Journal for Geometry and Graphics
Volume 24 (2020), No. 1, 35–47.

Mappings from the Plane of a Triangle to a Circumconic

Clark Kimberling[1], **Peter J. C. Moses**[2]

[1] *Department of Mathematics, University of Evansville,*
1800 Lincoln Avenue, Evansville, Indiana, 47722 USA
email: `ck6@evansville.edu`

[2] *Engineering Division, Moparmatic Co.,*
Astwood Bank, nr. Redditch, Worcestershire, B96 6DT, UK
email: `moparmatic@gmail.com`

Abstract. Barycentric coordinates are used to define mappings M from the plane of a reference triangle ABC to a circumconic. Accordingly, the M-image of a triangle is a triangle inscribed in the conic. Certain triangles T defined in this manner have relatively simple barycentrics and appear to be new to the literature. Cases are examined in which T is perspective to ABC, or to a family of cevian, or anticevian, or cocevian triangles. Of particular interest are criteria for perspectivity that involve cubic curves.

Key Words: triangle geometry, triangle center, circumconic, barycentric coordinates, circumcircle, perspective, area.

MSC 2010: 51N20, 51N15

1. Introduction

Suppose that P is a point in the plane of a triangle ABC, extended to include the line L^∞ at infinity. We represent P by barycentric coordinates, $P = p : q : r$, so that the circumconic with perspector P is given by

$$pyz + qzx + rxy = 0, \tag{1}$$

or by

$$\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 0, \tag{2}$$

which is understood to mean (1) even if $xyz = 0$; abbreviations like (2) will appear freely in the sequel in order to save space, as in (5).

The conic (2) passes through A, B, C, and the lines tangent to the conic at A, B, C are given by

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$$ry + qz = 0, \quad pz + rx = 0, \quad qx + py = 0,$$

respectively. These lines intersect in pairs to form the triangle $A'B'C'$ given by

$$A' = -p : q : r, \quad B' = p : -q : r, \quad C' = p : q : -r.$$

The triangle $A'B'C'$ is perspective to ABC with perspector P (see [3, p. 114] and Figure 1).

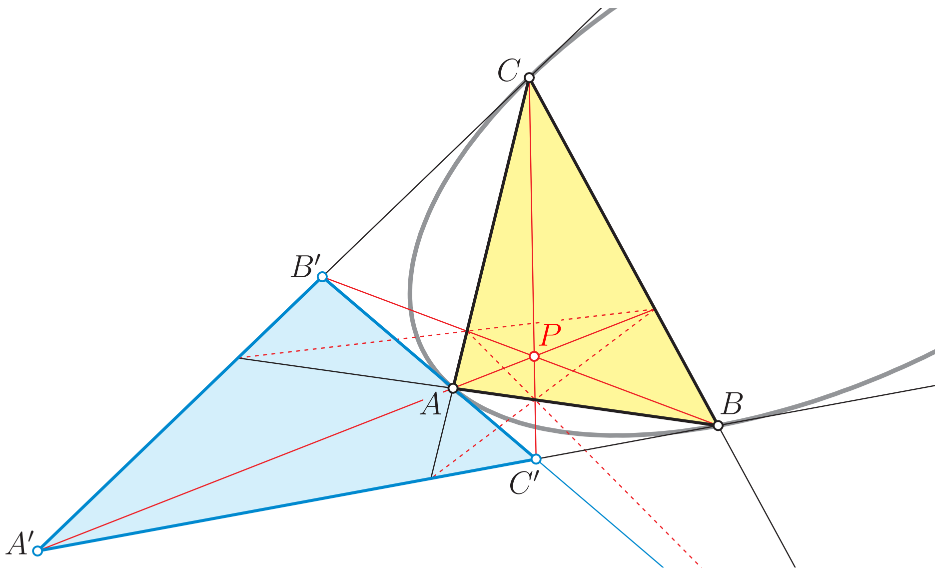

Figure 1: The triangles ABC, $A'B'C'$, and the circumconic with perspector P

In this paper, points, and therefore sets of points, are regarded as functions of the side-lengths a, b, c of ABC, so that we may write

$$P = P(a,b,c) = p(a,b,c) : q(a,b,c) : r(a,b,c). \tag{3}$$

Likewise, if U is a point, then it is understood that

$$U = U(a,b,c) = u(a,b,c) : v(a,b,c) : w(a,b,c).$$

Now suppose that $U = u : v : w$ be a point on the line L^∞ at infinity, which is to say that

$$u + v + w = 0. \tag{4}$$

We call a point $D = d : e : f$ *acceptable* if $u(d,e,f)$, $v(d,e,f)$, $w(d,e,f)$ are defined and not all 0, and if D is acceptable, we define

$$U(D) = u(d,e,f) : v(d,e,f) : w(d,e,f).$$

By (4), we have $U(D) \in L^\infty$, so that for any point P, the point

$$M(P) = \frac{p}{u(d,e,f)} : \frac{q}{v(d,e,f)} : \frac{r}{w(d,e,f)} \tag{5}$$

lies on the circumconic (1).

