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A Generalization of Pascal's Hexagon Theorem to Real Hilbert Spaces

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Abstract. We present a generalization of the classical Pascal Hexagon Theorem to real Hilbert spaces. The key ingredient is a Cone Lemma which allows a reformulation of the problem in terms of vertices of cones with spherical cross-section bases.

Key Words: Pascal's hexagon theorem, Hilbert space, Linear variety, Cone
MSC 2010: 46C05, 51A20

1. Introduction

Pascal's Hexagon Theorem [5, 7] is one of the marvels of elementary Euclidean geometry. It asserts that if $ABCDEF$ is a cyclic hexagon (not necessarily convex) in some Euclidean plane, and if the opposite sides intersect, say $M = \overleftrightarrow{AB} \cap \overleftrightarrow{DE}$, $N = \overleftrightarrow{BC} \cap \overleftrightarrow{EF}$, and $P = \overleftrightarrow{CD} \cap \overleftrightarrow{FA}$, then M, N, and P are collinear (see also Figure 1). Over the years it has been generalized in various ways to conics, to projective planes [3], and in higher dimensions [1, 2, 4, 6]. We present here a generalization to real Hilbert spaces. Such thing is possible by noticing first that the points M, N, and P appear as vertices of cone regions supported by disk cross-sections ($\overline{AD}$ and $\overline{BE}$ for M, $\overline{BE}$ and $\overline{CF}$ for N (cf. Figure 2), and $\overline{CF}$ and $\overline{AD}$ for P). Typically, pairs of cross-sections generate two cone regions (in Figure 2, $\overline{BE}$ and $\overline{CF}$ generate cone regions with vertices $V^+ = N$ and $V^- \notin \{M, N, P\}$), and then the challenge is to choose judiciously the right cones. The whole task is possible due to a key Cone Lemma (see below). The Cone Lemma already appears in [4], where a Pascal generalization to the Euclidean 3-space is proposed. However, there the implementation and execution of cone methods are lacking.

2. Cone Lemma

Let H be a real Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$ and associated norm $\| \cdot \|$. Denote by $S(H)$ the *unit sphere* of H, $S(H) = \{x \in H \mid \|x\| = 1\}$. In connection with $S(H)$ we will consider two types of geometric subsets of H, *linear varieties* and *cones*.

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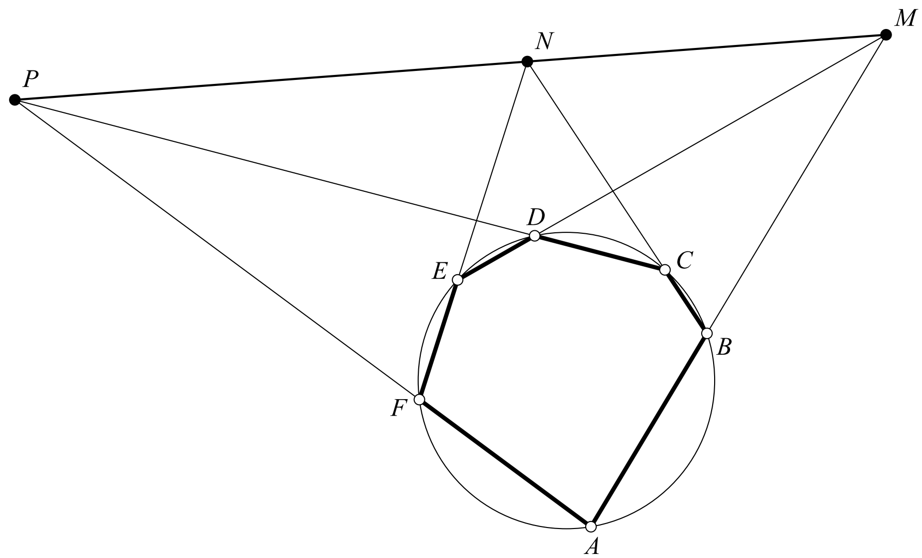

Figure 1: Pascal hexagon: The points of intersection of opposite sides of cyclic hexagon $ABCDEF$ are collinear

