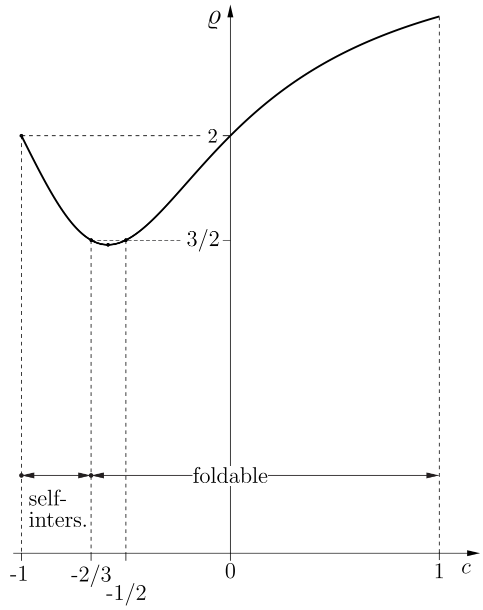

Figure 3: Graph of the function
$\varrho(c)$ for $n = 4$.

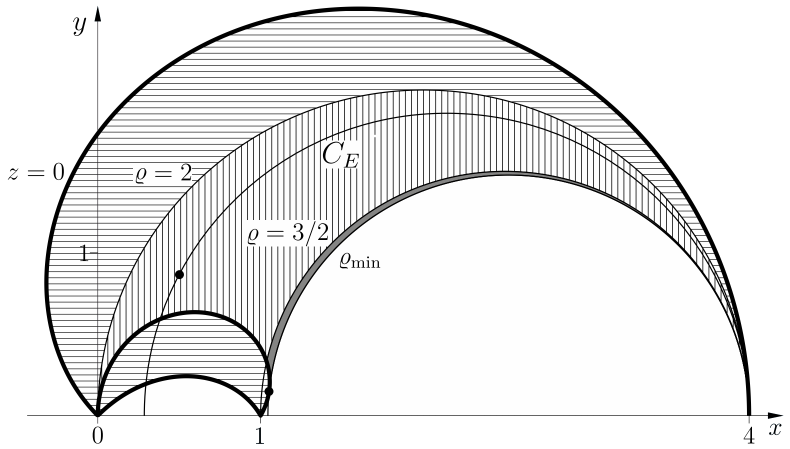

Figure 4: Admissible domains for V_4.

manner. This explains the line segments V_k–A_k, V_k–S_k, V_k–A'_k and V_k–S'_k ($k = 0, \pm 1, \ldots$). In the assembled state S'_3 coalesces with A'_3. From this it follows that the line segment $S'_3 - A_0$ coalesces with the line segment A'_3–S_0. Hence the conclusion:

The triangle $(V_0V_1V_4)$ *is intersected*
by the triangle $(V_1V_2V_5)$ *in the line segment* V_1–A_1,
by the triangle $(V_{-1}V_0V_3)$ *in the line segment* V_0–S_0,
by the triangle $(V_{-1}V_2V_3)$ *in the line segment* A'_3–S_0 *and at* S_0 *by the edge* V_{-1}–V_3.

In terms of the lengths u and v shown in the figure, the lengths ℓ_1 and ℓ_2 are calculated from the formulas $m = vy/[b + v(x-1)]$ (this is the slope of V_0–B_0),

$$\ell_1^2 = 1 + u^2 - \frac{2xu}{a}, \qquad \ell_2^2 = \ell_1^2 + (a-u)^2 - 2\ell_1\Big(1 - \frac{u}{a}\Big)\frac{x + my}{\sqrt{1+m^2}}\,. \tag{16}$$

The calculation of u and v themselves is based on the following formulas (Wittenburg [1, pp. 64–67]):

The normal vector $\vec{n}$ of a plane and an arbitrary point $\vec{r}_A$ in this plane define the vector $\vec{m} = -\vec{n}/(\vec{r}_A \cdot \vec{n})$ of the equation $\vec{m} \cdot \vec{r} = -1$ governing the plane of the triangle. The vectors

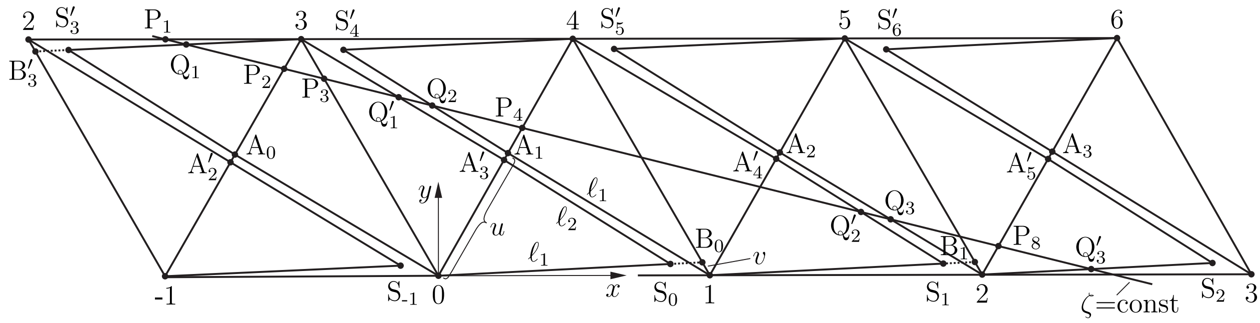

Figure 5: $n = 4$. Lines of intersection and line $\zeta = $const.