Example 1. Let $P = a^2 : b^2 : c^2 = X(6)$ in the Encyclopedia of Triangle Centers - ETC [1], and let $U = b - c : c - a : a - b = X(514)$, on L^∞. Then for every acceptable point

$D = d : e : f$, the point

$$M(D) = \frac{a^2}{e-f} : \frac{b^2}{f-d} : \frac{c^2}{d-e}$$

lies on the circumcircle, given by

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = 0.$$

Example 2. Let $P = 1 : 1 : 1 = X(2)$ in [1], and let $U = 2a - b - c : 2b - c - a : 2c - a - b = X(519)$. Then for every acceptable point $d : e : f$, the point

$$\frac{1}{2d - e - f} : \frac{1}{2e - f - d} : \frac{1}{2f - d - e}$$

lies on the Steiner circumellipse, given by

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$$

Returning to the general circumconic (1) and U on L^∞ as in (3), if $d_i : e_i : f_i$ are three points, then $M(d_i : e_i : f_i)$ is a (possibly degenerate) triangle inscribed in (1). The main purpose of this paper is to examine such triangles and the manner in which their properties were discovered and confirmed with the help of computers. First, though, we restrict our attention to the circumcircle and examine the set of points $M(X)$ as X ranges through a line.

2. Wrapping a line around the circumcircle

In this section, we take $P = X(6)$, as in Example 1, so that the conic (1) is the circumcircle, and we take $U = b - c : c - a : a - b$. Suppose L is a line and that $d_1 : e_1 : f_1$ and $d_2 : e_2 : f_2$ are distinct points on L. Then L consists of all points of the form

$$X(t) = d_1 + td_2 : e_1 + te_2 : f_1 + tf_2, \tag{6}$$

where t may be regarded as a real variable or as a function of (a, b, c), depending on context. The image of X, on the circumcircle, is given by

$$M(X(t)) = \frac{a^2}{e_1 - f_1 + t(e_2 - f_2)} : \frac{b^2}{f_1 - d_1 + t(f_2 - d_2)} : \frac{c^2}{d_1 - e_1 + t(d_2 - e_2)}. \tag{7}$$

Lemma 1. *Suppose that $U = u : v : w$ and $X = x : y : z$ are points satisfying these conditions:*

$$(v - w)(w - u)(u - v)(y - z)(z - x)(x - y) \neq 0; \quad \frac{y - z}{v - w} = \frac{z - x}{w - u} = \frac{x - y}{u - v}.$$

Then U and X are collinear with the centroid, G.

Proof. Write the common nonzero value as k, so that

$$y - z = k(v - w), \quad z - x = k(w - u), \quad x - y = k(u - v).$$

Regarding z as a parameter, t, we then have

$$x = t - k(w - u), \quad y = t + k(v - w), \quad z = t,$$

so that the point $X = x : y : z$ clearly lies on the line through $G = t : t : t = 1 : 1 : 1$ and $u - w : v - w : 0$. This second point is on the line GU, so that X is also on GU. $\square$

Theorem 2. *Let $X(t_1))$ and $X(t_2)$ be distinct points on the line L in (6), so that the points $M(X(t_1))$ and $M(X(t_2))$ are points on the circumcircle, as in (7). If $M(X(t_1)) = M(X(t_2))$, then L passes through G.*

Proof. Assume that $M(X(t_1)) = M(X(t_2))$. Then by (7),

$$\frac{a^2}{e_1-f_1+t_1(e_2-f_2)} ::= \frac{a^2}{e_1-f_1+t_2(e_2-f_2)} ::, \quad \frac{a^2}{e_1+t_1e_2-(f_1+t_2f_2)} ::= \frac{a^2}{e_1+t_2e_2-(f_1+t_2f_2)} ::$$

