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Constructions in the Locus of Isogonal Conjugates for a Quadrilateral

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Abstract. Given fixed distinct points A, B, C, D, we examine properties of and constructions involving the cubic plane curve locus of points X for which (XA, XC), (XB, XD) are isogonal. We establish a necessary and sufficient condition for all cubics $\mathcal{C} \in \mathbb{R}^2$ such that there exist corresponding $A, B, C, D \in \mathcal{C}$ for which $\mathcal{C}$ is this exact locus.

Key Words: isogonal cubic, focal cubic, isogonal conjugate, excellent point, spiral center of quadrangle, Miquel point

MSC 2010: 51N20, 51N35, 51N15

1. Introduction

In this paper, we characterize the locus of all points P with an isogonal conjugate in a given quadrilateral $ABCD$. This turns out to be a cubic plane curve, which we will call the *isogonal cubic* of $ABCD$. The isogonal cubic is a well-established figure in geometry. However, its properties are often considered with respect to the base quadrilateral $ABCD$, without considering the isogonal cubic as an individual curve, and often neglecting degenerate cases of $ABCD$.

The first half of the paper is dedicated to discovering geometric properties of non-degenerate isogonal cubics, and also providing constructions on the isogonal cubic with straightedge and compass. This sets up the second half, which characterizes all possible non-degenerate cubics $\mathcal{C} \in \mathbb{RP}^2$ such that there exist $A, B, C, D \in \mathbb{R}^2$ for which $\mathcal{C}$ is the isogonal cubic of $ABCD$. We also establish the notion of the spiral center and isogonal conjugation purely with respect to a valid cubic $\mathcal{C}$.

The following is the main result we aim to prove throughout this paper:

Theorem 1. *Let $\mathcal{C}$ be a non-degenerate cubic in $\mathbb{R}^2$, and let $\mathcal{C}_0$ denote its embedding in $\mathbb{CP}^2$. Then the following two conditions are equivalent:*

(1) *There exist distinct $A, B, C, D \in \mathcal{C}$ such that $\mathcal{C}$ is the isogonal cubic of $ABCD$.*

(2) *The circular points at infinity [8] I, J lie on $\mathcal{C}_0$, and the tangents at I, J meet on $\mathcal{C}_0$.*

In particular, the assertion that (2) is a *sufficient* condition requires special care in $\mathbb{R}^2$ and $\mathbb{CP}^2$.

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1.1. Definitions and conventions

Definition 1 (Isogonality in $\mathbb{RP}^2$)**.** For points $P, A, B, C, D \in \mathbb{R}^2$, pairs of lines (PA, PC), (PB, PD) are called *isogonal* if they share the same pair of angle bisectors (Figure 1). If P is a real point at infinity while A, B, C, D remain in $\mathbb{R}^2$, we slightly modify our definition of isogonality to mean that for any line ℓ intersecting PA, PB, PC, PD at points $E, F, G, H \in \mathbb{R}^2$, directed lengths EF and HG will be equal.

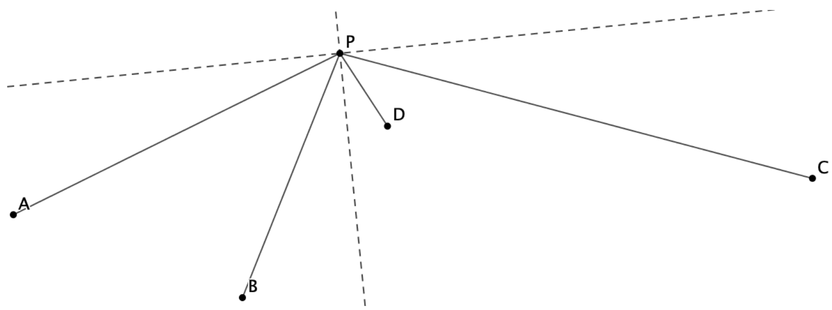

Figure 1: Definition of isogonality of the pairs (PA, PC) and (PB, PD)

The definition for points at infinity is equivalent to the midline of parallel lines PA, PC being the same as the midline of parallel lines PB, PD, which complies with the idea of angles as a conceptual measure of "distance" between two lines.

Definition 2 (Quadrilateral conventions and isogonal conjugates)**.** We use the term "quadrilateral" throughout this paper to refer to possibly self-intersecting quadrilaterals, whose vertices are distinct but possibly collinear. Points P, Q are called *isogonal conjugates* in a (possibly self-intersecting) polygon $A_1A_2 \ldots A_n$ if and only if (A_iP, A_iQ), (A_iA_{i-1}, A_iA_{i+1}) are isogonal for all i, where indices are cycled mod n (Figure 2).

We will exclusively work in directed angles. For points X, Y, the notation XY denotes the line XY if X and Y are distinct, while XY denotes the tangent at X if $X \equiv Y$ and the context of the curve containing X is clear (usually the isogonal cubic). The notation (XYZ) denotes the circumcircle of XYZ provided X, Y, Z are distinct.

Definition 3 (Notation for intersection)**.** We will let $\mathcal{S} \cap \mathcal{T}$ denote the intersection of sets of points $\mathcal{S}$ and $\mathcal{T}$, which is unique when $\mathcal{S}$ and $\mathcal{T}$ are distinct lines. When $\mathcal{S}$ is a cubic and $\mathcal{T}$ is a line XY such that $X, Y \in \mathcal{S}$, we will use the notation $XY \cap \mathcal{S}$ to denote

- If either X or Y is a singular point, whichever one of X, Y is singular
- If X, Y are distinct and XY is not tangent to $\mathcal{S}$, the third intersection of XY with $\mathcal{S}$
- If X, Y are distinct and XY is tangent to $\mathcal{S}$, the tangency point of XY with $\mathcal{S}$
- If X, Y are not distinct and X is not an inflection point of $\mathcal{S}$, the intersection of the tangent to $\mathcal{S}$ at X with $\mathcal{S}$
- If X, Y are not distinct and X is an inflection point, the point X

These are essentially equivalent to the *third intersection* of XY with $\mathcal{C}$ counting multiplicity.

We begin with this well-known characterization of all points with isogonal conjugates:

Theorem 2. *For fixed distinct points $A, B, C, D \in \mathbb{R}^2$ not all collinear, a point $P \in \mathbb{RP}^2$ is called* excellent *if (PA, PC) and (PB, PD) are isogonal. Then P is excellent if and only if it has an isogonal conjugate in $ABCD$.*

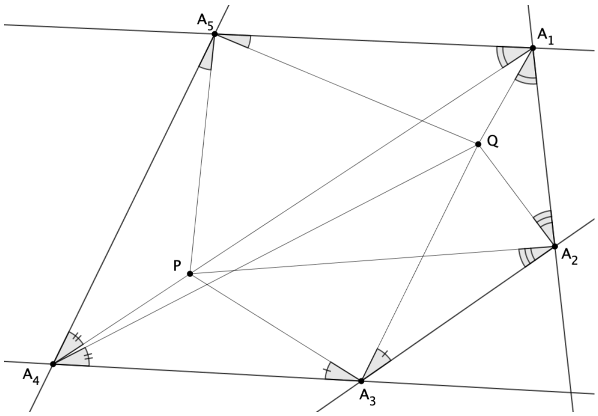

Figure 2: Definition of isogonal conjugates

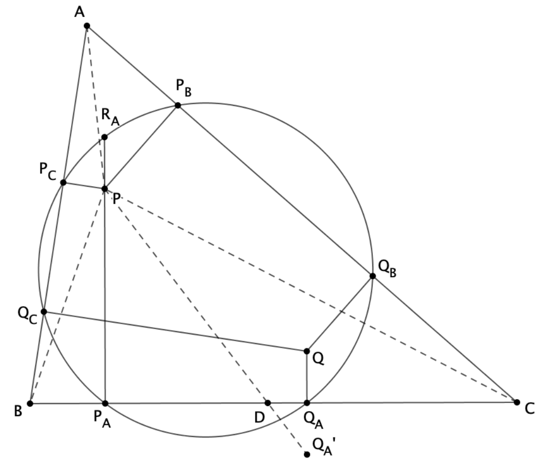

Figure 3: Collinear case of Theorem 1.5

Most proofs for this fact do not address the case when three of A, B, C, D are collinear, so we will provide the full proof of this lemma for the sake of rigor.

Proof. The first case is when, without loss of generality, B, C, D are collinear. In this case, we need to prove the following: For triangle ABC and $D \in BC$ and point P, the isogonal conjugate Q of P satisfies that BC is a bisector of angle $\angle PDQ$ if and only if (PA, PD), (PB, PC) are isogonal.
This is, in turn, equivalent to the following: For isogonal conjugates P, Q in ABC, if Q'_A be the reflection of Q over BC, then (PA, PQ'_A), (PB, PC) are isogonal. To prove this, let P, Q have pedal triangles $P_AP_BP_C$, $Q_AQ_BQ_C$ respectively; by [2], these share the same circumcircle ω centered at the midpoint of PQ. Let PP_A meet ω at $R_A \neq P_A$; $PR_AQ_AQ'_A$ is a parallelogram, so (Figure 3)

$$\begin{aligned}
\angle Q'_APC &= \angle Q'_APP_A + \angle P_APC = \angle Q_AR_AP_A + 90^\circ - \angle PCB \\
&= \angle Q_AP_CP_A + 90^\circ - \angle PCB = \angle Q_AP_CP_B + \angle BP_CP_A + 90^\circ - \angle PCB \\
&= \angle Q_AQ_BQ_C + \angle BPP_A + 90^\circ - \angle PCB \\
&= \angle Q_AQ_BQ + \angle QQ_BQ_C + 90^\circ - \angle CBP + 90^\circ - \angle PCB \\
&= \angle BCQ + \angle QAB - \angle CBP - \angle PCB \\
&= \angle PCA + \angle CAP + \angle BPC = \angle BPC + \angle CPA = \angle BPA
\end{aligned}$$

as desired.