$\vec{m}_1$ and $\vec{m}_2$ of two planes determine the Plücker vectors $\vec{v} = \vec{m}_1 \times \vec{m}_2$ and $\vec{w} = \vec{m}_2 - \vec{m}_1$ of the line of intersection g of these planes. A line through points $\vec{r}_{\mathrm{B}}$ and $\vec{r}_{\mathrm{C}}$ in one of the planes has the second Plücker vector $\vec{w}_1 = \vec{r}_{\mathrm{B}} \times \vec{r}_{\mathrm{C}}$. The point of intersection of this line with g is $\vec{r} = \vec{w}_1 \times \vec{w}/(\vec{w}_1 \cdot \vec{v})$. The distance u of A_1 from V_0 and the distance v of B_1 from V_2 are

$$u = a\,\frac{\mathrm{n^{th}\ coordinate\ of\ }(\vec{r}_{\mathrm{A}_1} - \vec{r}_0)}{\mathrm{n^{th}\ coordinate\ of\ }(\vec{r}_4 - \vec{r}_0)}, \qquad v = b\,\frac{\mathrm{n^{th}\ coordinate\ of\ }(\vec{r}_{\mathrm{B}_1} - \vec{r}_2)}{\mathrm{n^{th}\ coordinate\ of\ }(\vec{r}_5 - \vec{r}_2)} \tag{17}$$

for $n = 1, 2, 3$. Evaluation with $n = 2$ results in the expressions $u/a = N_1/D_1$, $v/b = (N_2/D_2 - s_2)/(s_5 - s_2)$ with

$$\begin{aligned}
N_1 &= (s_4 - 4s_1)(4s_2 + 2s_4 - 8s_1 - s_3 - s_5),\\
D_1 &= s_4(26s_1 - 8s_2 - 3s_3 - 4s_4 + 3s_5) - 4(s_1 + s_3 - s_4)(-7s_1 + 4s_2 + s_4 - s_5),\\
N_2 &= -s_3N_1 + (5s_2 - 2s_5)(s_1 + s_3 - s_4)^2,\\
D_2 &= (s_1 + s_3 - s_4)(37s_1 - 10s_2 + 3s_3 - 14s_4 + 6s_5) - s_3(26s_1 - 8s_2 - 3s_3 - 4s_4 + 3s_5).
\end{aligned}$$

With addition theorems this is reduced to

$$\frac{u}{a} = \frac{c^2 + (c+1)^2}{2c^2} \geq \frac{1}{2}, \qquad \frac{v}{b} = \frac{c+1}{(2c+1)(3c+1)}. \tag{18}$$

From Figure 5 it was learned that the line segments V_0–A_0 and V_0–S_0 and the line segments V_3–S'_3 and V_3–A'_3 coalesce pairwise when the PC is formed. This means that the congruent triangles ($\mathrm{V}_0\mathrm{A}_0\mathrm{V}_3$) and $\mathrm{V}_0\mathrm{A}'_3\mathrm{V}_3$) with side lengths ℓ_1, $a - u$, b and the congruent triangles ($\mathrm{V}_0\mathrm{S}_0\mathrm{A}'_3$) and ($\mathrm{V}_3\mathrm{S}'_3\mathrm{A}_0$) with side lengths ℓ_1, $a - u$, ℓ_2 are the faces of a tetrahedron labeled $\mathrm{T}(0, 3)$.

In like manner tetrahedra $\mathrm{T}(k, k + 3)$ are defined for $k = \pm 1, \pm 2, \ldots$. Every face of a tetrahedron is a section of a triangle of vertices. Every triangle of vertices is divided into two triangles which are faces of (different) tetrahedra and a nonconvex pentagon, for example, the pentagon ($\mathrm{S}_0\mathrm{V}_0\mathrm{V}_1\mathrm{A}_1\mathrm{A}'_3$) on the triangle ($\mathrm{V}_0\mathrm{V}_1\mathrm{V}_4$) and the pentagon ($\mathrm{S}'_4\mathrm{V}_4\mathrm{V}_3\mathrm{A}'_3\mathrm{A}_1$) on the triangle ($\mathrm{V}_0\mathrm{V}_3\mathrm{V}_4$). The congruent pentagons are the faces of a new PC referred to as *core body*.

The core body is in edge-to-edge contact with each tetrahedron along three edges, for example, with the tetrahedron $\mathrm{T}(0, 3)$ along the edges V_0–A_0 = V_0–S_0, A_0–S'_3 = S_3–A'_3 and V_3–S'_3 = V_3–A'_3. The pentagon has side lengths ℓ_1, 1, ℓ_1, $2u - a = a(1 + c)^2/c^2$, ℓ_2.

Let α_1, β_1, α_1, β_4, α_2 be the fold angles at these sides. β_1 and β_4 are the fold angles at the edges V_0–V_1 and V_0–V_4, respectively. α_1 and α_2 are the angles made by the triangle ($\mathrm{V}_0\mathrm{V}_1\mathrm{V}_4$) with the intersecting triangles ($\mathrm{V}_{-1}\mathrm{V}_0\mathrm{V}_3$) and ($\mathrm{V}_{-1}\mathrm{V}_2\mathrm{V}_3$), respectively. The cosine of every angle is the scalar product of the unit normal vectors of the two triangles involved. With (2) the expressions are obtained (β_3 is the angle at the edge V_0–V_3):

$$\left.\begin{aligned}
\cos\beta_1 &= (r/y)^2[z^2(-25c_1 - 6c_3 + 8c_4 - c_7 + 24) - r^2(s_1 + s_3 - s_4)^2],\\
\cos\beta_4 &= (r/y)^2[z^2(8c_1 - 16c_2 + 24c_3 - 10c_4 - 6) - r^2(s_1 + s_3 - s_4)^2],\\
\cos\beta_3 &= (r/y)^2[z^2(-6c_1 - 17c_3 + 24c_4 - 9c_5 + 8) - r^2(s_1 + s_3 - s_4)^2],\\
\cos\alpha_1 &= (r/y)^2[z^2(26c_1 - 16c_2 + 3c_3 - 4c_4 + 3c_5 - 12) + r^2(s_1 + s_3 - s_4)^2],\\
\cos\alpha_2 &= (r/y)^2[z^2(22c_1 - 32c_2 + 9c_3 + c_5) + r^2(s_1 + s_3 - s_4)^2].
\end{aligned}\right\} \tag{19}$$