Consequently,

$$\frac{f_1+t_2f_2-(d_1+t_2d_2)}{f_1+t_1f_2-(d_1+t_1d_2)} = \frac{d_1+t_2d_2-(e_1+t_2e_2)}{d_1+t_1d_2-(e_1+t_1e_2)} = \frac{e_1+t_2e_2-(f_1+t_2f_2)}{e_1+t_1e_2-(f_1+t_1f_2.)}$$

The asserted conclusion now follows by Lemma 1. □

From (7), we find that if the line L misses G, then the four points

$$A,\ B,\ C,\ \frac{a^2}{e_1-f_1} : \frac{b^2}{f_1-d_1} : \frac{c^2}{d_1-e_1}$$

are the images $M(X(t))$ for these four values of t:

$$-\frac{e_2-f_2}{e_1-f_1},\ -\frac{f_2-d_2}{f_1-d_1},\ -\frac{d_2-e_2}{d_1-e_1},\ 0,$$

respectively, and that the point

$$\frac{a^2}{e_2-f_2} : \frac{b^2}{f_2-d_2} : \frac{c^2}{d_2-e_2}$$

is approached as a limit as $t \to \pm\infty$. Thus, there are two possibilities for any line L: if L passes through G, then every point on L is mapped to a single point on the circumcircle; otherwise, L is wrapped exactly once around the circumcircle.

3. Method for discovering perspectivities

The results after this section depend on computer-based searching for perspectivities of triangles T_1 and $M(T_2)$, where the mapping M is as in (5) for some choice of P and U. The method can be summarized as follows: Let

$$T_1 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}, \quad T_2 = \begin{pmatrix} x_4 & y_4 & z_4 \\ x_5 & y_5 & z_5 \\ x_6 & y_6 & z_6 \end{pmatrix}, \quad M(T_2) = \begin{pmatrix} x_7 & y_7 & z_7 \\ x_8 & y_8 & z_8 \\ x_9 & y_9 & z_9 \end{pmatrix}.$$

Let

$$\Psi = \begin{pmatrix} y_1z_7 - z_1y_7 & z_1x_7 - x_1z_7 & x_1y_7 - y_1x_7 \\ y_2z_8 - z_2y_8 & z_2x_8 - x_2z_8 & x_2y_8 - y_2x_8 \\ y_3z_9 - z_3y_9 & z_3x_9 - x_3z_9 & x_3y_9 - y_3x_9 \end{pmatrix}.$$

The triangles T_1 and $M(T_2)$ are perspective if and only if the determinant $|\Psi|$ equals 0. In the sequel, either T_1 or T_2 is a function of $X = x : y : z$, so that the equation $|\Psi| = 0$ takes the form $|\Psi(x, y, z)| = 0$; we call the set of such points X the *perspectivity locus* of T_1 and $M(T_2)$.

We shall consider cases in which this locus is the whole plane (excluding G) or a cubic, while noting here that in many cases the locus is an algebraic curve of degree greater than 3.

Given that $|\Psi(x,y,z)| = 0$, the method continues by computing the perspector as the point

$$\begin{aligned} X^* &= (z_2x_8 - x_2z_8)(x_3y_9 - y_3x_9) - (x_2y_8 - y_2x_8)(z_3x_9 - x_3z_9) \\ &: (x_2y_8 - y_2x_8)(y_3z_9 - z_3y_9) - (y_2z_8 - z_2y_8)(x_3y_9 - y_3x_9) \\ &: (y_2z_8 - z_2y_8)(z_3x_9 - x_3z_9) - (z_2x_8 - x_2z_8)(y_3z_9 - z_3y_9). \end{aligned}$$

Typically, the determinant $|\Psi|$ and the perspector X^*, when factored, can be reduced considerably in length upon cancelation of common factors. If T_1 and T_2 are central triangles, then X^* is a triangle center, thus having barycentrics of the form $g(a,b,c) : g(b,c,a) : g(c,a,b)$, where $g(a,c,b) = g(a,b,c)$.

In the sequel, there are four families of triangles to be represented as matrices. Given a point $X = x : y : z$, the cevian triangle of X has vertices $0 : y : z$, $x : 0 : z$, $x : y : z$, represented as the matrix

$$\begin{pmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{pmatrix}.$$

Likewise, the anticevian triangle of X is

$$\begin{pmatrix} -x & y & z \\ x & -y & z \\ x & y & -z \end{pmatrix},$$

the cocevian triangle of X is

$$\begin{pmatrix} 0 & y & -z \\ -x & 0 & z \\ x & -y & 0 \end{pmatrix},$$

and the cocevian triangle of $yz : zx : xy$, this being the isotomic conjugate of X, is

$$\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}.$$

Regarding the last two triangles, they are degenerate, as the vertices are collinear.