Next, we prove this fact when no three of A, B, C, D are collinear. While the proof for the general case is well-known, we will provide it for the sake of completion.

Lemma 1. *For quadrilateral $ABCD$ and point P, let the projections of P onto AB, BC, CD, DA be E, F, G, H. Then $EFGH$ is cyclic if and only if $\angle APB = \angle DPC$.*

Proof. With the various cyclic quadrilaterals,

$$\begin{aligned}
\angle EFG + \angle GHE &= \angle EFP + \angle PFG + \angle GHP + \angle PHE \\
&= \angle EBP + \angle PCG + \angle GDP + \angle PAE = \angle APB + \angle CPD,
\end{aligned}$$

which directly implies the desired statement (Figure 4). ∎

Back to the main problem, drop perpendiculars E, F, G, H from P to AB, BC, CD, DA. Let AB meet CD at X, and AD meet BC at Y. First, we prove that if isogonal conjugate then $\angle APB = \angle DPC$. Let P have isogonal conjugate P' in $ABCD$. Then P' is the isogonal conjugate of P in both YAB and XAD. Drop from P' perpendiculars E', F', G', H'; by [2] on YAB, $EFHE'F'H'$ is cyclic, and on XAD we get $EGHE'G'H'$ is cyclic. In other words, F, F', G, G' all lie on $(EE'HH')$, implying that $EFGH$ is cyclic, hence $\angle APB = \angle DPC$ as desired.
Now, we prove that if $\angle APB = \angle DPC$ then it has an isogonal conjugate P'. Then $EFGH$ is cyclic; let its circumcircle meet AB, BC, CD, DA at E', F', G', H' (Figure 4). By [2], the perpendiculars to AB, BC, CD at E', F', G' concur at a single point P', the isogonal conjugate of P in XBC. Analogously, the perpendiculars to AB, CD, DA at E', G', H' concur at the isogonal conjugate of P in XAD. In other words, $P'H' \perp DA$ and is the isogonal conjugate of P in both XAD and XBC, implying that P' is the desired isogonal conjugate of P in $ABCD$. □

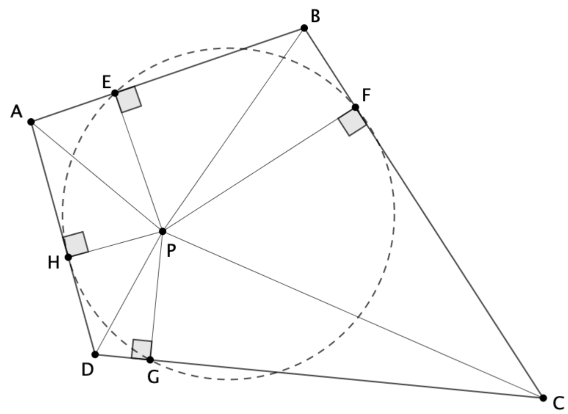

Figure 4: Lemma 1 – projections onto sides of quadrilateral

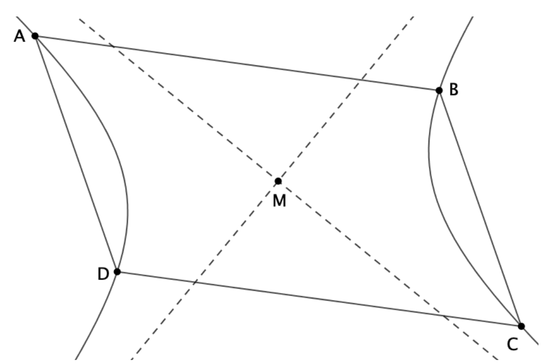

Figure 5: Locus of excellent points in a parallelogram

1.2. Degenerate cases

One case where the locus of isogonal conjugates becomes degenerate is when A, B, C, D are collinear on a line ℓ. In this case, the locus becomes the line ℓ along with the circle centered on ℓ whose inversion swaps A with C and B with D, if this circle exists.

Another case is when $ABCD$ is a parallelogram, where we have the following characterization:

Theorem 3. *If $ABCD$ is a parallelogram, the locus of excellent points is the line of infinity, along with the circumhyperbola passing through the points of infinity on the two angle bisectors of $\angle ABC$* (Figure 5).

Proof. By our extension of isogonality to $\mathbb{RP}^2$, the line of infinity is part of this locus. Then for all points $P \in \mathbb{R}^2$, by the dual of Desargues' involution theorem ([5], 133), (PA, PC), (PB, PD) are isogonal if and only if angle APC has the same angle bisectors as the pair of lines through P parallel to AB and AD. Thus if P_1 and P_2 are the points of infinity along with these angle bisectors of $\angle BAD$, we essentially need to find the locus $P \in \mathbb{R}^2$ for which the angle bisectors of APC are parallel to ℓ_1 and ℓ_2.

We claim that this locus is the hyperbola $\mathcal{H}$ centered at the midpoint M of AC passing through P_1, P_2, A, C. For any point $P \in \mathcal{H}$, $\mathcal{H}$ becomes the circumrectangular hyperbola of triangle PAC centered at M, which is the isogonal conjugate of the perpendicular bisector of AC wrt PAC. The isogonal conjugates of P_1, P_2 in PAC then become the two arc midpoints of AC in (PAC), so P_1, P_2 are indeed the points of infinity along the angle bisectors of PAC. For the other direction, take any point P such that $\angle APC$ has angle bisectors passing through P_1, P_2. Then the isogonal conjugate $\mathcal{H}'$ of the perpendicular bisector of AC wrt PAC will also pass through P_1, P_2, implying that $\mathcal{H} \equiv \mathcal{H}'$, so $P \in \mathcal{H}$, as desired. It is now clear that $B, D \in \mathcal{H}$, which completes the proof. □

For the rest of the paper, we will assume quadrilateral $ABCD$ does not fall under either of these cases. In particular, A, B, C, D are not all collinear, and the midpoints of AC, BD are distinct.

2. Preliminary lemmas

Up until Section 6, we will work in $\mathbb{RP}^2$. All angles are directed mod $180°$.

The following provides another well-known characterization of isogonal conjugates.

Definition 4. Let P be the Miquel Point of $ABCD$; we will also call the Miquel Point the *spiral center*, short for the center of spiral similarity. For any point Y, call the unique point Y' for which P is the spiral center of $AYCY'$ the *spiral inverse* of Y.

Theorem 4. *The spiral inverse X' of an excellent point X is also the isogonal conjugate of X.*

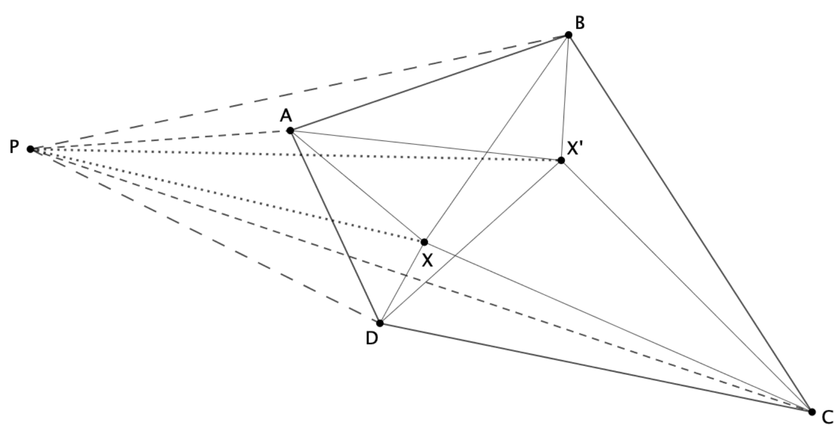

Figure 6: Spiral inverses equal isogonal conjugates

Proof. Let $E = AD \cap BC$ and $F = AB \cap CD$. If no three of A, B, C, D are collinear, we have no problems, and otherwise we will assume without loss of generality that A, B, D are collinear. Either way, the following relation is true:

$$\angle DX'C = \angle DX'P + \angle PX'C = \angle XBP + \angle PAX = \angle APB + \angle BXA = \angle DEC + \angle CXD.$$

Note that if A, B, D are collinear, then we would have $B \equiv E$ and $D \equiv F$, but the above angle chase would still hold. Similarly, $\angle EX'C = \angle EDC + \angle CXE$, implying that X, X' are isogonal conjugates in CDE.

If no three of A, B, C, D are collinear (Figure 6), then analogously, X, X' are isogonal conjugates in BCF, implying X, X' are isogonal conjugates in $ABCD$, so we are done. Otherwise, under our assumption that A, B, D are collinear, X, X' will be isogonal conjugates in BCD. Since X is excellent, this implies X, X' are isogonal conjugates in $ABCD$, as desired. □

The following proof reveals that the locus of excellent points X for a given quadrangle is cubic (Figure 7).