As usual, a linear variety l of H is any subset of H with the property that if x, y are distinct points of l then the *line* through x and y, $\{tx + (1-t)y \mid t \in \mathbf{R}\}$, is contained in l. Important examples of linear varieties are the *hyperplanes* $h = h_c(z)$, where for fixed $c \in \mathbf{R}$ and $z \in S(H)$, $h_c(z) = \{x \in H \mid \langle x, z\rangle = c\}$. We will be interested primarily in hyperplanes that do intersect $S(H)$, and do it *generically*, i.e., they are not tangent to $S(H)$. Equivalently, they are hyperplanes of type $h_c(z)$, where $z \in S(H)$ and $c = \cos\theta$, for some $0 < \theta \leq \pi/2$. We will call these intersections, $S(H) \cap h_{\cos\theta}(z)$, $z \in S(H)$, $0 < \theta \leq \pi/2$, *spherical cross-sections* in standard form. Notice that in a spherical cross-section, θ and z are unique, unless $\theta = \pi/2$ in which case z is determined up to a $\pm$ ambiguity.

A cone in H, more precisely the cone with vertex $v \in H$ and base $\emptyset \neq B \subset H$, $v \notin B$, denoted $C(v, B)$, is by definition the union of all the lines through v and the points of B, i.e., $C(v, B) = \{tv + (1-t)b \mid t \in \mathbf{R},\, b \in B\}$. We will only be concerned with cones $C(v, B)$ for which $v \notin S(H)$ and B is a spherical cross-section. It is obvious that if two such cones coincide (equal as sets) then their vertices are the same point. Not so for the bases.

Cone Lemma. *Let $S(H) \cap h_{\cos\theta_1}(z_1)$ and $S(H) \cap h_{\cos\theta_2}(z_2)$ be two spherical cross-sections in standard form such that $\theta_1 \neq \theta_2$ and $\langle z_1, z_2\rangle \neq \pm\cos(\theta_1 - \theta_2)$. Of all the cones $C(v, S(H) \cap h_{\cos\theta_1}(z_1))$, $v \in H \setminus S(H)$, there are exactly two such that*

$$C(v, S(H) \cap h_{\cos\theta_1}(z_1)) = C(v, S(H) \cap h_{\cos\theta_2}(z_2)). \tag{1}$$

They have vertices

$$v^{\pm} = \frac{\sin\theta_2}{\sin(\theta_2 \pm \theta_1)} z_1 \pm \frac{\sin\theta_1}{\sin(\theta_2 \pm \theta_1)} z_2, \tag{2}$$

where the signs $\pm$ correspond.

Proof. The proof hinges on the observation that suitable involutions of $S(H)$ take spherical cross-sections to spherical cross-sections.

Indeed, for a fixed point $p \in H \setminus S(H)$ define $T_p\colon S(H) \to S(H)$ by

$$\begin{aligned} T_p(x) = {} & \text{the other point of intersection, besides } x, \text{ of the} \\ & \text{line through } p \text{ and } x \text{ with } S(H), \text{ for } x \in S(H). \end{aligned} \tag{3}$$