4. Inscribed triangles perspective to the reference triangle

Let $P = p : q : r$ and $U = u : v : w$, and define

$$M(U) = \frac{p}{1/v - 1/w} : \frac{q}{1/w - 1/u} : \frac{r}{1/u - 1/v}.$$

The method outlined in the preceding section leads to the following results.

Theorem 3. *Suppose that k is a constant and that*

$$T_k(X) = \begin{pmatrix} kx + y + z & y & z \\ x & x + ky + z & z \\ x & y & x + y + kz \end{pmatrix}. \tag{8}$$

Then $T_k(X)$ is perspective to ABC if and only if $k = -1$, and if $k = -1$, the perspector is the point

$$pyz(y - z) : qzx(z - x) : rxy(x - y).$$

Theorem 4. *If*

$$M(U) = \frac{p}{2u - v - w} : \frac{q}{2v - w - u} : \frac{r}{2w - u - v},$$

and $T_k(X)$ is as in (8), then $T_k(X)$ is perspective to ABC if and only if $k = -1$ or $k = 2$. If $k = -1$, the perspector is

$$p(2x - y - z) : q(2y - z - x) : r(2z - x - y),$$

and if $k = 2$, the perspector is

$$\frac{p}{x - 2y - 2z} : \frac{q}{y - 2z - 2x} : \frac{r}{z - 2x - 2y}.$$

5. Inscribed triangles perspective to anticevian triangles

In this section, let $P = p : q : r$, and let T_1 be the anticevian triangle of P, given by the matrix

$$T_1 = \begin{pmatrix} -p & q & r \\ p & -q & r \\ p & q & -r \end{pmatrix}.$$

The method described in Section 3 gives three more theorems involving the mapping

$$M(U) = \frac{p}{v - w} : \frac{q}{w - u} : \frac{r}{u - v}.$$

Theorem 5. *For arbitrary $X = x : y : z$, let*

$$T_2 = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}.$$

Then $M(T_2)$ is perspective to T_1, and the perspector is

$$p(y^2 + z^2 - x^2) : q(z^2 + x^2 - y^2) : r(x^2 + y^2 - z^2).$$

The same result holds for

$$T_2 = \begin{pmatrix} \pm yz & z^2 & y^2 \\ z^2 & \pm zx & x^2 \\ y^2 & z^2 & \pm xy \end{pmatrix}.$$

Theorem 6. *For arbitrary $X = x : y : z$, let*

$$T_2 = \begin{pmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{pmatrix}.$$

Then $M(T_2)$ is perspective to T_1, and the perspector is

$$p(x^2y^2 + x^2z^2 - y^2z^2) : q(y^2z^2 + y^2x^2 - z^2x^2) : r(z^2x^2 + z^2y^2 - x^2y^2).$$

The same result holds for

$$T_2 = \begin{pmatrix} \pm yz & y^2 & z^2 \\ x^2 & \pm zx & z^2 \\ x^2 & y^2 & \pm xy \end{pmatrix}.$$

Theorem 7. *For arbitrary $X = x : y : z$, let*

$$T_2 = \begin{pmatrix} -x & y & z \\ x & -y & z \\ x & y & -z \end{pmatrix}.$$

Then $M(T_2)$ is perspective to T_1, and the perspector is

$$p(x^2 + xy + xz - yz) : q(y^2 + yz + yx - zx) : r(z^2 + zx + zy - xy).$$

6. Perspectivity cubics and perspectors

The method in Section 3 readily gives two theorems about triangles inscribed in an arbitrary circumconic,

$$dyz + ezx + fxy = 0.$$

Define

$$M(X) = \frac{d}{y-z} : \frac{e}{z-x} : \frac{f}{x-y}.$$

Theorem 8. *Let T_1 be the cevian triangle of X and let T_2 be the cocevian triangle of a point $U = u : v : w$. Then the locus of $X = x : y : z$ such that T_1 is perspective to $M(T_2)$ is the cubic given by*

$$\sum_{cyclic} \Big(\big(d^2u^2(fw(u+w)y^2z - ev(u+v)yz^2\big) \Big) = 0.$$

If X lies on this cubic, then the associated perspector is given by

$$g(d,e,f,u,v,w,x,y,z) : g(e,f,d,v,w,u,y,z,x) : g(f,d,e,w,u,v,z,x,y),$$

where

$$\begin{aligned} g(d,e,f,u,v,w,x,y,z) = {} & efvw(vw + wu + uv)x^2 \\ & + d^2u(u^3 + u^2(v+w) + 3uvw + v^2w + vw^2)yz \\ & - dfw(u^3 + u^2(v+w) + uv(v+2w) + vw^2)xy \\ & - dev(u^3 + u^2(v+w) + uw(w+2v) + v^2w)zx. \end{aligned}$$