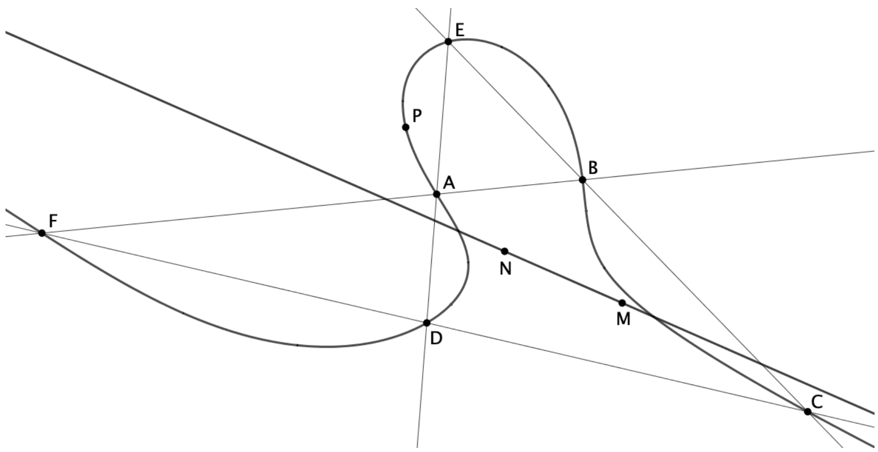

Figure 7: Locus of excellent points: the isogonal cubic

Proof. Proving that the locus is cubic amounts to examining the equation

$$\frac{d-x}{a-x} \Big/ \overline{\frac{d-x}{a-x}} = \frac{c-x}{b-x} \Big/ \overline{\frac{c-x}{b-x}}$$

in the complex plane ([1], 6.1). Expanding this gives the desired third-degree equation in x. Note that the coefficients of 3^{rd} degree coefficients $x^2\overline{x}$, $x\overline{x}^2$ in the expansion are both zero if and only if $a + c = b + d$, which corroborates our earlier finding that if $ABCD$ is a parallelogram, then the locus would be degenerate. □

Definition 5. Denote by $\mathcal{C}$ the cubic which is the locus of all excellent points X. We will sometimes call $\mathcal{C}$ the *"isogonal cubic"* throughout this paper.

For the rest of this paper, we will assume that $\mathcal{C}$ is non-degenerate. Now, we may also recall the following well-known fact.

Theorem 5 (Isogonal conjugate at infinity)**.** *Let M, N be the midpoints of AC, BD. Then the isogonal conjugate of P is the point of infinity along MN.*

Proof. Consider the parabola $\mathcal{P}$ tangent to the sides of $ABCD$. Considering $\mathcal{P}$ as an inscribed parabola of ABC, by [12], the focus of $\mathcal{P}$ lies on (ABC) and the directrix passes through the orthocenter of ABC. Applying the same logic to triangles BCD, CDA, DAB, we conclude that the focus of P is the Miquel Point P, and its directrix is the Gauss-Bodenmiller line ([11]), which is perpendicular to MN.
It is well known ([2]) that for any conic with foci X_1, X_2 and any point X for which tangents from X exist, XX_1 and XX_2 are isogonal in the angle formed by the tangents from X to the conic. Applying this to $X \equiv A$ and conic $\mathcal{P}$, we conclude that AP and the perpendicular from A to the directrix are isogonal in $\angle BAD$. Similar relations with B, C, D imply the desired result. □

Theorem 6. *Consider two pairs $(X, X'), (Y, Y')$ of isogonal conjugates. Then A, C are isogonal conjugates in $XYX'Y'$.*

Proof. Since $(AX, AX'), (AY, AY')$ are isogonal, A, B, C, D are excellent in $XYX'Y'$. Since A, C are spiral inverses in $XYX'Y'$, they are isogonal conjugates, as desired. □

The following corollary immediately follows.

Corollary 1. *For isogonal conjugates X, X' and excellent point Y, (YX, YX') and (YA, YC) are isogonal.*

This directly implies the following critical characterization:

Corollary 2 (Generalization of isogonal cubic)**.** *Consider two pairs $(X, X'), (Y, Y')$ of isogonal conjugates. Then the isogonal cubic of $ABCD$ is the isogonal cubic of $XYX'Y'$. Furthermore, any pair of isogonal conjugates (K, L) in $ABCD$ are also isogonal conjugates in $XYX'Y'$.*

Proof. By Corollary 1, for any point Z, if pairs of lines $(ZA, ZC), (ZB, ZD)$ are isogonal, then lines $(ZX, ZX'), (ZY, ZY')$ are also isogonal, so $ABCD$ and $XYX'Y'$ indeed share the same isogonal cubic. The second part then directly follows from Corollary 1. □

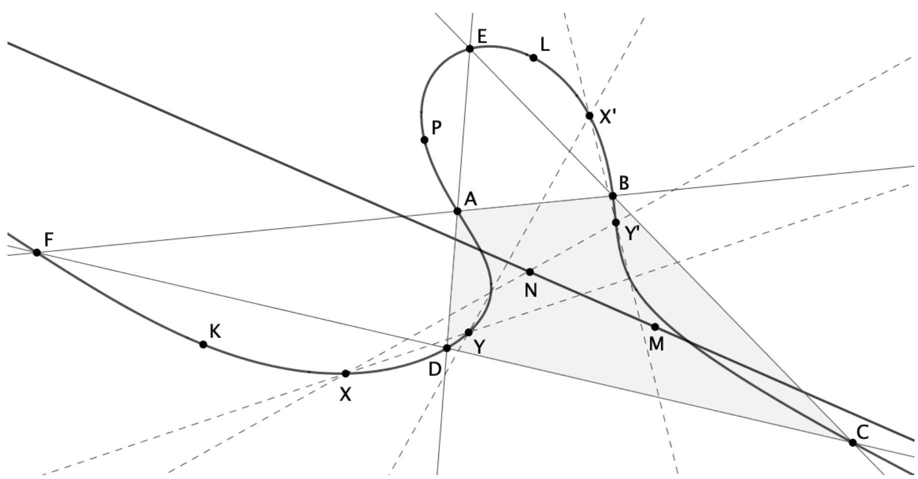

Figure 8: Quadrilateral completeness

Thus the following is true by the dual of Desargues' Involution Theorem on $XYX'Y'$:

Corollary 3 (Quadrilateral Completeness)**.** *For two pairs $(X, X'), (Y, Y')$ of isogonal conjugates, $XY \cap X'Y'$ and $XY' \cap X'Y$ lie on $\mathcal{C}$* (Figure 8).

We now illustrate the relationship between isogonality and inconics.

Theorem 7. *Let $ABCD$ have inconic ω and isogonal conjugates X, X'. Then the tangents to ω from X and X' intersect at two pairs of isogonal conjugates.*

Proof. Denote by $IJKL$ the quadrilateral formed by these two pairs of tangents, such that IJ, JK, KL, LI are tangent to ω. Since ω is an inconic of $XIX'K$ and the tangents from A to ω (AB and AD) are isogonal in $\angle XAX'$, by the dual of Desargues' Involution on $XIX'K$ with respect to point A, (AI, AK) are also isogonal in $\angle BAD$. Similar arguments imply $(I, K), (J, L)$ are isogonal conjugates as desired. □

Corollary 4. *For excellent point X and isogonal conjugates Y, Y', the line XY' is tangent to the inconic of lines AB, BC, CD, DA, XY.*

3. Relationship of inconics with excellent points

Consider any excellent point X. Let ω be the inconic of AB, BC, CD, DA, PX. It is well-known that the center of ω lies on MN ([9]).

By Corollary 4, since P and the point of infinity P_∞ along MN are isogonal conjugates, the line ℓ_1 through X parallel to MN must also be tangent to ω. Reflect ℓ_1 over the center of ω to get the second tangent ℓ_2 from the point of infinity ∞_{MN} along MN to ω; let ℓ_2 meet the second tangent from P to ω (other than PX) at X' (Figure 9).

Applying the proof of Corollary 7 on isogonal conjugates P and P_∞, points X and X' must also be isogonal conjugates. Let PX meet ℓ_2 at Y, and let PX' meet ℓ_1 at Y'; then Y, Y' are isogonal conjugates as well. We immediately get the following two corollaries:

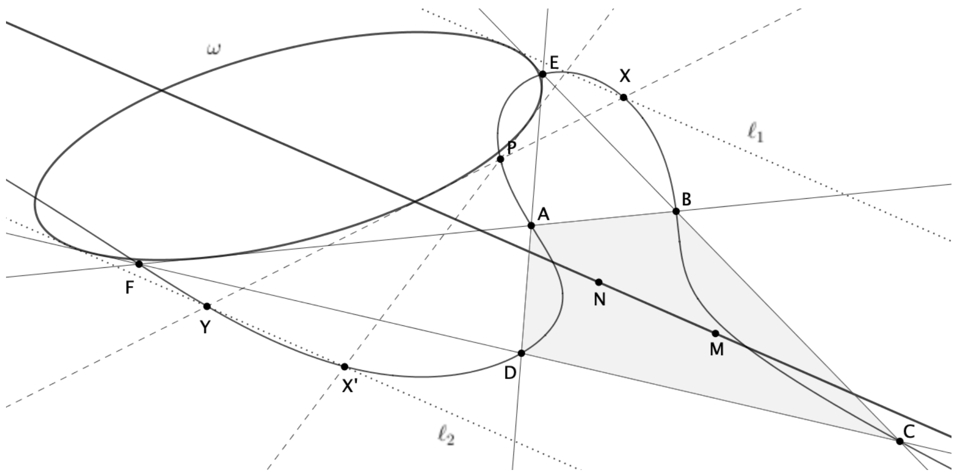

Figure 9: Tangents to an inconic

Theorem 8. *The midpoint of any two isogonal conjugates lies on MN.*

Theorem 9. *For $X \in \mathcal{C}$ and P_∞ the point of infinity along MN, let $Y = PX \cap \mathcal{C}$ and $X' = P_\infty Y \cap \mathcal{C}$. Then X, X' are isogonal conjugates.*

We may also note that the midpoint of XY lies on MN. Since $\ell_1 \parallel \ell_2$, the bisectors of $\angle PXP_\infty, \angle PYP_\infty$ are parallel (perpendicular) to each other, with the perpendicular pairs of bisectors intersecting on MN. Thus, the following is a direct result.