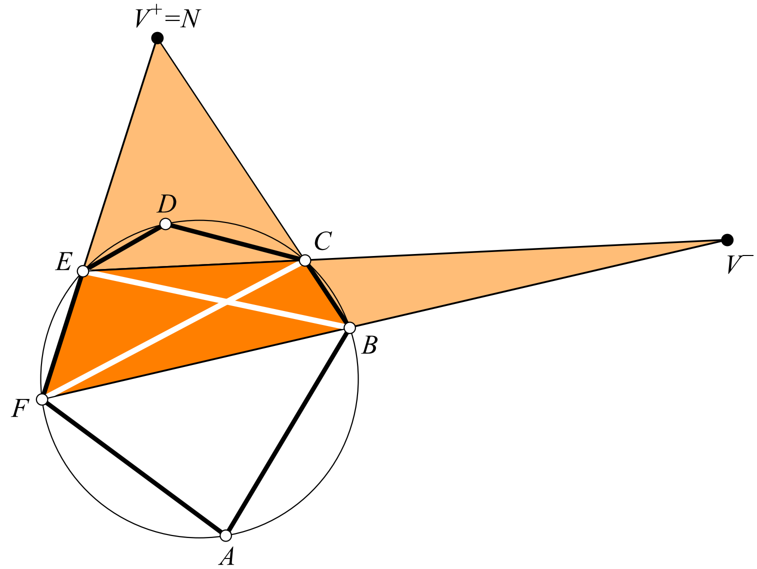

Figure 2: The two cone regions with vertices $V^{\pm}$ generated by cross-sections $\overline{BE}$ and $\overline{CF}$ in disk

$T_p(x) = p + t(x-p)$ for some $t \in \mathbf{R}$, and from the power of a point property,

$$\|T_p(x) - p\| \cdot \|x - p\| = \left| \|p\|^2 - 1 \right|,$$

we get

$$t = \frac{\|p\|^2 - 1}{\|x-p\|^2} \quad \text{and} \quad T_p(x) = p + \frac{\|p\|^2 - 1}{\|x-p\|^2}(x-p), \quad x \in S(H). \tag{4}$$

T_p is an involution, $T_p \circ T_p = \mathrm{Id}_{S(H)}$, or equivalently $T_p^{-1} = T_p$, or further

$$x = p + \frac{\|p\|^2 - 1}{\|T_p(x) - p\|^2}(T_p(x) - p), \quad x \in S(H). \tag{5}$$

Now, if $x \in S(H) \cap h_c(z)$ for some fixed points $z \in S(H)$ and $c \in \mathbf{R}$, $0 \le c < 1$, we get, via the substitution $y = T_p(x)$,

$$\begin{aligned}
&c = \langle x, z \rangle = \left\langle p + \frac{\|p\|^2 - 1}{\|y-p\|^2}(y-p),\ z \right\rangle \quad \text{or} \\
&(\langle p, z\rangle - c)\|y-p\|^2 + (\|p\|^2 - 1)\langle y-p,\ z\rangle = 0, \quad \text{or} \\
&(\langle p, z\rangle - c)\left(1 - 2\langle y, p\rangle + \|p\|^2\right) + (\|p\|^2-1)\langle y, z\rangle - (\|p\|^2-1)\langle p, z\rangle = 0, \ \text{or} \\
&\langle y,\ (\|p\|^2-1)z - 2(\langle p, z\rangle - c)p\rangle = c\left(\|p\|^2+1\right) - 2\langle p, z\rangle.
\end{aligned}$$

The point $(\|p\|^2-1)z - 2(\langle p,\ z\rangle - c)p \in H$ is non-zero. To see this immediately, it is a nice exercise to verify first that

$$\|(\|p\|^2-1)z - 2(\langle p, z\rangle - c)p\|^2 = \left(c\left(\|p\|^2+1\right) - 2\langle p, z\rangle\right)^2 + \left(\|p\|^2-1\right)^2\left(1-c^2\right), \tag{6}$$

from which it also follows that

$$\left|c\left(\|p\|^2+1\right) - 2\langle p, z\rangle\right| < \|(\|p\|^2-1)z - 2(\langle p, z\rangle - c)p\|.$$

Setting now

$$z^* = \begin{cases} \dfrac{|c\,(\|p\|^2+1) - 2\langle p,z\rangle|}{c\,(\|p\|^2+1) - 2\langle p,z\rangle} \cdot \dfrac{(\|p\|^2-1)z - 2(\langle p,z\rangle - c)p}{\|(\|p\|^2-1)z - 2(\langle p,z\rangle - c)p\|}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad c\,(\|p\|^2+1) - 2\langle p,z\rangle \neq 0, \\ \dfrac{(\|p\|^2-1)z - 2(\langle p,z\rangle - c)p}{\|(\|p\|^2-1)z - 2(\langle p,z\rangle - c)p\|}, \quad \text{if} \quad c\left(\|p\|^2+1\right) - 2\langle p,z\rangle = 0, \end{cases} \tag{7}$$