Theorem 9. *Let T_1 be the anticevian triangle of X and let T_2 be the cocevian triangle of a point $U = u : v : w$. Then the locus of X such that T_1 is perspective to $M(T_2)$ is the cubic given by*

$$\sum_{cyclic} \big(\big(d^2(f(u+w)y^2z - e(u+v)yz^2\big) \big) = 0.$$

If X lies on this cubic, then the associated perspector is given by

$$h(d,e,f,u,v,w,x,y,z) : h(e,f,d,v,w,u,y,z,x) : h(f,d,e,w,u,v,z,x,y),$$

where

$$\begin{aligned} h(d,e,f,u,v,w,x,y,z) = {} & ef(vw + wu + uv)(v^2 + 3vw + w^2)x^2 \\ & + 2d^2uvw(2u + v + w)yz \\ & - de\left(v^3(u+w) + uv^2(3u+4w) + uv(u^2 + 2uw + 2w^2) - u^3w\right)zx \\ & - df\left(w^3(u+v) + uw^2(3u+4v) + uw(u^2 + 2uv + 2v^2) - u^3v\right)yz. \end{aligned}$$

Regarding Theorems 8 and 9, we conjecture that the locus of the perspector is also a cubic.

Example 3. Let

$$T_1 = \begin{pmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & v & -w \\ -u & 0 & w \\ u & -v & 0. \end{pmatrix},$$

$$M(d:e:f) = \frac{1}{e-f} : \frac{1}{f-d} : \frac{1}{d-e} \text{ (on the Steiner circumellipse)},$$

$$U = u:v:w = X(69) = b^2+c^2-a^2 : c^2+a^2-b^2 : a^2+b^2-c^2.$$

Here, the triangles T_1 and $M(T_2)$ are perspective if and only if $X = x:y:z$ lies on the cubic K099 (see GIBERT [2] for a catalogue of cubics using codes of the form $K\ldots$ and $pK(\ldots,\ldots)$, with presentations of the properties of the coded cubics). In the following list of associations $X \to X'$, the triangle center X lies on K099 and the associated perspector, also a triangle center, X', is as shown:

$$X(69) \to X(69), \quad X(76) \to X(264), \quad X(304) \to X(322),$$
$$X(305) \to X(14615), \quad X(3718) \to X(33672), \quad X(3926) \to X(6527).$$

Example 4. Using $U = u:v:w = X(75) = bc:ca:ab$ with the same triangles and mapping M as in Example 3, we obtain these associations:

$$X(75) \to X(75), \quad X(76) \to X(69), \quad X(274) \to X(86), \quad X(310) \to X(314),$$
$$X(314) \to X(8822), \quad X(6064) \to X(7), \quad X(7182) \to X(33673).$$

Examples 3 and 4 appear to generalize as follows: the perspectivity cubic, for the stated triangles and mapping, arises from U as $pK(U', U'')$, where

$$U' = a^4u^2/(v+w) :: \text{ and } U'' = a^2u/(v+w) :: .$$

Example 5. Let T_1 and T_2 be as in Example 3, and let

$$M(d:e:f) = \frac{a^2}{e-f} : \frac{b^2}{f-d} : \frac{c^2}{d-e} \text{ (on the circumcircle)},$$

$$U = u:v:w = X(69) = b^2+c^2-a^2 : c^2+a^2-b^2 : a^2+b^2-c^2.$$

Here, the triangles T_1 and $M(T_2)$ are perspective if and only if $X = x:y:z$ lies on the Thomson cubic, K002 in [2]. In the following list, the triangle center X lies on K002 and the associated perspector, also a triangle center, X', lies on the Darboux cubic, K004:

$$X(1) \to X(1490), \quad X(2) \to X(20), \quad X(3) \to X(3),$$
$$X(4) \to X(3183), \quad X(6) \to X(1498), \quad X(9) \to X(40),$$
$$X(57) \to X(3182), \quad X(223) \to X(1), \quad X(282) \to X(3353),$$
$$X(1073) \to X(2130), \quad X(1249) \to X(4), \quad X(3341) \to X(84),$$
$$X(3342) \to X(3472), \quad X(3343) \to X(64), \quad X(3344) \to X(3355),$$
$$X(3350) \to X(3346).$$