Theorem 10 (Parallel bisectors)**.** *If X, Y lie on $\mathcal{C}$ with $P \in XY$, then the midpoint of XY lies on MN, and the bisectors of $\angle AXC$ and $\angle AYC$ are parallel to each other.*

This produces a neat construction: if we are given an excellent point X, we may construct an excellent point Y such that the bisectors of $\angle AXC$ and $\angle AYC$ are parallel to each other. By Corollary 10, this is done by letting lines PX and MN intersect at a point O and setting Y to be the reflection of X over O (Figure 10).

4. Constructing elements of the isogonal cubic

We begin by noting that since $\mathcal{C}$ has real coefficients, for any two non-singular points X and Y on $\mathcal{C}$ in $\mathbb{RP}^2$, line XY is either tangent to $\mathcal{C}$ at X or Y, or XY intersects $\mathcal{C}$ at a third point in $\mathbb{RP}^2$.

We begin with a well-known general lemma about cubics.

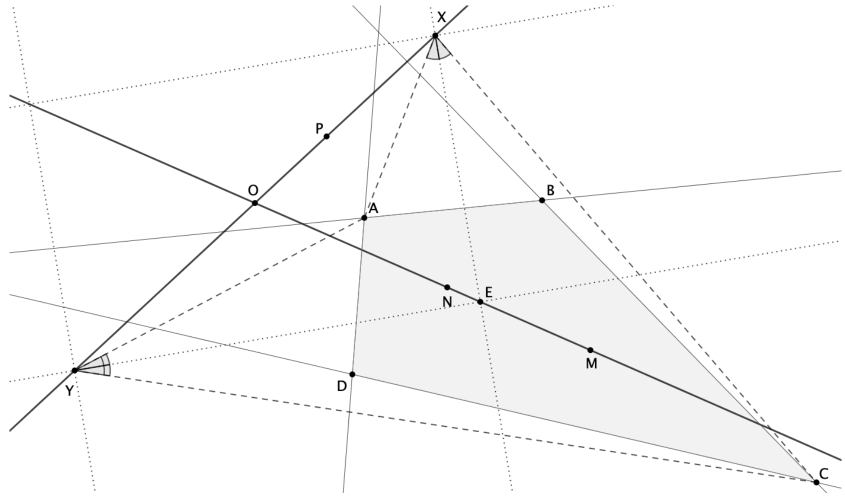

Figure 10: Angles with parallel bisectors

Lemma 2. *From any point X on general non-degenerate cubic $\mathcal{C}$, there are at most 4 points $Y \in \mathcal{C}$ other than X for which XY intersects $\mathcal{C}$ at Y with multiplicity 2.*

Proof. Note that if X is singular, there are no such points Y, or else line XY would intersect $\mathcal{C}$ at both X and Y with multiplicity 2, yielding $2+2=4$ total intersections. Assuming that X is non-singular now, consider the embedding of $\mathcal{C}$ in $\mathbb{CP}^2$ with equation $F(x,y,z)=0$. For any point $Y=(p:q:r)$ on $\mathcal{C}$, Y is either a singular point, or the equation of the tangent at Y is given by

$$\frac{\partial F}{\partial x}(p,q,r)\cdot x + \frac{\partial F}{\partial y}(p,q,r)\cdot y + \frac{\partial F}{\partial z}(p,q,r)\cdot z = 0$$

We want $X=(x_0:y_0:z_0)$ to satisfy the above equation, so fixing X gives us an equation in p,q,r with degree $3-1=2$. Let $g(x,y,z)$ denote the expression

$$\frac{\partial F}{\partial x}(x,y,z)\cdot x_0 + \frac{\partial F}{\partial y}(x,y,z)\cdot y_0 + \frac{\partial F}{\partial z}(x,y,z)\cdot z_0$$

Regardless of whether Y is a singular point of $\mathcal{C}$ or XY is tangent to $\mathcal{C}$ at Y, all such points Y will be solutions to the cubic $F(x,y,z)=0$ and the conic $g(x,y,z)=0$, which by Bezout's Theorem ([4], Section 5.3) gives at most $3(3-1)$ total solutions.

Note that X itself also satisfies both equations; we now claim that X is actually a solution with multiplicity at least 2. Let ℓ be the tangent to $\mathcal{C}$ at X; then ℓ has equation

$$\frac{\partial F}{\partial x}(x_0,y_0,z_0)\cdot x + \frac{\partial F}{\partial y}(x_0,y_0,z_0)\cdot y + \frac{\partial F}{\partial z}(x_0,y_0,z_0)\cdot z = 0$$

To prove that X is a solution to both F and g with multiplicity 2, we use the fact that $I_X(F,g) \geq m_X(F)m_X(g)$, where I_X denotes the multiplicity of the intersection of curves F and g at X, and $m_X(F), m_X(g)$ denoting the multiplicity of point P on curves F,g ([4], Section 3.3).

If X is a singular point of F, then $I_X \geq 2$ as desired. Otherwise, equality holds iff the tangents at X to F and g are distinct. Thus it suffices to show that ℓ is tangent to the conic $\mathcal{H}$ formed by g. To prove this, the tangent to $\mathcal{H}$ at X is given by equation $Ax+By+Cz=0$, where

$$A = \frac{\partial^2 F}{\partial x^2}(x_0,y_0,z_0)\cdot x_0 + \frac{\partial^2 F}{\partial y^2}(x_0,y_0,z_0)\cdot y_0 + \frac{\partial^2 F}{\partial z^2}(x_0,y_0,z_0)\cdot z_0$$

and B, C are defined similarly. By Euler's Homogeneous Function Theorem ([10]),

$$2 \cdot \frac{\partial F}{\partial x}(x_0, y_0, z_0) = \frac{\partial^2 F}{\partial x^2}(x_0, y_0, z_0) \cdot x_0 + \frac{\partial^2 F}{\partial y^2}(x_0, y_0, z_0) \cdot y_0 + \frac{\partial^2 F}{\partial z^2}(x_0, y_0, z_0) \cdot z_0$$

which implies that

$$A = 2 \cdot \frac{\partial F}{\partial x}(x_0, y_0, z_0), \qquad B = 2 \cdot \frac{\partial F}{\partial y}(x_0, y_0, z_0), \qquad C = 2 \cdot \frac{\partial F}{\partial z}(x_0, y_0, z_0)$$

so the tangent to $\mathcal{H}$ at X indeed has the same equation as ℓ, as desired.
Thus X is a solution to $\mathcal{C}$ and $\mathcal{H}$ with multiplicity at least 2, so there are at most $3(3-1)-2 = 4$ such points Y, as desired. $\square$

The following lemma also better characterizes $\mathcal{C}$.

Lemma 3. *In $\mathbb{RP}^2$, $\mathcal{C}$ contains exactly one point at infinity.*

Proof. The embedding of $\mathcal{C}$ in $\mathbb{CP}^2$ will contain the circular points at infinity I, J by virtue of being isogonal conjugates, so back in $\mathbb{RP}^2$ there can only be one real point at infinity. On the other hand, given $ABCD$ the point of infinity along the Newton-Gauss line will lie on $\mathcal{C}$, so there is exactly one. $\square$

To better establish tangencies in $\mathcal{C}$, we first need to examine singular points.

Theorem 11 (Singular points on the isogonal cubic)**.** *A point $I \in \mathcal{C}$ is a singular point if and only if the isogonal conjugate of I is itself.*

Proof. We remind our readers of our assumption in Section 1 that $\mathcal{C}$ is not degenerate.
First, we prove that if I is its own isogonal conjugate, then it is a singular point. Assume the contrary, that I is not singular; then there are at most five non-singular points $X \in \mathcal{C}$ such that XI is tangent to $\mathcal{C}$ at either X or I. For all points X such that XI is *not* tangent to $\mathcal{C}$, line XI will intersect $\mathcal{C}$ at a point $Y \neq I, X$. By Corollary 1 this means that the line through I, X, Y bisects angles $\angle AXC$, $\angle AYC$.
In particular, this means line CX is the reflection of line AX over line XY, and line CY is the reflection of line AY over line XY, which implies that C is the reflection of A over XY. Thus XI is the perpendicular bisector of AC. But line XI rotates around I as we vary X along the cubic, contradicting the uniqueness of the perpendicular bisector of AC, the desired contradiction.
Next, we prove that if I is a singular point, then the isogonal conjugate of I is itself. Assume the contrary; then let $J \neq I$ be the isogonal conjugate of I. Choose any isogonal conjugates K, L distinct from I, J (though we can set $K \equiv A$ etc). By Corollary 3, $X = KI \cap LJ$ and $Y = KJ \cap LI$ will lie on $\mathcal{C}$. Since I is a singular point, KI will not intersect $\mathcal{C}$ at a point other than K or I, so X is either the same as K or I.
If $X \equiv I$, then I, L, J are collinear. But since I is singular, line ILJ intersects $\mathcal{C}$ at I with multiplicity 2, so our assumption that $L \neq I, J$ implies that $I \equiv J$, the desired contradiction.
Thus we must have $X \equiv K$, so K, L, J are collinear. Thus we conclude J lies on line KL for any isogonal conjugates K, L. Choosing another pair (R, S) of isogonal conjugates such that no three of K, L, R, S are collinear, let $T = KR \cap LS$ and $U = KS \cap LR$; by Corollary 3, T and U are isogonal conjugates in $\mathcal{C}$, so KL, RS, TU concur at a single point J.
Consider a conic $\mathcal{H}$ passing through K, L, R, S but not tangent to line TU. Then the pole of line TU in $\mathcal{H}$ is the intersection of KL and RS, which is precisely J, which lies on line TU. For the pole of TU in $\mathcal{H}$ to lie on TU itself, TU must be tangent to $\mathcal{H}$ at J, the desired contradiction. $\square$