and

$$c^* = \frac{|c\,(\|p\|^2+1) - 2\langle p,z\rangle|}{\|(\|p\|^2-1)z - 2(\langle p,z\rangle - c)p\|}, \tag{8}$$

we just proved that

$$T_p(S(H)\cap h_c(z)) = S(H)\cap h_{c^*}(z^*)).$$

Notice that above the choices of z, c, z^*, and c^* make $S(H)\cap h_c(z)$ and $S(H)\cap h_{c^*}(z^*)$ spherical cross-sections in standard form.

Assume now that $v \in H\setminus S(H)$ and the spherical cross-sections in standard form $S(H)\cap h_{\cos\theta_1}(z_1)$ and $S(H)\cap h_{\cos\theta_2}(z_2)$, $\theta_1 \neq \theta_2$, generate the same cone with vertex v. Without loss of generality we can assume $\theta_2 \neq \pi/2$. This guarantees that in $h_{\cos\theta_2}(z_2)$, z_2 and θ_2 are unique. Consequently, Equations (7) and (8) above yield

$$\begin{aligned} z_2 &= \frac{|\cos\theta_1\,(\|v\|^2+1) - 2\langle v,z_1\rangle|}{\cos\theta_1\,(\|v\|^2+1) - 2\langle v,z_1\rangle} \cdot \frac{(\|v\|^2-1)z_1 - 2(\langle v,z_1\rangle - \cos\theta_1)v}{\|(\|v\|^2-1)z_1 - 2(\langle v,z_1\rangle - \cos\theta_1)v\|}, \\ \cos\theta_2 &= \frac{|\cos\theta_1\,(\|v\|^2+1) - 2\langle v,z_1\rangle|}{\|(\|v\|^2-1)z_1 - 2(\langle v,z_1\rangle - \cos\theta_1)v\|}. \end{aligned} \tag{9}$$

Via Equation (6), the Equations (9) are equivalent to

$$\begin{aligned} \left(\cos\theta_1\left(\|v\|^2+1\right) - 2\langle v,z_1\rangle\right) z_2 &= \cos\theta_2\left((\|v\|^2-1)z_1 - 2(\langle v,z_1\rangle - \cos\theta_1)v\right), \\ \left|\cos\theta_1\left(\|v\|^2+1\right) - 2\langle v,z_1\rangle\right| &= \frac{\sin\theta_1\cos\theta_2}{\sin\theta_2}\left|\|v\|^2-1\right|. \end{aligned} \tag{10}$$

After some routine algebra, from (10) we get

$$2(\langle v,z_1\rangle - \cos\theta_1)\sin\theta_2 v = \begin{cases} (\|v\|^2-1)\,(\sin\theta_2 z_1 - \sin\theta_1 z_2), \\ \qquad\qquad \text{if} \quad \dfrac{2(\langle v,z_1\rangle - \cos\theta_1)}{\|v\|^2-1} < \cos\theta_1, \\ (\|v\|^2-1)\,(\sin\theta_2 z_1 + \sin\theta_1 z_2), \\ \qquad\qquad \text{if} \quad \dfrac{2(\langle v,z_1\rangle - \cos\theta_1)}{\|v\|^2-1} > \cos\theta_1. \end{cases} \tag{11}$$

Notice that since $\theta_1 \neq \theta_2$ and $z_1, z_2 \in S(H)$, in either form of (11) we must have $\langle v,z_1\rangle - \cos\theta_1 \neq 0$. This means that v must be of the form

$$v = \frac{1}{\lambda}(\sin\theta_2 z_1 - \sin\theta_1 z_2) \quad \text{or} \quad v = \frac{1}{\mu}(\sin\theta_2 z_1 + \sin\theta_1 z_2), \tag{12}$$

for suitable non-zero real scalars λ and μ. A straightforward calculation shows that for the expressions (12) of v to satisfy (11) it is necessary that both λ and μ satisfy the same quadratic equation $\sigma^2 - 2\sin\theta_2\cos\theta_1 + (\sin^2\theta_2 - \sin^2\theta_1) = 0$, whose roots are $\sin(\theta_2 \pm \theta_1)$.