If X is on K002, then the perspector, on K004, is the point $g(a,b,c) : g(b,c,a) : g(c,a,b)$, where

$$\begin{aligned} g(a,b,c) = {} & (a-b-c)(a+b-c)(a-b+c)(a+b+c) \\ & \quad (a^4+2a^2b^2-3b^4+2a^2c^2+6b^2c^2-3c^4)x^2 \\ & + 8a^2(a^2-b^2-c^2)(a^2+b^2-c^2)(a^2-b^2+c^2)yz \\ & - (3a^8-8a^6b^2+6a^4b^4-b^8+8a^6c^2+4a^4b^2c^2-16a^2b^4c^2 \\ & \quad +4b^6c^2-26a^4c^4-6b^4c^4+16a^2c^6+4b^2c^6-c^8)xy \\ & - (3a^8+8a^6b^2-26a^4b^4+16a^2b^6-b^8-8a^6c^2+4a^4b^2c^2+4b^6c^2 \\ & \quad +6a^4c^4-16a^2b^2c^4-6b^4c^4+4b^2c^6-c^8)xz\,. \end{aligned}$$

Example 5 appears to generalize as follows: the perspectivity cubic, for the stated triangles and mapping, arises from U as $pK(U', U'')$, where

$$U' = a^4/(v+w) :: \text{ and } U'' = a^2/(v+w) ::$$

Theorem 10. *Let*

$$T_1 = \begin{pmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix},$$

The triangles T_1 and $M(T_2)$ are perspective if and only if $X = x : y : z$ lies on the cubic given by

$$\sum_{cyclic} \left(a^4c^2v^3w(u+w)y^2z - a^4b^2w^3v(u+v)yz^2\right) = 0. \tag{9}$$

If $x : y : z$ lies on the perspectivity cubic (9), then the perspector is given by

$$g(a,b,c,u,v,w,x,y,z) : g(b,c,a,v,w,u,y,z,x) : g(c,a,b,w,u,v,z,x,y), \tag{10}$$

where

$$\begin{aligned} g(a,b,c,u,v,w,x,y,z) = {} & b^2c^2u^3(u+v+w)x^2 \\ & + a^4vw(u^2+v^2+w^2+vw+2uv+2uw)yz \\ & - a^2b^2uw(u^2+v^2+2uv+2vw+us)zx \\ & - a^2c^2uv(u^2+w^2+2uw+2vw+uv)xy \end{aligned}$$

Theorem 11. *Let*

$$T_1 = \begin{pmatrix} -x & y & z \\ x & -y & z \\ x & y & -z \end{pmatrix},$$

and let T_2 be the cocevian triangle of $vw : wu : uv$, given by

$$T_2 = \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix}.$$

The triangles T_1 and $M(T_2)$ are perspective if and only if $X = x : y : z$ lies on the cubic given by

$$\sum_{cyclic} \left(a^4c^2v(u+w)y^2z - a^4b^2w(u+v)yz^2\right) = 0. \tag{11}$$

If $x : y : z$ lies on the perspectivity cubic (11), then the perspector is given by the barycentrics

$$h(a,b,c,u,v,w,x,y,z) : h(b,c,a,v,w,u,y,z,x) : h(c,a,b,w,u,v,z,x,y),$$

where

$$h(a,b,c,u,v,w,x,y,z) = N(a,b,c,u,v,w,x,y,z)/D(a,b,c,u,v,w,x,y,z),$$

where the numerator N is given by

$$N = x\big(b^2c^2u^2(u+v+w)x - a^2c^2v(u+w)(u+v-w)y - a^2b^2w(u+v)z\big),$$

and the denominator D by

$$\begin{aligned} D = {} & b^2c^2u^2(u+v+w)x^2 + a^2c^2v(u+w)(u+v+w)y^2 + a^2b^2w(u+v)(u+v+w)z^2 \\ & - a^2(u+v+w)\big(b^2w(u+v) + c^2v(u+w)\big)yz \\ & + b^2\big(u^2(w(-a^2+2b^2-c^2) - c^2(u+v) + 2(b^2-c^2)uvw + a^2w(v^2-uw-vw)\big)xz \\ & + c^2\big(u^2(v(-a^2+2c^2-b^2) - b^2(u+w) + 2(c^2-b^2)uvw + a^2v(w^2-uv-vw)\big)xy. \end{aligned}$$

7. Other perspective pairs

The method described in Section 3 applies to many pairs T_1 and T_2. Here, were take $T_1 = ABC$ and vary T_2 and the mapping M.