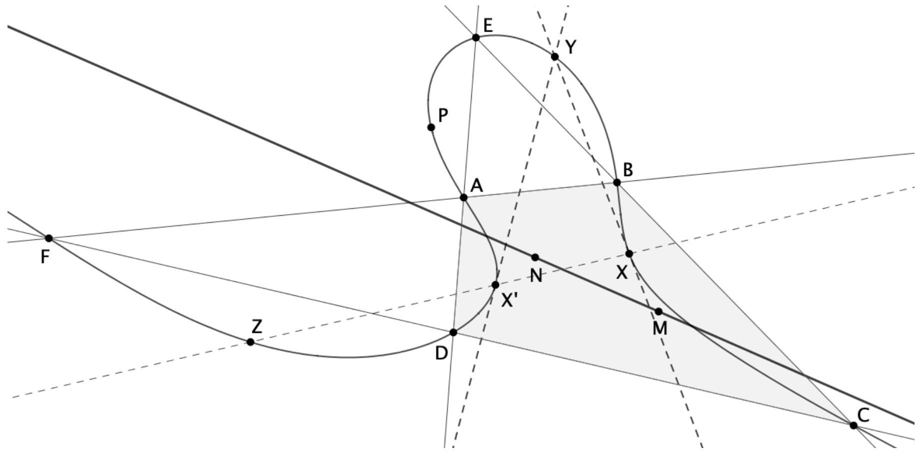

Figure 11: Tangents to the cubic

In this case where $\mathcal{C}$ contains a singular point, it becomes a *strophoid*. The paper [6] by H. Stachel explores this specific case in detail.

Now, we will construct the tangent to $\mathcal{C}$ at any non-singular point X as follows (Figure 11).

Theorem 12 (Tangent to isogonal cubic)**.** *For isogonal conjugates X, X', the isogonal ℓ to XX' in $\angle AXC$ is tangent to $\mathcal{C}$ at X.*

Proof. As we move a point $Y \in \mathcal{C}$ with isogonal conjugate Y', XY and XY' are isogonal in $\angle AXC$. Therefore, as Y approaches X', Y' will approach X, eventually letting XY' intersect $\mathcal{C}$ with multiplicity 2 at X. □

In particular, the tangent to $\mathcal{C}$ at the point of infinity along MN is given by the unique (by Corollary 3) asymptote of $\mathcal{C}$.

Theorem 13. *Let the tangents to $\mathcal{C}$ at isogonal conjugates X, X' meet at Y, and let XX' meet $\mathcal{C}$ at $Z \neq X, X'$. Then Y, Z are isogonal conjugates.*

Proof. Let Z^* be the isogonal conjugate of Z. By Corollary 6 (XZ, XZ^*) are isogonal in $\angle AXC$, and $(X'Z, X'Z^*)$ are isogonal in $\angle AX'C$, so $Z^* \equiv Y$, as desired. □

Lemma 4. *The bisectors of $\angle AZC$ are perpendicular and parallel to XX'.*

Proof. Examining isogonal conjugates $(A, C), (X, X')$, this follows from Corollary 1. □

Theorem 14. *Let PZ meet $\mathcal{C}$ at $W \neq Z$. Then $WX = WX'$.*

Proof. By Corollary 10, the bisectors of XWX' are perpendicular and parallel to XX', which gives the desired result. □

Corollary 5. *If we denote P by 0 on the conic, then $W = X + X'$ under cubic addition ([4], Proposition 5.6.4). Thus, the cubic sum of any two isogonal conjugates X, X' is equidistant from X, X'.*

We thus obtain the following construction, if we desire to find all pairs of isogonal conjugates (Y, Y') such that YY' passes through a given excellent point X.

Theorem 15. *For $X \in \mathcal{C}$, let distinct $Y, Z \neq X$ lie on $\mathcal{C}$ such that X, Y, Z are collinear and XY bisects $\angle AXC$. Then Y, Z are isogonal conjugates* (Figure 12).

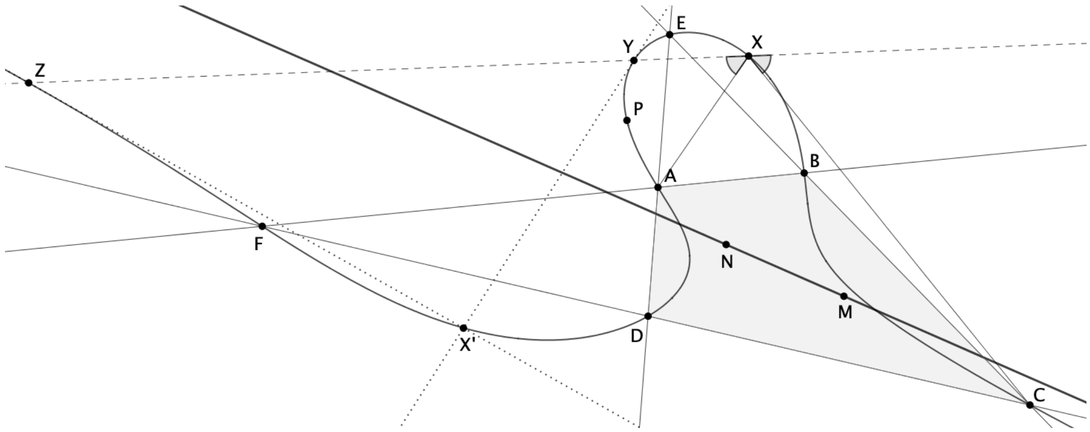

Figure 12: Isogonal conjugates collinear with a given point

Proof. Let Y' be the isogonal conjugate of Y; then by Corollary 1, (XY, XY') and (XA, XC) are isogonal. Since XY bisects $\angle AXC$, XY' and XY must be the same line, implying that either $Y \equiv Y'$ or $Z \equiv Y'$.

In the latter case we are done. In the former case, Y must be an incenter or excenter of $ABCD$, so by Corollary 11 Y is a singular point. But X, Y, Z are collinear and distinct despite line XYZ intersecting Y with multiplicity 2, the desired contradiction. □

Note that this also gives us a construction of points Y on $\mathcal{C}$ such that ZY is tangent to $\mathcal{C}$ at Y, where Z is a fixed point on $\mathcal{C}$. This is done by letting X be the isogonal conjugate of Z and intersecting the angle bisectors of $\angle AXC$ with $\mathcal{C}$. By the above, there will be up to four such intersections X_1, X_2, X_3, X_4 on $\mathcal{C}$ for which ZX_1, ZX_2, ZX_3, ZX_4 are tangent to $\mathcal{C}$ at X_1, X_2, X_3, X_4.

5. Constructing intersections with lines and circles

Theorem 16 (Line intersection)**.** *Consider excellent points X, Y. Denote by Z the intersection of the reflections of XY over the bisectors of $\angle AXC$ and $\angle AYC$. Then the intersection of XY with $\mathcal{C}$ other than X, Y is also the isogonal conjugate of Z. Furthermore, $PXYZ$ is cyclic.*

Proof. Let $W = XY \cap \mathcal{C}$, and let W have isogonal conjugate W'. By Corollary 1, XW and XW' are isogonal in $\angle AXC$, so line XW' is the reflection of XY over the bisectors of $\angle AXC$, implying that $W' \equiv Z$, proving that $XY \cap \mathcal{C}$ is indeed the isogonal conjugate of Z.
To prove $PXYZ$ is cyclic, let X', Y' be the isogonal conjugates of ABC. By Corollary 3, $XY \cap \mathcal{C}$ lies on $X'Y'$, hence $W \in \mathcal{X}'Y'$. Then under spiral inversion, line $X'Y'W$ is mapped to the circumcircle of XYZ, which must pass through P, as desired. □

As a direct corollary, we have the following well-known theorem:

Corollary 6 (Spiral center of isogonal conjugates lies on circumcircle)**.** *For isogonal conjugates (A, C), (B, D) in $\triangle XYZ$, the spiral center of $ABCD$ lies on (XYZ).*

We may remark that this provides a construction of the intersection of $\mathcal{C}$ with any line XY, provided that X and Y lie on $\mathcal{C}$ themselves. Next, we characterize intersections of $\mathcal{C}$ with circles.