We now return to Eqs. (18). Intersection of triangles occurs under the conditions $u/a < 1$ and $0 < v/b < 1$. This is the condition $-1 < c < -2/3$. In Figure 3 this interval is marked *self-intersecting*. PCs with either $c = -1$ or with $-2/3 \le c < 1$ are foldable. PCs with $c = -1$ or with $c = -2/3$ are both foldable and self-intersecting. In Figure 4 the circles $c = -1$ ($\varrho = 2$) and $c = -2/3$ ($\varrho = 3/2$) are shown. Hence the conclusions:

If the number of PCs for a given triangle is one (V_4 *in the horizontally hatched domain*)*, then this PC is foldable. If the number is two, then two cases have to be distinguished: Two foldable PCs exist if* V_4 *is located in the dark-shaded domain. Two PCs exist, one foldable and one self-intersecting, if* V_4 *is located in the vertically hatched domain. Every self-intersecting PC consists of a core body with congruent nonconvex pentagonal faces and of an infinite number of congruent tetrahedra.*

The two-parametric family of core bodies is one of the most interesting results of this investigation.

Example 1:

For equilateral triangles two PCs with the parameters (14) (foldable) and (15) (self-intersecting) were obtained. Both PCs have the following properties in common. A single cluster of six triangles reaches more than half around the circular cylinder. Since φ/π is irrational no two vertices are located on one and the same generator of the circular cylinder.

Equations (19) can be given the forms

$$\cos\beta_1 = (49c+50)/48,\ \cos\beta_4 = (16c+5)/12,\ \cos\beta_3 = (4c+3)/3,\ \cos\alpha_1 = -1/12$$

(the same for both PCs), $\cos\alpha_2 = (8c+7)/12$. The foldable PC in Figure 6 has the fold

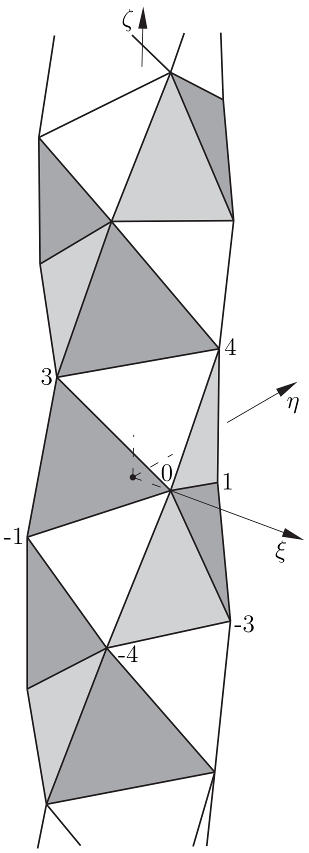

Figure 6: $n = 4$; foldable PC with equilateral triangles.

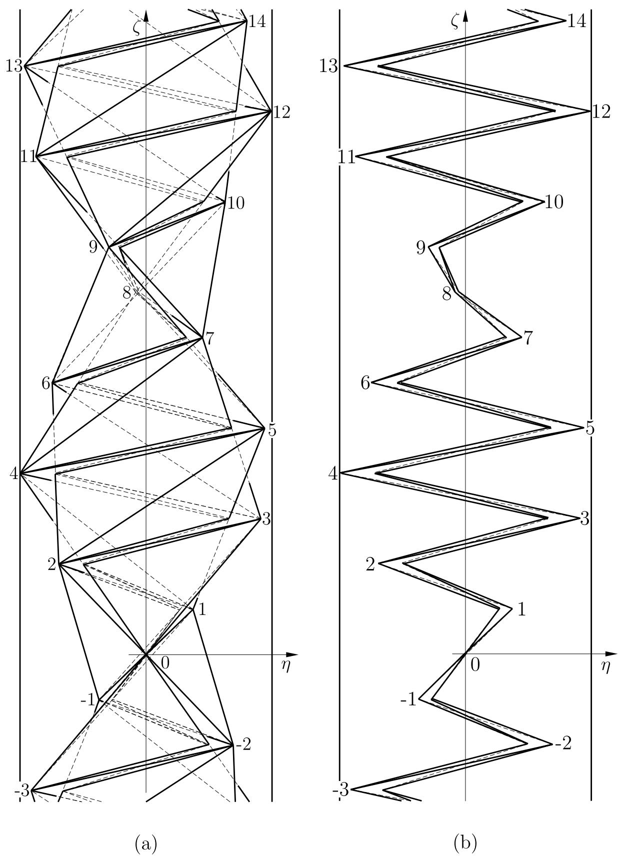

Figure 7: $n = 4$; self-intersecting PC with equilateral triangles (a) and core body (b).

angles $\beta_1 \approx 25.29°$, $\beta_4 \approx 76.29°$, $\beta_3 \approx 34.88°$.