Example 6. Let

$$T_2 = \begin{pmatrix} x & z-y & y-z \\ z-x & y & x-z \\ y-x & x-y & z \end{pmatrix},$$

and define

$$M(U) = p/(1/v - 1/w) : q/(1/w - 1/u) : r/(1/u - 1/v).$$

Then $M(T_2)$ is perspective to ABC for all acceptable points $X = x : y : z$, and the perspector is

$$\frac{p}{-x+y+z} : \frac{q}{x-y+z} : \frac{r}{x+y-z}.$$

Example 7. Let M be as in Example 6, and let

$$T_2 = \begin{pmatrix} x & y-z & z-y \\ x-z & y & z-x \\ x-y & y-x & z \end{pmatrix},$$

Then $M(T_2)$ is perspective to ABC for all acceptable points $X = x : y : z$, and the perspector is

$$p(-x+y+z) : q(x-y+z) : r(x+y-z).$$

Example 8. Let

$$T_2 = \begin{pmatrix} 2x+y+z & y & z \\ x & x+2y+z & z \\ x & y & x+y+2z \end{pmatrix},$$

and define

$$M(U) = \frac{p}{2u-v-w} : \frac{q}{2v-w-u} : \frac{r}{2w=u-v}.$$

Then $M(T_2)$ is perspective to ABC for all acceptable points $X = x : y : z$, and the perspector is

$$\frac{p}{x-2y-2z} : \frac{q}{y-2z-2x} : \frac{r}{z-2x-2y}.$$

Example 9. Let

$$T_2 = \begin{pmatrix} 2x+y+z & -x & -x \\ -y & x+2y+z & -y \\ -z & -z & x+y+2z \end{pmatrix},$$

and define $M(U)$ as in the preceding example. Then $M(T_2)$ is perspective to ABC for all acceptable points $X = x : y : z$, and the perspector is

$$p(3x+y+z) : q(x+3y+z) : r(x+y+3z).$$

8. Perspective triangles from a single point

For Theorems 12–16 in this section, we take $U = b-c : c-a : a-b$ and

$$M(U) = \frac{a^2}{v-w} : \frac{b^2}{w-u} : \frac{c^2}{u-v}.$$

The two triangles T_1 and T_2 are, in each case, derived from a single point X.

Theorem 12. *Let $X = x : y : z$ be a point, and let*

$$T_1 = \begin{pmatrix} 0 & zx & xy \\ yz & 0 & xy \\ yz & zx & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -x & y & z \\ x & -y & z \\ x & y & -z \end{pmatrix}.$$

Then $M(T_1)$ and $M(T_2)$ are perspective, and their perspector is the point

$$\frac{a^2(-3x+y+z)}{y+z} : \frac{b^2(x-3y+z)}{z+x} : \frac{c^2(x+y-3z)}{x+y}.$$

For examples, see $X(33628)-X(33636)$ in [1].

Theorem 13. *Let $X = x : y : z$ be a point, and let*

$$T = \begin{pmatrix} -x & y & z \\ x & -y & z \\ x & y & -z \end{pmatrix}.$$

Then T and $M(T)$ are perspective if and only if X lies on the cubic K141, which is $pK(X(2), X(76))$, given by

$$\sum_{cyclic} a^2(b^2y^2z - c^2yz^2) = 0.$$

Theorem 14. *Let $X = x : y : z$ be a point, and let*

$$T = \begin{pmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{pmatrix}.$$

Then T and $M(T)$ are perspective if and only if X lies on the cubic

$$\sum_{cyclic} \left(b^2(a^2+c^2)y^2z - c^2(a^2+b^2)yz^2 \right) = 0,$$

which is the isotomic conjugate of K836.