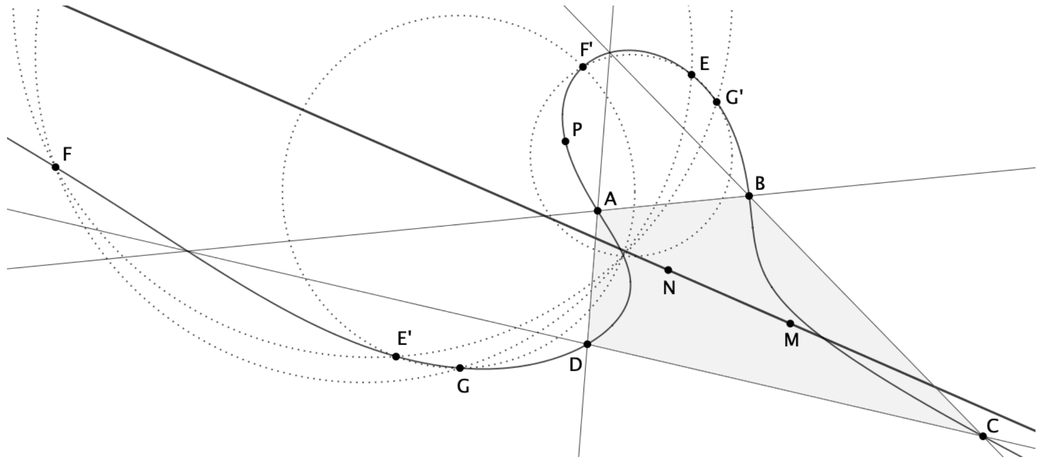

Figure 13: Intersecting with circles

Theorem 17 (Circle intersection)**.** *Consider excellent points E, F, G with isogonal conjugates E', F', G'. Then (EFG) meets $\mathcal{C}$ at one other point which lies on $(EF'G'), (E'FG'), (E'F'G)$.*

Proof. There are two parts to this. First, we prove that if H is a point on $\mathcal{C}$ such that $EFGH$ is cyclic, then H lies on $(E'F'G)$ (which would imply that it lies on $(EF'G'), (E'FG')$ by symmetry). To prove this, by Corollary 6 with $EGE'G'$, since $H \in \mathcal{C}$, $\angle F'GE' = \angle EGF = \angle EHF = \angle F'HE'$.
Next, we prove that if $H \equiv (EFG) \cap (E'F'G)$, then $\angle F'HE' = \angle F'GE' = \angle EGF = \angle EHF$, which implies that $E \in \mathcal{C}$ as desired (Figure 13). □

We may remark that this provides a construction for all points on (EFG) lying on $\mathcal{C}$, provided E, F, G lie on $\mathcal{C}$ themselves. The following is in fact true.

Theorem 18. *All circles intersect $\mathcal{C}$ in the real plane at at most 4 points.*

Proof. The circular points at infinity lie on $\mathcal{C}$ by virtue of being isogonal conjugates. The result follows from Bezout's Theorem, where curves of degree 2 and 3 meet for at most six points in $\mathbb{CP}^2$. □

6. Characterizing the isogonal cubic

For this section, we will work in $\mathbb{CP}^2$ and let I, J denote the circular points at infinity. We must first extend the definition of isogonality to $\mathbb{CP}^2$ as follows:

Definition 6. For distinct points $P, A, B, C, D \in \mathbb{CP}^2$, call the two pairs of lines (PA, PB) and (PC, PD) *isogonal* if and only if the three pairs of lines (PA, PB), (PC, PD), (PI, PJ) comprise a single involution, where I, J are the circular points at infinity.

One can check this complies with the angular definition of isogonality if $P, A, B, C, D \in \mathbb{RP}^2$.

Corollary 7. *For distinct points A, B, C, D such that neither of I, J lie on any of the lines AB, BC, CD, DA, the locus of points X for which $(XA, XB), (XC, XD)$ are isogonal is a cubic (or curve of lesser degree) through A, B, C, D, I, J in $\mathbb{CP}^2$.*

Proof. For any six points A, B, C, D, E, F, the locus of points X for which (XA, XB), (XC, XD), (XE, XF) comprise a single involution is a cubic through A, B, C, D, E, F. Setting E, F as the circular points at infinity gives the desired result. □

Hence we will call a non-degenerate cubic $\mathcal{C}$ the "isogonal cubic" of quadrilateral $ABCD$ if it is the locus of all points X for which $(XA, XC), (XB, XD)$ are isogonal (using the new definition).

Corollary 8 (Loci of isogonality). *If the locus of points X for which $(XA, XC), (XB, XD)$ are isogonal is a non-degenerate cubic, then neither I nor J cannot lie on any of the lines AB, BC, CD, DA.*

Proof. Assume the contrary, that WLOG $I \in AB$. Then for any point P on line AB, pairs (XA, XC), (XB, XD), (XI, XJ) are part of a single degenerate involution. Thus the locus of points X for which (XA, XC), (XB, XD) are isogonal includes line AB, contradicting the proposition that the locus is a non-degenerate cubic. □

In other words, if $ABCD$ has a non-degenerate isogonal cubic $\mathcal{C}$, then I and J will not lie on AB, BC, CD, DA.

Now, the main result of this paper is the following:

Theorem 19 (Characterization of all isogonal cubics). *Let $\mathcal{C}$ be a non-degenerate cubic in $\mathbb{CP}^2$ containing circular points at infinity I and J at non-singular points. Then the following two conditions are equivalent:*

(1) *There exist non-singular $A, B, C, D \in \mathcal{C}$ such that $\mathcal{C}$ is the isogonal cubic of $ABCD$.*

(2) *The tangents to $\mathcal{C}$ at I, J intersect each other on $\mathcal{C}$.*

We begin with the following direct result of Cayley-Bacharach ([7]).

Lemma 5 (Cubics containing complete quadrilateral). *For P, Q on non-degenerate cubic $\mathcal{C}$, consider $T \in \mathcal{C}$ and let $U = PT \cap \mathcal{C}$, $V = QT \cap \mathcal{C}$ such that P, Q, T, U, V are non-singular. Then $PV \cap QU \in \mathcal{C}$ iff $PP \cap QQ \in \mathcal{C}$.*

Proof. Let $X = PP \cap QQ, Y = PV \cap QU$. Applying the Cayley-Bacharach Theorem on triples of lines (XPP, QTV, QUY), (XQQ, PTU, PVY) completes both directions. □

Lemma 6 (Locus of involution). *Consider distinct points A, B, C, D, E, F in general position and $G = AC \cap BD, H = AD \cap BC, I = AE \cap BF, J = AF \cap BE$, such that none of the ten points are the circular points at infinity. Then there is a unique cubic $\mathcal{C}$ through these ten points. Furthermore, for every $P \in \mathcal{C}$, we have (PA, PB), (PC, PD), (PE, PF), (PG, PH), and (PI, PJ) are part of a fixed involution* (Figure 14).

Proof. By the dual of Desargues' Involution Theorem, the locus $\mathcal{C}$ of all points P for which (PA, PB), (PC, PD), (PE, PF) are part of a single involution is a cubic through $A, B, C, D, E, F, G, H, I, J$. Thus, there exists a cubic through these 10 points. Since A, B, C, D, E, F are in general position, no four of the 10 constructed points are collinear. Since $\mathcal{C}$ passes through these 10 fixed points, the cubic through these 10 points must be unique, as desired. □

We are finally set up to prove the main result.

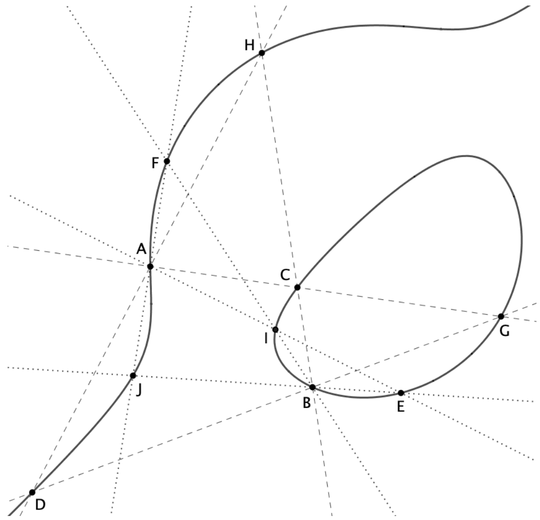

Figure 14: Two complete quadrilaterals

Theorem 20 (Characterization in $\mathbb{CP}^2$, condition (2) $\Rightarrow$ (1))**.** *Let $\mathcal{C}$ be a non-degenerate cubic through I, J such that I and J are non-singular, and II intersects JJ at a point X on $\mathcal{C}$. Then there exist non-singular points $A, B, C, D \in \mathcal{C}$ apart from I, J such that $\mathcal{C}$ is the isogonal cubic of $ABCD$.*

Proof. Choose any point $A \in \mathcal{C}$. Let $B = IA \cap \mathcal{C}, D = JA \cap \mathcal{C}$; by Corollary 5, ID and JB meet a point C on $\mathcal{C}$. Construct four points $A', B', C', D' \in \mathcal{C}$ distinct from A, B, C, D analogously, where $I = A'B' \cap C'D'$ and $J = A'D' \cap B'C'$. We may select A, A' such that none of $A, A', B, B', C, C', D, D'$ are singular.
By Corollary 6, $\mathcal{C}$ is the locus of points P for which $(PI, PJ), (PA, PC), (PA', PC')$ are part of a single involution. But since this involution concerns the circular points at infinity, it follows that $\mathcal{C}$ is the locus for which $(PA, PC), (PA', PC')$ are isogonal. We are done by taking quadrilateral $AA'CC'$. □

Theorem 21 (Characterization in $\mathbb{CP}^2$, condition (1) $\Rightarrow$ (2))**.** *Let $\mathcal{C}$ be the non-degenerate isogonal cubic of $ABCD$ where A, B, C, D are non-singular. Suppose that I, J lie on $\mathcal{C}$ at non-singular points distinct from A, B, C, D. Then $II \cap JJ \in \mathcal{C}$.*

Proof. Let $X = AI \cap CJ$ and $Y = AJ \cap CI$; then X, Y are non-singular. Note that (XA, XC) and (XI, XJ) are the same pair of lines, so (XA, XC), (XB, XD), (XI, XJ) form an involution. (If X is the same point as either A, C, I, or J, we instead use the tangent to $\mathcal{C}$ at X when necessary.) So by definition, $X \in \mathcal{C}$; similarly, $Y \in \mathcal{C}$. By Corollary 5, this implies $II \cap JJ \in \mathcal{C}$ as desired. □