For the self-intersecting PC the angles are

$$\beta_1 \approx 101.11°,\ \beta_4 \approx 145.32°,\ \beta_3 \approx 103.81°,\ \alpha_1 \approx 94.78°,\ \alpha_2 \approx 95.97°.$$

Equation (18) yields $u \approx .5030$, $v \approx .0473$, $(u/v = 85/8)$. In Figure 5 the length v is shown correctly. Not so the side length A_1–A'_3 of the pentagon. It is very small: $2u - 1 = (c+1)^2/c^2 \approx .006$. Equations (16) yield the side lengths $\ell_1 = (3\sqrt{23} - 5\sqrt{7})\sqrt{143}/16$, $\ell_2 = [(6209 - 489\sqrt{161})/8]^{1/2}$. Figure 7a shows the self-intersecting PC inside the circular cylinder in projection along the ξ-axis onto the η,ζ-plane. In Figure 7b the core body alone is shown. On every edge V_k–V_{k+4} (k arbitrary) the points A'_{k+3} closer to V_k and A_{k+1} closer to V_{k+4} are located. The line segments of intersection are V_k–A_k–A'_{k+3}–V_{k+3} $(k = 0, \pm 1, \ldots)$.

Example 2:

Given are $x = 6/5$ and $b = 1$. This determines $a^2 = 12/5$, $y^2 = 24/25$, $\varrho = 11/7$ and two PCs with $c = -3/4$ (self-intersecting) and $c = -2/5$ (foldable). For the self-intersecting PC Eqs. (13), (18), (16) and (19) determine

$$z^2 = 3/35,\ r^2 = 64/245,\ u = 2\sqrt{15}/9,\ v = 2/5,\ \ell_1^2 = 11/27,\ \ell_2^2 = 7/135,$$
$$\cos\beta_1 = 17/32,\ \beta_4 = 90°,\ \cos\beta_3 = -7/8,\ \cos\alpha_1 = 3/56,\ \cos\alpha_2 = -51/175.$$

Figure 5 has the form shown in Figure 8. The pentagons constituting the faces of the core body are shaded. The short side A_1–A'_3 of the pentagon has the length $2u - a = a/9$. The four parallel lines are explained later. The foto in Figure 9 shows on the far left a model of the core body with three tetrahedra in place (red, green and blue).

The parameters of the foldable PC with $c = -2/5$ are

$$z^2 = 3/28,\ r^2 = 125/392,\ \cos\beta_1 = 24/25,\ \cos\beta_4 = 7/8,\ \beta_3 = 90°.$$

The second from the left in Figure 9 is a model of this PC.

Example 3:

The vertex V_4 is an arbitrary point on the circle $\varrho = 2$. As independent parameter the distance a from V_0 is chosen $(0 < a < 4)$. In Figure 4 the line connecting the vertex V_4 in

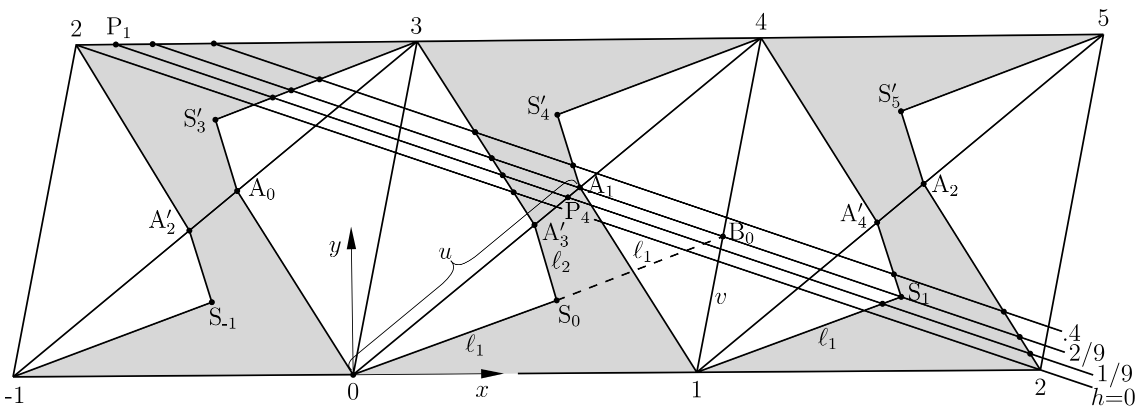

Figure 8: V_4 at $x = 6/5$, $b = 1$. Triangular faces of tetrahedra, pentagonal faces of the core body (shaded), lines $h =$ const.

Figure 9: From left to right: Core body and foldable PC of Example 2; two foldable PCs of Example 4.

the triangle ($V_0V_1V_4$) with the vertex V_4 on the x-axis is orthogonal to the edges V_0–V_4, V_1–V_5, ... of triangles (Thales' circle). From this it follows that vertices of equal name come into coincidence in two ways, namely, by folding the equidistant edges V_0–V_4, V_1–V_5, ... (distance $d = \sqrt{16-a^2}/4$) either with fold angles $\beta_4 = 90°$ or with $\beta_4 = 180°$. In either case $\beta_1 = \beta_3 = 0$ and $z = a/4$. The resulting PCs are a straight cylinder with square cross section of side length d ($\beta_4 = 90°$) and a flatfolded strip of width d ($\beta_4 = 180°$). The cylinder with square cross section is a foldable four-bar mechanism if the edges are revolute joints. In deformed positions the vertices of triangles are not on the surface of a straight circular cylinder. The same results are obtained from Eqs. (1), (12), (13) and (19).