For the final two theorems in this section, let

$$M(U) = \frac{a}{v-w} : \frac{b}{w-u} : \frac{c}{u-v},$$

so that M maps U to a point on the circumconic

$$ayz + bzx + cxy = 0.$$

Theorem 15. *Let $X = x : y : z$ be a point, and let*

$$T = \begin{pmatrix} 0 & y & z \\ x & 0 & z \\ x & y & 0 \end{pmatrix}, \quad M(T) = \begin{pmatrix} a/(y-z) & b/z & -c/y \\ -a/z & b/(z-x) & c/x \\ a/y & -b/x & c/(x-y) \end{pmatrix},$$

Then T and $M(T)$ are perspective if and only if X lies on the cubic

$$\sum_{cyclic} \left(b(a+c)y^2z - c(a+b)yz^2 \right) = 0,$$

which is the isotomic conjugate of K345.

To illustrate Theorem 15, in the following list of associations $X \to X'$, the triangle center X lies on the isotomic conjugate of K345, and the perspector, X', also a triangle center, is as shown:

$$\begin{array}{lll} X(2) \to X(2), & X(75) \to X(63), & X(85) \to X(57), \\ X(86) \to X(81), & X(274) \to X(333), & X(333) \to X(1817), \\ X(348) \to X(18623), & X(31623) \to X(27). & \end{array}$$

Theorem 16. *Let $X = x : y; z$ be a point, and let*

$$T_1 = \begin{pmatrix} -x & y & z \\ x & -y & z \\ x & y & -z \end{pmatrix}, \quad T_2 = \begin{pmatrix} -a/(y-z) & b/(z+x) & -c/(x+y) \\ -a/(y+z) & -b/(z-x) & c/(x+y) \\ a/(y+z) & -b/(z+x) & -c/(x-y) \end{pmatrix}.$$

Then T_1 and T_2 are perspective if and only if X lies on the circumconic

$$ayz + bzx + cxy = 0$$

or the quartic curve given by

$$\sum_{cyclic} a(by^2z^2 - cy^2z^2) = 0. \tag{12}$$

To illustrate Theorem 16, in the following list of associations $X \to X'$, the triangle center X lies the circumconic $ayz + bzx + cxy = 0$, and the perspector, X', also a triangle center, is as shown:

$$X(88) \to X(2226),\ \ X(100) \to X(1016),\ \ X(162) \to X(23964),$$
$$X(190) \to X(1016),\ \ X(651) \to X(1262),\ \ X(662) \to X(249).$$

To continue illustrating Theorem 16, in the following list of associations $X \to X'$, the triangle center X lies the quartic curve (12), and the perspector, X', also a triangle center, is as shown:

$$X(2) \to X(2),\quad X(7) \to X(3160).$$

9. Miscellany

It is easy to confirm that the mapping M defined by

$$M(X) = \frac{a^2}{y-z} : \frac{b^2}{z-x} : \frac{c^2}{x-y}$$

has no fixed points—that is, points X such that $M(X) = X$. However, there is evidence to support the following conjecture.

Conjecture. *The mapping $M^2(X) = M(M(X))$ has a fixed point if ABC is obtuse.*

We have

$$M^2(X) = \frac{a^2(x-y)(z-x)}{b^2(x-y)-c^2(z-x)} : \frac{b^2(y-z)(x-y)}{c^2(y-z)-a^2(x-y)} : \frac{c^2(z-x)(y-z)}{a^2(z-x)-b^2(y-z)}\,.$$

Abbreviating $M^2(X)$ as $\alpha : \beta : \gamma$ and setting $M^2(X)$ equal to $x : y : z$ yields $x/y = \alpha/\beta$ and $x/z = \alpha/\gamma$, leading to the following system of two equations:

$$\begin{aligned} a^2(x-y)(z-x)\big(c^2(y-z)-a^2(x-y)\big)y &= b^2(y-z)(x-y)\big(b^2(x-y)-c^2(z-x)\big)x,\\ a^2(x-y)(z-x)\big(a^2(z-x)-b^2(y-z)\big)z &= c^2(z-x)(y-z)\big(b^2(x-y)-c^2(z-x)\big)x. \end{aligned} \tag{13}$$

Putting $z = 1$ and solving the system for (x, y) gives a result too elaborate to appear here. We offer, instead, numerical sampling. As a representative example, if $(a, b, c) = (13, 6, 9)$, then $M^2(X) = X$ for two points X:

$$-3.1236506\ldots : -1.3384571\ldots : 1 \ \text{ and } \ 1.5927864\ldots : -0.19240701\ldots : 1.$$

Sampling suggests that there are always two real roots if ABC is obtuse, and none otherwise.

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Received September 12, 2019; final form May 5, 2020