Returning to $\mathbb{R}^2$, we derive the complete characterization of all non-degenerate isogonal cubics:

Theorem 22 (Characterization of all isogonal cubics in $\mathbb{R}^2$)**.** *Let $\mathcal{C}$ be a non-degenerate cubic in $\mathbb{R}^2$, and let $\mathcal{C}_0$ denote its embedding in $\mathbb{CP}^2$. Then the following two conditions are equivalent:*

(1) *There exist distinct $A, B, C, D \in \mathcal{C}$ such that $\mathcal{C}$ is the isogonal cubic of $ABCD$.*
(2) *The circular points at infinity I, J lie on $\mathcal{C}_0$, and the tangents to $\mathcal{C}_0$ at I, J intersect each other on $\mathcal{C}_0$.*

Proof. The only aspects of the proof we need to modify for $\mathbb{R}^2$ are to prove that:

1. Under the conditions of (1), if $\mathcal{C}$ is the isogonal cubic of $ABCD$ where A, B, C, D are distinct, then A, B, C, D cannot be singular points of $\mathcal{C}_0$.
2. Under the conditions of (2), if any cubic $\mathcal{C}$ in $\mathbb{R}^2$ satisfies that its embedding $\mathcal{C}_0$ in $\mathbb{CP}^2$ passes through I and J, then I and J are not singular.
3. Under the conditions of (2), for $A \in \mathcal{C}$ such that $K = AI \cap \mathcal{C}_0, L = AJ \cap \mathcal{C}_0, A' = IL \cap JK$ all lie on distinct points of $\mathcal{C}_0$, then the point A' will be contained in $\mathcal{C}$ as well.

Let $\mathcal{C}$ have Cartesian equation $ax^3 + bx^2y + cxy^2 + dy^3 + G(x, y) = 0$, where G is a second-degree polynomial in x, y. Then a, b, c, d must be real, and since $\mathcal{C}$ is not degenerate, they cannot all be zero. Thus $\mathcal{C}_0$ has equation $F(x, y, z) = 0$ where $F(x, y, z) = ax^3+bx^2y+cxy^2+dy^3 + zP(x, y, z)$, where $P(x, y, z)$ is a second-degree homogeneous polynomial in x, y, z.

Proof. For (a), assume that $\mathcal{C}$ is the isogonal cubic of $ABCD$. Note that A is a singular point in $\mathcal{C}$ iff it is a singular point in $\mathcal{C}_0$, because both are equivalent to $\frac{\partial F}{\partial x}(A) = \frac{\partial F}{\partial y}(A) = \frac{\partial F}{\partial z}(A) = 0$, the same equation in both $\mathbb{RP}^2$ and $\mathbb{CP}^2$. We just need to show that A is not a singular point in $\mathbb{R}^2$.

By Corollary 11, A is singular if and only if A is the isogonal conjugate of itself in $ABCD$. But the isogonal conjugate of A is C, and since A and C are distinct, this cannot happen. Therefore, A and similarly B, C, D cannot be singular points of $\mathcal{C}_0$. This proves part (a). ■

Proof. For (b), we consider general cubic $\mathcal{C}$ which contains I, J. Plugging in $I = (1 : i : 0)$ yields equation $a + bi - c - di = 0$, which implies that $a = c$ and $b = d$ because a, b, c, d are all real. Thus $\mathcal{C}$ has equation $(x^2 + y^2)(ax + by) + zP(x, y, z)$. We get $\frac{\partial F}{\partial x}(1 : i : 0) = 3ax^2 + 2xby + ay^2$.

Assume, for the sake of contradiction, that I is a singular point. We require $\frac{\partial F}{\partial x} = 0$ for $(1 : i : 0)$, which rearranges to $2a + 2bi = 0$. Since a, b are real, this implies that $a = b = 0$, so a, b, c, d are all zero - the desired contradiction. This proves part (b). ■

Proof. For (c), it suffices to show that $A' \in \mathbb{R}^2$. Note that I, J, A, K, L, A' all lie on $\mathcal{C}_0$, which has all real coefficients. Now, K, L do not lie on the line of infinity, or else A would lie on the line of infinity, which would imply $\mathcal{C}_0$ containing four points on a line and thus be degenerate, a contradiction. Thus K and L are contained in $\mathbb{C}^2$ and thus can be expressed in Cartesian coordinates (k_x, k_y) and (l_x, l_y) respectively. Since $A \in \mathbb{R}^2$, note that K and L cannot lie in $\mathbb{R}^2$ — otherwise, the lines AK and AL would be entirely contained in $\mathbb{RP}^2$, and would therefore never intersect the line of infinity at complex points I and J.

From part (b), $\mathcal{C}$ must have equation of the form $ax^3 + bx^2y + axy^2 + by^3 + G(x, y) = 0$ where a, b and the coefficients of G are real numbers. For K to satisfy this equation, the point K' whose Cartesian coordinates are the complex conjugates of K — that is, $K' = \left(\overline{k_x}, \overline{k_y}\right)$ in Cartesian coordinates — must also satisfy this equation, and thus lie on $\mathcal{C}_0$. Having started with A, I, K collinear, we now claim that A, J, K' are collinear. It suffices to show that

$$\begin{vmatrix} 1 & -i & 0 \\ a_x & a_y & 1 \\ \overline{k_x} & \overline{k_y} & 1 \end{vmatrix} = 0 \quad \text{given that} \quad \begin{vmatrix} 1 & i & 0 \\ a_x & a_y & 1 \\ k_x & k_y & 1 \end{vmatrix} = 0$$

Letting $k_x = p + qi$ and $k_y = r + si$ for $p, q, r, s \in \mathbb{R}$, the second determinant equation yields

$$0 = \begin{vmatrix} 1 & i & 0 \\ a_x & a_y & 1 \\ p + qi & r + si & 1 \end{vmatrix} = -r - si - q + pi + a_y - ia_x$$

where (a_x, a_y) are the Cartesian coordinates of A. Equating the real and imaginary parts yields $a_y = q + r, a_x = p - s$. Similarly, the first determinant equation gives us

$$0 = \begin{vmatrix} 1 & -i & 0 \\ a_x & a_y & 1 \\ p - qi & r - si & 1 \end{vmatrix} = -r + si - q - pi + a_y + ia_x$$

and equating the real and imaginary parts yields $a_y = q + r, a_x = p - s$ - the exact same conditions. Therefore, given A, I, K are collinear, we indeed conclude A, J, K' are collinear. In other words, K' is the unique intersection of $\mathcal{C}_0$ with AJ, hence $K' \equiv L$. Thus $A' = IK' \cap JK$, and A' will not be a point at infinity (otherwise K will also be a point at infinity). Hence, in quadrilateral $AKA'K' \in \mathbb{C}^2$, we derive that AK meets $A'K'$ at a point of infinity, and AK' meets $A'K$ at a point of infinity, so complex segments AA' and KK' share the same midpoint M. Letting A have Cartesian coordinates (m, n) in $\mathbb{C}^2$, this means that M has Cartesian coordinates

$$\left(\frac{a_x + m}{2}, \frac{a_y + n}{2}\right) = \left(\frac{k_x + \overline{k_x}}{2}, \frac{k_y + \overline{k_y}}{2}\right)$$

But $\frac{k_x + \overline{k_x}}{2}$ is just the real part of k_x, so the coordinates of M are real as well. Hence m and n are real, so $A' = (m, n)$ lies in $\mathbb{R}^2$, proving part (c). ■

With (a), (b), (c) proven, for the sake of completion we will show how this fully finishes our characterization. For the direction (1) ⇒ (2), we start with $ABCD$, and by (a) none of A, B, C, D are singular points. Then the result directly follows from Corollary 21.
For the direction (2) ⇒ (1), we start with circular points at infinity I, J lying on $\mathcal{C}_0$, which by (b) implies that I, J are not singular points. Assuming that the tangents to $\mathcal{C}_0$ at I and J intersect each other on $\mathcal{C}$, we can choose any point $A \in \mathcal{C}$ and letting $K = AI \cap \mathcal{C}_0$, $L = AJ \cap \mathcal{C}_0$, and $A' = IL \cap JK$ where $A' \in \mathcal{C}_0$, and then choose another point $B \in \mathcal{C}$ and define $B' \in \mathcal{C}_0$ the same way, such that all points formed by these intersections are distinct. By Corollary 20, $\mathcal{C}_0$ will be the isogonal cubic of $ABA'B'$. In addition, (c) implies that A' and B' will in fact lie in $\mathbb{R}^2$ as well. Therefore, $ABA'B'$ is fully contained in $\mathbb{R}^2$, so $\mathcal{C}$ is indeed the isogonal cubic of $ABA'B'$. This completes the solution. □

7. Uniqueness in the isogonal cubic

With this algebraic characterization of all isogonal cubics in $\mathbb{R}^2$ in mind, in this section, we prove that given an isogonal cubic $\mathcal{C} \in \mathbb{RP}^2$, there is only one possible spiral center P, and for any $X \in \mathcal{C}$, there is only one possible point that could be the isogonal conjugate of X.

Theorem 23 (Uniqueness of the spiral center)**.** *Consider non-degenerate $\mathcal{C} \in \mathbb{RP}^2$ such that there exist $A, B, C, D \in \mathbb{R}^2$ for which $\mathcal{C}$ is the isogonal cubic of $ABCD$. Let $ABCD$ have spiral center P. Let $\mathcal{C}_0$ denote the embedding of $\mathcal{C}$ in $\mathbb{CP}^2$. Then PI and PJ are respectively tangent to $\mathcal{C}_0$ at I and J.*

Proof. Assume, for the sake of contradiction, that PI is not tangent to $\mathcal{C}_0$ at I; then PJ cannot be tangent to $\mathcal{C}_0$ at J either, so by part (c) of Corollary 22, PI and PJ intersect $\mathcal{C}_0$ at $K, L \in \mathbb{C}^2$ respectively, distinct from I, J, P, and IL and JK intersect $\mathcal{C}_0$ at $Q \in \mathbb{R}^2$. Then by Corollary 20, $\mathcal{C}$ is the non-degenerate isogonal cubic of the three quadrilaterals $ABCD$, $APCQ$, $BPDQ$.