Example 4:

The vertex V_4 is an arbitrary point on the circle $\varrho = 3/2$. As independent parameter the coordinate x is chosen ($1 < x < 4$). The equation $(x-5/2)^2 + y^2 = (3/2)^2$ of the circle determines $y^2 = (4-x)(x-1)$, $a^2 = 5x-4$, $b^2 = 3(x-1)$. The roots associated with the

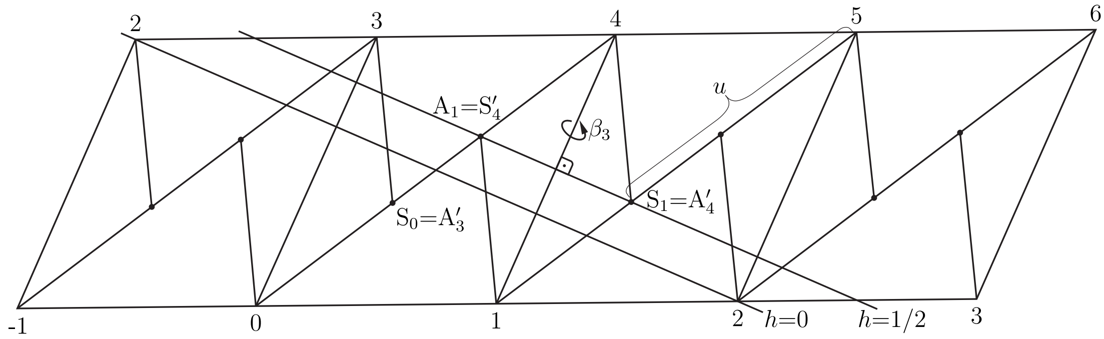

Figure 10: V_4 at $x = 3/2$, $y = \sqrt{5}/2$. Lines of intersection and lines $h =$ const.

circle are $c = -1/2$ and $c = -2/3$. The PC with $c = -1/2$ is foldable. It is characterized by the parameters $z^2 = \frac{1}{3}(x-1), r^2 = \frac{1}{9}(4-x)$ and, independent of x, $\beta_1 = \beta_4 = 0, \beta_3 = 120°$. This PC is a straight cylinder with equilateral triangular cross section. The second from the right in Figure 9 is a model of this PC in the special case $x = 3/2, y = \sqrt{5}/2$.

The PC with $c = -2/3$ is both foldable and self-intersecting. The parameters are $z^2 = (27x-28)/80$, $r^2 = (81/800)(4-x)$. The condition $z^2 \geq 0$ requires that $x \geq 28/27$. At $x = 28/27$ the circle $\varrho = 3/2$ intersects the line $z = 0$. Furthermore, $\cos\beta_1 = (27x-29)/[27(x-1)]$, $\cos\beta_4 = (11x-12)/[16(x-1)]$ and, independent of x, $\beta_3 = 180°$, $u = \frac{5}{8}a$, $v = b$, $\ell_1 = \frac{3}{8}a$, $\ell_2 = 0$.

In the special case $x = 3/2$, $y = \sqrt{5}/2$, Figure 5 has the form shown in Figure 10. The results for v, ℓ_1 and ℓ_2 show that for an arbitrary k S_k coincides with A'_{k+3} and S'_{k+3} with A_k. In the folded state A_k coalesces with A'_{k+3}, triangles sharing the edge V_k–V_{k+3} are coplanar ($\beta_3 = 180°$), the edge V_k–V_{k+4} is intersected by the edge V_{k+1}–V_{k+5} at A_{k+1} and by the edge V_{k-1}–V_{k+3} at A'_{k+3}. The tetrahedra are degenerate with volume zero. Figure 9 shows on the far right a model of this PC. Seen in projection along the ζ-axis the infinitely many triangles cover the circular ring with outer radius r and with inner radius $r_i = r\cos(2\varphi - \pi) = r/9$.

2.2.1. Cross sections

In this section, cross sections of foldable and of self-intersecting PCs in planes $\zeta =$ const. are investigated. As is shown further below, the lines parallel to V_2–V_2 in Figure 5 and in Figure 8 are mapped into polygonal contours of cross sections. Since the shape of cross sections is periodically repeated with period length z, it suffices to investigate planes $\zeta = hz$ with $0 \leq h \leq 1$.

A line $h =$const. intersects eight edges of triangles in points $P_1, \ldots, P_8$ as shown in Figure 5. If the PC is foldable, the cross section is the octogon $(P_1, \ldots, P_8)$ (a symmetric hexagon if $h = 0$ or $h = 1$). If the PC is self-intersecting, a line $h =$const. intersects, in addition to eight edges of triangles, edges connecting the core body to three tetrahedra. In Figure 5 the points of intersection are Q_1 coalescing with Q'_1, Q_2 coalescing with Q'_2 and Q_3 coalescing with Q'_3. In the assembled state these three points are determined by $P_1, \ldots, P_8$. Q_1, for example, is the point of intersection of the lines P_1–P_2 and P_3–P_4.