It is easy to check now that when $v = \dfrac{\sin\theta_2}{\sin(\theta_2 \pm \theta_1)} z_1 - \dfrac{\sin\theta_1}{\sin(\theta_2 \pm \theta_1)} z_2$ (the signs correspond), then

$$\frac{2(\langle v, z_1\rangle - \cos\theta_1)}{\|v\|^2 - 1} = \frac{\sin(\theta_2 \pm \theta_1)}{\sin\theta_2}.$$

Obviously, only the minus sign meets the restrictions imposed on acceptable vertices by the first half of Equation (11), which leads to the correct vertex

$$v^- = \frac{\sin\theta_2}{\sin(\theta_2 - \theta_1)} z_1 - \frac{\sin\theta_1}{\sin(\theta_2 - \theta_1)} z_2.$$

Likewise, when $v = \dfrac{\sin\theta_2}{\sin(\theta_2 \pm \theta_1)} z_1 + \dfrac{\sin\theta_1}{\sin(\theta_2 \pm \theta_1)} z_2$, again it follows that

$$\frac{2(\langle v, z_1\rangle - \cos\theta_1)}{\|v\|^2 - 1} = \frac{\sin(\theta_2 \pm \theta_1)}{\sin\theta_2}.$$

In the equation directly above only the plus sign works now, and so the valid vertex becomes

$$v^+ = \frac{\sin\theta_2}{\sin(\theta_2 + \theta_1)} z_1 + \frac{\sin\theta_1}{\sin(\theta_2 + \theta_1)} z_2.$$

Finally, the hypotheses $\langle z_1, z_2\rangle \neq \pm\cos(\theta_1 - \theta_2)$ guarantee that $v^\pm \notin S(H)$. Also, it is easy to see that $v^+ \neq v^-$. The proof of the lemma is complete. $\square$

3. Main results

Classical Pascal Theorem in Hilbert Spaces. *Let $S(H) \cap h_{\cos\theta_1}(z_1)$, $S(H) \cap h_{\cos\theta_2}(z_2)$, and $S(H) \cap h_{\cos\theta_3}(z_3)$ be three spherical cross-sections in standard form, pairwise satisfying the hypotheses of the Cone Lemma. Let $v_{12}^\pm$, $v_{13}^\pm$, and $v_{23}^\pm$ be the vertices of the cones guaranteed to exist by the Cone Lemma, where the indices indicate the hyperplanes associated to them. Then for any choice of two vertices $w_{12} \in \{v_{12}^\pm\}$ and $w_{13} \in \{v_{13}^\pm\}$ there is a vertex $w_{23} \in \{v_{23}^\pm\}$ such that w_{12}, w_{13}, and w_{23} are collinear.*

Proof. There are four possible choices for the vertices w_{12} and w_{13}. One of the great computational strengths of the Cone Lemma is that the vertices $v^\pm$ are interchangeable: If one allows for an angle θ to be negative ($-\pi/2 \leq \theta < 0$) then the cross-section does not change since $\cos(-\theta) = \cos\theta$, however the vertices are flipped. This observation allows to prove the Theorem by assuming, without loss of generality, that

$$\begin{aligned} w_{12} = v_{12}^+ &= \frac{\sin\theta_2}{\sin(\theta_2 + \theta_1)} z_1 + \frac{\sin\theta_1}{\sin(\theta_2 + \theta_1)} z_2, \\ w_{13} = v_{13}^+ &= \frac{\sin\theta_3}{\sin(\theta_3 + \theta_1)} z_1 + \frac{\sin\theta_1}{\sin(\theta_3 + \theta_1)} z_3. \end{aligned}$$