By Corollary 3, there is one point of infinity $P_\infty \in \mathcal{C}$, which is the point of infinity along the Newton-Gauss lines of $ABCD$, $APCQ$, $BPDQ$. Let AP meet $\mathcal{C}$ at E; then by Corollary 9, C, E, P_∞ are collinear. Since $\mathcal{C}$ is the isogonal cubic of $APCQ$, it follows that $AP \cap CQ \in \mathcal{C}$, so in fact $Q = CE \cap \mathcal{C}$. Since C, E lie in $\mathbb{R}^2$ they are distinct from P_∞.
If C, E, P_∞ are distinct, then $Q \equiv P_\infty$, contradicting $Q \in \mathbb{R}^2$, as desired. So line CE intersects $\mathcal{C}$ with multiplicity 2. We assumed that $ABCD \in \mathbb{R}^2$, so we cannot have $C \equiv P_\infty$, hence either $C \equiv E$ or $E \equiv P_\infty$. In either case, we cannot have $Q \equiv P_\infty$ else $Q \in \mathbb{R}^2$ is contradicted; thus $Q \equiv C$. But considering A, P are the respective isogonal conjugates of C, Q in $APCQ$, so this implies $A \equiv P$. Now, the isogonal conjugates of A, P in $ABCD$ are C, P_∞, which implies that $C \equiv P_\infty$ — the desired contradiction. □

In other words, a given non-generate isogonal cubic can only have one possible spiral center — we may now call this *the* spiral center of a given isogonal cubic $\mathcal{C}$. The point P is called the *singular focal point* of $\mathcal{C}$, conventionally defined as the intersection of the tangents to $\mathcal{C}$ at I and J. This leads to the following result, allowing us to define isogonal conjugation on any given isogonal cubic without having to construct a base quadrilateral $ABCD$:

Theorem 24 (Uniqueness of the isogonal conjugate). *Consider non-degenerate $\mathcal{C} \in \mathbb{RP}^2$ such that there exist $A, B, C, D \in \mathbb{R}^2$ for which $\mathcal{C}$ is the isogonal cubic of $ABCD$. Then for any point $X \in \mathcal{C}$, there is only one possible point $X' \in \mathcal{C}$ which could be the isogonal conjugate of X in $ABCD$.*

Proof. Let P be the spiral center of $\mathcal{C}$, and let P_∞ be the point of infinity of $\mathcal{C}$. Consider any $X \in \mathcal{C}$. If $X \equiv P$, its isogonal conjugate is P_∞, and vice versa.
If X is neither P nor P_∞, let $Y = PX \cap \mathcal{C}$ and $X' = P_\infty Y \cap \mathcal{C}$. Then by Corollary 9, X' is the isogonal conjugate of X in $ABCD$ no matter which $ABCD$ we choose. Since P is fixed, X' depends only on X, as desired. □

Therefore, given any non-degenerate isogonal cubic $\mathcal{C} \in \mathbb{R}^2$ and any point $X \in \mathcal{C}$, the spiral center P and the isogonal conjugate of X with respect to $\mathcal{C}$ are well-defined. Thus, we may now revisit our constructions of intersections and tangents, this time with a general isogonal cubic.

Theorem 25 (Tangents to the Isogonal Cubic). *For non-singular $X \in \mathcal{C}$, let X' be its isogonal conjugate. Let ℓ be the isogonal of XX' wrt lines XP, XP_∞. Then ℓ is tangent to $\mathcal{C}$ at X.*

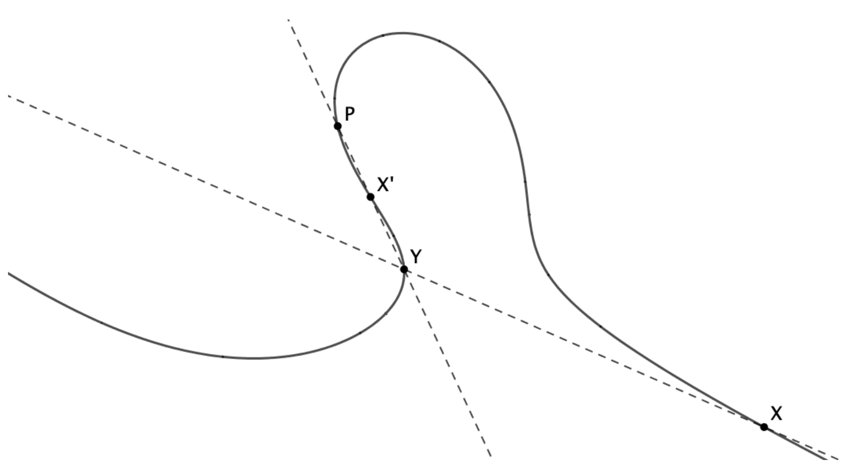

Figure 15: Construction of the isogonal conjugate in a general cubic

Theorem 26 (Line intersections in the isogonal cubic)**.** *For distinct $X, Y \in \mathcal{C}$, let ℓ_X be the isogonal of XY wrt lines XP, XP_∞; define ℓ_Y analogously. Then $XY \cap \mathcal{C}$ is the isogonal conjugate of $\ell_X \cap \ell_Y$.*

We note that with these results, the isogonal cubic is now equivalent to a *focal curve*, as mentioned by F. Dingeldey in [3, Sect. 61]. It is also stated here that $\mathcal{C}$ is the locus of the foci of all conics tangent to AB, BC, CD, DA.

8. Algebraic characterization in the cartesian plane

To conclude the paper, we present a purely algebraic characterization of all possible isogonal cubics in $\mathbb{R}^2$ for the sake of completion.

Theorem 27. *A non-degenerate cubic $\mathcal{C} \in \mathbb{R}^2$ is an isogonal cubic of some quadrilateral $ABCD$ if and only if it has the form $f(x, y) = f(p, q)$, where*

$$f(x, y) = Ax^3 + Bx^2y + Axy^2 + By^3 + Cx^2 + Dxy + Ey^2 + Fx + Gy$$

such that all coefficients are real and $(A, B) \neq (0, 0)$, and

$$p = \frac{AE - AC - BD}{2(A^2 + B^2)}, \qquad q = \frac{BC - AD - BE}{2(A^2 + B^2)}.$$

Furthermore, the spiral center of $\mathcal{C}$ is (p, q), and the unique real asymptote of $\mathcal{C}$ is given by

$$(A^3 + AB^2)x + (A^2B + B^3)y + (A^2E - ABD + B^2C) = 0.$$

Proof. Let the embedding $\mathcal{C}_0$ of $\mathcal{C}$ in $\mathbb{CP}^2$ have equation $g(x, y, z) = 0$, where

$$g(x, y, z) = Ax^3 + Bx^2y + Axy^2 + By^3 + Cx^2z + Dxyz + Ey^2z + Fxz^2 + Gyz^2 + Hz^3$$

where the equality of the coefficients of x^3 with xy^2 and x^2y with y^3 is given by part (b) of Corollary 22. Let g denote the left-hand side of the above equation. We compute

$$\frac{\partial g}{\partial x} = 3Ax^2 + 2Bxy + 2Fxz + Ay^2 + Dyz + Fz^2, \quad \frac{\partial g}{\partial y} = 3By^2 + 2Axy + 2Eyz + Bx^2 + Dxz + Gz^2$$
$$\frac{\partial g}{\partial z} = 3Hz^2 + 2Fxz + 2Gyz + Cx^2 + Dxy + Ey^2$$

Plugging in the partial derivatives for $(1 : i : 0)$, the tangent to $\mathcal{C}_0$ at $(1 : i : 0)$ has equation

$$(2A + 2Bi)x + (-2B + 2Ai)y + (C + Di - E)z = 0$$

and similarly the tangent to $\mathcal{C}_0$ at $(1 : -i : 0)$ has equation

$$(2A - 2Bi)x + (-2B - 2Ai)y + (C - Di - E)z = 0$$

The spiral center P of $\mathcal{C}$ is then given by the solution to these two equations. Solving yields

$$(x : y : z) \;=\; \big(AE - AC - BD : \; BC - AD - BE : \; 2(A^2 + B^2)\big)$$

Since A and B are not both 0, converting back to Cartesian coordinates implies that P indeed has coordinates given by (p, q). In Cartesian coordinates, the value

$$Ax^3 + Bx^2y + Axy^2 + By^3 + Cx^2 + Dxy + Ey^2 + Fx + Gy$$

must be a constant, particularly $-H$. Plugging in (p, q) immediately gives the equation for $\mathcal{C}$ to be $f(x, y) = f(p, q)$ as desired.
To determine the asymptote, we find that points of infinity on $\mathcal{C}_0$ are given by

$$0 = Ax^3 + Bx^2y + Axy^2 + By^3 = (x^2 + y^2)(Ax + By)$$

so the real point of infinity is given by $P_\infty = (B : -A : 0)$. Plugging this into the equations for the partial derivatives yields that the tangent to $\mathcal{C}_0$ at P_∞, and by extension the unique real asymptote of $\mathcal{C}$, indeed takes the above equation. This completes the proof. $\square$

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