The coordinates of $P_1, \ldots, P_8$ in the assembled state are determined as follows. The edge V_{-3}–V_1, for example, has the equation $\vec{r}(\lambda_2) = \vec{r}_{-3} + \lambda_2(\vec{r}_1 - \vec{r}_{-3})$ with parameter λ_2. The condition $\zeta = hz$ is $-3z + 4\lambda_2 z = hz$. It determines $\lambda_2 = \frac{1}{4}(3+h)$ and the ξ, η-coordinates of $P_2:$ $\xi_2 = r[c_3 + \lambda_2(c_1 - c_3)] = rc(hs^2 + c^2)$, $\eta_2 = r[-s_3 + \lambda_2(s_1 + s_3)] = rs(hc^2 + s^2)$. In like manner the coordinates of the other points are determined (Table 1). In the same way

Table 1: Coordinates at cross sections

i	*edge*	λ_i	ξ_i/r	η_i/r
1	V_0–V_1	h	$h(c-1)+1$	sh
2	V_{-3}–V_1	$\frac{1}{4}(3+h)$	$c(hs^2+c^2)$	$s(hc^2+s^2)$
3	V_{-2}–V_1	$\frac{1}{3}(2+h)$	$\frac{1}{3}[2c^2+2c-1+h(1-c)(2c+1)]$	$\frac{1}{3}s[h(2c+1)+2(1-c)]$
4	V_{-2}–V_2	$\frac{1}{4}(2+h)$	$2c^2-1$	shc
5	V_{-1}–V_2	$\frac{1}{3}(1+h)$	$\frac{1}{3}[2c^2+2c-1-h(1-c)(2c+1)]$	$\frac{1}{3}s[h(2c+1)-2(1-c)]$
6	V_{-1}–V_3	$\frac{1}{4}(1+h)$	$c(-hs^2+c^2)$	$s(hc^2-s^2)$
7	V_0–V_3	$\frac{1}{3}h$	$1+\frac{1}{3}h(c-1)(2c+1)^2$	$\frac{1}{3}sh(4c^2-1)$
8	V_0–V_4	$\frac{1}{4}h$	$1-2hc^2s^2$	$shc(2c^2-1)$

the coordinates of $P_1, \ldots, P_8$ in the unfolded state are determined. They prove that in the unfolded state lines $h=$const. are straight lines.

Let $\vec{\varrho}_i$ $(i=1,\ldots,8)$ be the vector with coordinates $[\xi_i,\ \eta_i,\ 0]$. In Figure 11 cross sections of the foldable PC with equilateral triangles in the planes $h=0$ and $h=.5$ are shown. Points on the circle of radius r denote the equidistant vertices $V_{-3}, \ldots, V_4$. The cross section in the plane $h=0$ is the symmetric hexagon drawn in solid lines with vectors $\vec{\varrho}_i(0)$. The cross section in the plane $h=.5$ is the symmetric octogon drawn in dashed lines. The cross sectional area $A(h)$ is

$$A(h)=\frac{1}{2}\left|\vec{\varrho}_2\times(\vec{\varrho}_3-\vec{\varrho}_1)+\vec{\varrho}_4\times(\vec{\varrho}_5-\vec{\varrho}_3)+\vec{\varrho}_6\times(\vec{\varrho}_7-\vec{\varrho}_5)+\vec{\varrho}_8\times(\vec{\varrho}_1-\vec{\varrho}_7)\right|. \tag{20}$$

It turns out that $A(h)$ is, independent of h, $A=\frac{2}{3}r^2s(1-c)^2(4c+3)$. Hence

$$\frac{\text{volume PC}}{\text{volume circular cylinder}}=\frac{2}{3\pi}\,s(1-c)^2(4c+3)\ . \tag{21}$$

In Figure 12 the cross section of the self-intersecting PC with equilateral triangles in the plane $h=.4$ is shown. The shaded areas are the cross sections of the tetrahedra $T(-2,1)$, $T(-1,2)$, $T(0,3)$, and of the core body. Figure 8 shows that the line $h=$const passing through the coalescing points A_1 and S_1 is a significant line. The value of h on this line is

$$h^*=2(c+1)^2/c^2. \tag{22}$$

In the interval $h<h^*$ the cross section of the core body is the pentagon $(P_1Q_1P_4Q_2Q_3)$. In the interval $h^*\geq h\leq 1-h^*$ it is the quadrilateral $(P_1Q_1Q_2Q_3)$. As examples see the cross sections $h=1/9<h^*$ and $h=.4>h^*$ in Figure 13.

In Figure 11 all cross products $\vec{\varrho}_i\times\vec{\varrho}_{i+1}$ have the same sense of direction. In contrast, in Figure 12 the cross products $\vec{\varrho}_1\times\vec{\varrho}_2$ and $\vec{\varrho}_8\times\vec{\varrho}_1$ have one sense of direction, and all other cross products have the opposite sense of direction. This has the consequence that the expression in (20) is the difference *cross sectional area* A_c *of the core body* minus *total cross sectional area* A_t *of the three tetrahedra*. Equation (21) is replaced by

$$\frac{A_c(h)-A_t(h)}{\pi r^2}\equiv\frac{2}{3\pi}\,s(1-c)^2(4c+3)\qquad\text{independent of } h. \tag{23}$$

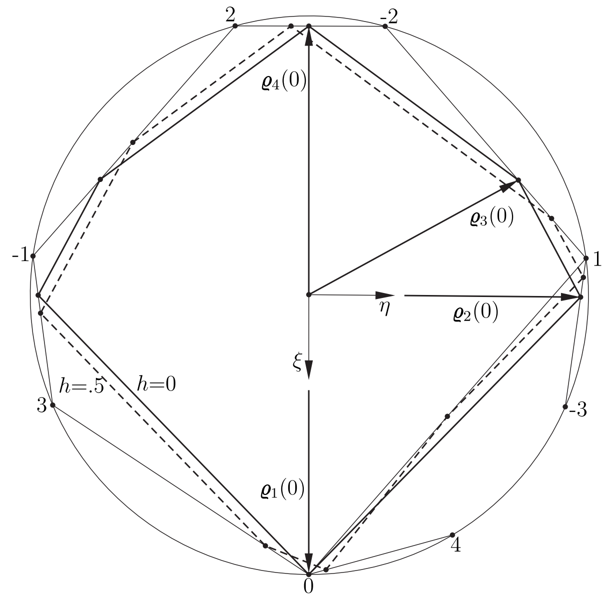

Figure 11: Cross sections in Figure 6 in planes $h = 0$ and $h = .5$.