We will show then that

$$w_{23} := v_{23}^- = \frac{\sin\theta_3}{\sin(\theta_3 - \theta_2)} z_2 - \frac{\sin\theta_2}{\sin(\theta_3 - \theta_2)} z_3$$

belongs to the line through w_{12} and w_{13}, i.e., there is $t \in \mathbf{R}$ such that

$$w_{23} = tw_{12} + (1-t)w_{13}$$

or equivalently,

$$\begin{aligned}\frac{\sin\theta_3}{\sin(\theta_3-\theta_2)}z_2 - \frac{\sin\theta_2}{\sin(\theta_3-\theta_2)}z_3 = {} & t\left(\frac{\sin\theta_2}{\sin(\theta_2+\theta_1)}z_1 + \frac{\sin\theta_1}{\sin(\theta_2+\theta_1)}z_2\right) \\ & + (1-t)\left(\frac{\sin\theta_3}{\sin(\theta_3+\theta_1)}z_1 + \frac{\sin\theta_1}{\sin(\theta_3+\theta_1)}z_3\right).\end{aligned} \tag{13}$$

Since generically z_1, z_2 and z_3 can be chosen to be linearly independent in H if $\dim_{\mathbf{R}} H \geq 3$, the only chance for (13) to hold true is to have

$$t\frac{\sin\theta_2}{\sin(\theta_2+\theta_1)} + (1-t)\frac{\sin\theta_3}{\sin(\theta_3+\theta_1)} = 0,$$

or

$$t = \frac{\sin\theta_3 \cdot \sin(\theta_2+\theta_1)}{\sin\theta_1 \cdot \sin(\theta_3-\theta_2)}. \tag{14}$$

Amazingly, it works, as for the choice of t given by (14) it is easily seen that the coefficients of z_2, respectively z_3, on both sides of (13) are the same.

One can infer that the proof of Pascal's Theorem is more natural when $\dim_{\mathbf{R}} H \geq 3$, since in the truly classical Pascal case ($H = \mathbf{R}^2$) one might be led to believe that the linear dependence of z_1, z_2, and z_3 should play a role. As we have just seen, it does not. □

In preparation for our next result we remind the reader that if S is a non-empty subset of H, the *linear span* of S, $l(S)$, is by definition the intersection of all the linear varieties of H containing S. $l(S)$ is obviously a linear variety, and moreover

$$l(S) = \left\{\sum_{j=1}^{n} t_j s_j \;\middle|\; s_j \in S,\ t_j \in \mathbf{R},\ \sum_{j=1}^{n} t_j = 1,\ n = 1, 2, \ldots\right\}.$$

Generalization of Pascal Theorem. *For some arbitrary index set I such that $0 \notin I$ assume given $S(H) \cap h_{\cos\theta_0}(z_0)$, $S(H) \cap h_{\cos\theta_i}(z_i)$, $i \in I$, spherical cross-sections in standard form such that for any pair $(0,i)$, $i \in I$, respectively $(i,j) \in I \times I$, $i \neq j$, the associated cross-sections satisfy the hypotheses of the Cone Lemma, with corresponding vertices $v_{0i}^{\pm}$, respectively $v_{ij}^{\pm}$. Fix one vertex $w_{0i} \in \{v_{0i}^{\pm}\}$ for each $i \in I$. Then for each $(p,q) \in I \times I$, $p \neq q$, one of the two vertices $v_{pq}^{\pm}$ belongs to $l(\{w_{0i} \mid i \in I\})$, the linear span of $\{w_{0i} \mid i \in I\}$.*

Proof. The proof is trivial since by the previous theorem for any $(p,q) \in I \times I$, $p \neq q$, there is one of $v_{pq}^{\pm}$ which belongs to $l(\{w_{0p}, w_{0q}\}) \subset l(\{w_{0i} \mid i \in I\})$.

Obviously, the theorem is relevant only if $l(\{w_{0i} \mid i \in I\}) \neq H$. If $\dim_{\mathbf{R}} H = n$, $n \geq 2$, the largest cardinality of I for which the theorem is interesting (for generic choices of cross-sections) is n, for a total of $n+1$ cross-sections. In such a case an additional $n(n-1)/2$ points are found to belong to the linear span of n points. □

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