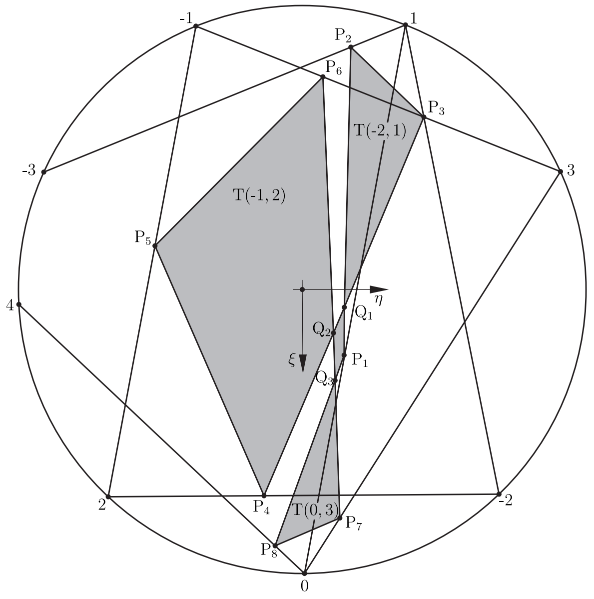

Figure 12: Cross section in Figure 7a in the plane $h = .4$.

In the case $c = -3/4$ treated in Example 2, we obtain $A_c(h) \equiv A_t(h)$. In Figure 13 this identity is demonstrated by the cross sections in the planes $h = 1/9$, $h = h^* = 2/9$, and $h = .4$ (shaded).

The ratios (21) and (23) are constant on the circle $c =$ const (arbitrary). The derivative with respect to φ is const.$\times(1-c)^2(16c^2+17c+2)$. It is zero on the circle $c = (-17\pm\sqrt{161})/32$.

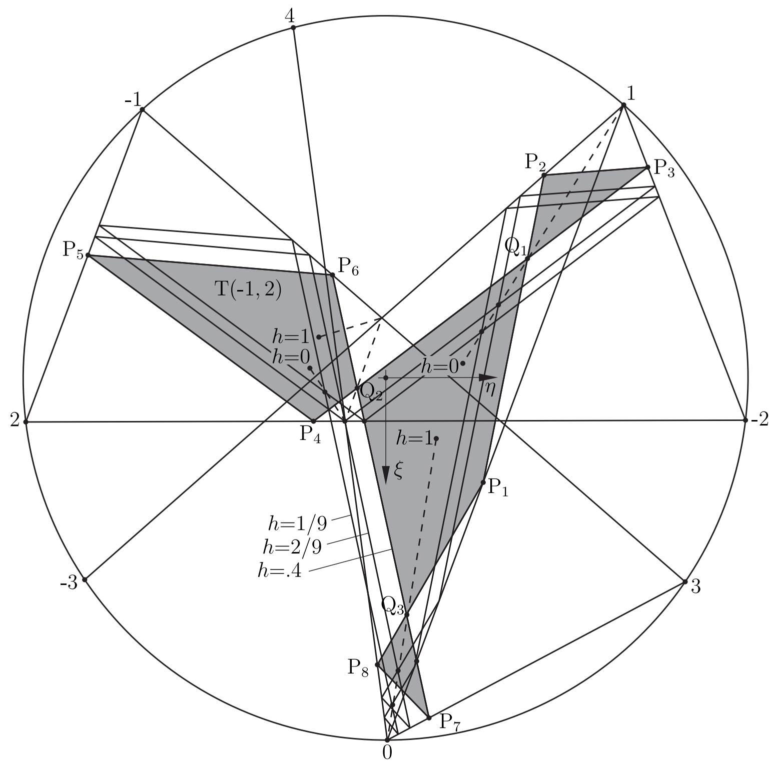

Figure 13: Three cross sections. Self-intersecting PC with $c = -3/4$, $h^* = 2/9$.

This is the circle C_E (see (14), (15) and Figure 4). The ratio has its maximum $\approx .6663$ for $c = (-17+\sqrt{161})/32$ and its minimum $\approx -.2093$ for $c = (-17-\sqrt{161})/32$. Hence the conclusion:

Among all foldable PCs the ones made of triangles having V_4 *on the circle* C_E *have the largest ratio* (21). *Among all self-intersecting PCs the ones made of triangles having* V_4 *on the circle* C_E *have the smallest ratio* (23).

On the basis of Table 1 the cross sectional area $A_c(h)$ of the core body is calculated:

$$\left.\begin{aligned} &2A_c(h) = \begin{cases} r^2(A_1 + B_1 h^2) & (0 \le h \le h^*), \\ r^2[A_2 + B_2 h(1-h)] & (h^* \le h \le 1-h^*), \end{cases} \\ &A_1 = \frac{8s^5(1+c)}{3c^2+4c+2}, \qquad B_1 = \frac{-2sc(1-c)^2(2c+1)(3c+2)}{3c^2+4c+2}, \\ &A_2 = \frac{-8s^5(1+c)(c^2+c+1)}{c(c+2)(3c^2+4c+2)}, \quad B_2 = \frac{8s^5 c(2c+1)}{(c+2)(3c^2+4c+2)}. \end{aligned}\right\} \tag{24}$$

Taking into account the symmetry $A_c(h) = A_c(1-h)$ this yields the ratio

$$\frac{\text{volume core body}}{\text{volume circular cylinder}} = \frac{2}{\pi r^2}\int_0^{1/2} A_c(h)\,\mathrm{d}h = \frac{2s^5}{3\pi c^3}\left(4c^2+7c+2\right). \tag{25}$$

This ratio increases monotonically from 0 at $c = -1$ to $50\sqrt{5}/(243\pi) \approx .1464$ at $c = -2/3$. With $c = -3/4$ (Figures 13 and 9) it is $49\sqrt{7}/(648\pi) \approx .06368$ and with $c = -(17+\sqrt{161})/32$ (Figures 12 and 7b) it is $\approx .00202$.

The difference ratio (25) minus ratio (23) is the ratio

$$\frac{\text{volume tetrahedra}}{\text{volume circular cylinder}} = \frac{2s}{3\pi c^3}(1-c)^2(2c+1)^2(3c+2). \tag{26}$$

This ratio is zero for $c = -1$ and for $c = -2/3$. The case $c = -2/3$ was the subject of Example 4 (Figure 10). With $c = (-17-\sqrt{161})/32$ (Figure 12) the ratio is $\approx .2113$. The maximum ratio is $\approx .2114$ at $c \approx -.9247$.

Let V be the volume of a single tetrahedron. Figures 5 and 8 show that between the planes $\zeta = 2z$ and $\zeta = 5z$ the entire tetrahedron $T(2,5)$ and parts of $T(0,3)$, $T(1,4)$, $T(3,6)$, and $T(4,7)$ are located. For reasons of symmetry the partial volumes of $T(0,3)$ and $T(1,4)$ and of $T(3,6)$ and $T(4,7)$ add up to V. The total volume between the two planes is $3V$. Hence $V/(\pi r^2 z)$ is the ratio (26). This yields

$$V = \frac{s(2c+1)^2(3c+2)}{12c^3(2+c)}(9-b^2)\sqrt{1-\frac{9-b^2}{4(1-c)(2+c)}}\,. \tag{27}$$

With the parameters $c = (-17-\sqrt{161})/32$, $b = 1$ of the equilateral triangle $V \approx .3637^3$ (Figure 7a) and with $c = -3/4$, $b = 1$ $V = 8\sqrt{3/5}/405 \approx .2483^3$ (the far left of Figure 9). Equation (27) has the form $V(\lambda,\varphi) = \lambda f(\varphi)\sqrt{1+\lambda g(\varphi)}$ with $\lambda = 9-b^2$ and with functions $f(\varphi)$ and $g(\varphi)$.

The stationarity conditions $\partial V/\partial\lambda = 0$ and $\partial V/\partial\varphi = 0$ are $\lambda = -2/(3g)$, $\lambda = -2f'/(fg' + 2f'g)$. With the first equation Eqs. (13) yield $r^2 = 1/[3(1-c)]$, $z^2 = 1/3$. Both equations together yield $fg' - f'g = 0$. This means that the derivative of f/g is zero. This is the equation

$(1-c)(2c+1)(12c^4+10c^3+5c^2+12c+6)=0$. The only root in the interval $-1<c<-2/3$ is $c\approx-.919800$. It determines the maximum volume $V_{\max}\approx.4051^3$, the radius $r\approx.4167$ and the parameters of the triangle $(\mathrm{V}_0\mathrm{V}_1\mathrm{V}_4)$: $a\approx2.387$, $b\approx1.863$, $x\approx1.613$, $y\approx1.759$.

Summary.
In the interval $0\le\varphi\le\pi$ the higher-order cycloid $z=0$ has $n-2$ branches connecting the points $x=0$ and $x=1$ on the x-axis. By these branches and by circles $c=$const. tangent to them the (x,y)-domain is divided into $N(n^2)$ domains differing in the number of solutions $-1\le c<1$ and in the number of solutions $z^2\ge0$. For every $k=1,\dots,n-2$ there is a domain for V_n admitting k different PCs. In the case $n=4$ self-intersecting PCs consist of a core body with congruent nonconvex pentagonal faces and of an infinite number of congruent tetrahedra.

Conjecture.
In the case $n=4$ the ratio $A/(\pi r^2)$ with A calculated from Eq. (20) was given the interpretations (21) and (23). The ratio has two stationary values if V_4 is located on the circle $\mathrm{C_E}$. Conjectures:

1. For arbitrary $n\ge3$ the area $A(h)$ calculated from an equation equivalent to Eq. (20) is independent of h.

2. The ratio $A/(\pi r^2)$ has $n-2$ stationary values if V_n is located on the circle $\mathrm{C_E}$.

3. The stationarity condition is Eq. (6) with $a=b=1$.

For $n=3,5$ and 6 this was verified. The ratio is $2s(1-c)^2/(3\pi)$ for $n=3$, $2s(1-c)^2(2c+1)^2/\pi$ for $n=5$ and $2s(1-c)^2(32c^3+36c^2+8c+1)/(3\pi)$ for $n=6$. For foldable PCs $A/(\pi r^2)$ is the ratio *volume* PC/*volume circular cylinder*. For self-intersecting PCs the interpretation needs clarification.

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