Journal for Geometry and Graphics
Volume 24 (2020), No. 1, 103–124.

# Optimal Cells in Crystallography and Arts

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**Abstract.** This is an overview on ball packings in different geometries. However, parallel to reported mathematical results we discuss various appearances of related phenomena in arts and architecture.
The ball (or sphere) packing problem with equal balls, without any symmetry assumption, in a 3-dimensional space of constant curvature ($\mathbf{E}^3$, $\mathbf{H}^3$, $\mathbf{S}^3$) is a very intensively researched area of geometry. In the Euclidean space $\mathbf{E}^3$ the famous Kepler conjecture was settled by Thomas Hales and in other spaces of constant curvatures ($\mathbf{H}^3$, $\mathbf{S}^3$) was partially solved by Böröczky and Florian. However, in the hyperbolic space $\mathbf{H}^3$ many open questions still need to be answered. In each considered geometry to every optimal ball packing configuration belongs a tiling, e.g., Dirichlet-Voronoi tiling. Their tile types play important roles in the crystallography.
Throughout human history the phenomenon of space and its orderliness has been intriguing artists as well, who, at many instances has intuited spatial configurations that bear striking resemblances with the outcomes of scientific research done in this field. In an attempt to blur a rather artificial boundary between the sciences and the arts, the ball packing problems unfolded in this article will be illustrated not only with mathematical visuals but with relatable artefacts and short introductions to their contexts.

*Key Words:* Thurston geometries, hyperbolic geometry, ball packings, geometry in architecture and arts

*MSC 2010:* 00A66, 52C15, 52C17, 51M10

## 1. Introduction: the separation of the sciences and the arts

Mathematics as we know it is a product of cultures which genesis histories is an account on how order conquers the primordial chaos. In such tales, the bright divine imposes predictable order

ISSN 1433-8157/$ 2.50 

over the dark and uncanny unruliness. The exploration into the nature of the triumphant laws is the most respectable activity in such cultures as it is believed that a cooperation with natural forces is the basis of the successful and lasting functioning of society as a whole. In the early history of civilizations with such values, there was a common understanding in society about the operation of the world and the place and purpose of humans within it. Creation was the enquiry, understanding and collaboration with forces personified by divine beings. Creative activity has taken many forms without differentiating it to be scientific or artistic. Today we know only a few material manifestations of such ancient intelligences, and can formulate only vague ideas about the nature of the intelligence itself.

With time, the study of the forces at play maintained its divine prestige, only the forces themselves were gradually deprived of their godlike status. ANAXIMANDER (∼610 – ∼546 BC) is the first recorded thinker who proposed a view on nature as the manifestation of underlying geometric harmonies and not as the caprice of Gods. He placed the origins of the world on a rational basis by introducing the abstract concept of apeiron, the source from which opposites were separated through which genesis gradually took place. The secret of creation thus became mathematical in nature and the privilege of the initiated few, divorced from the daily activities and needs of the common populace.

Physical creation (what we today call *art*) came to be called *'techné'*, a mimesis of tangible reality with a relatively law social status and was not included into the realm of the 'liberal arts' as opposed to arithmetic and geometry. During the Renaissance, considerable effort was invested to elevate the status of the arts to equal sciences. The pioneering role was played by the artisan-impresario Filippo BRUNELLESCHI (1377–1446) and the humanist Leon Battista ALBERTI (1404 –1472). BRUNELLESCHI introduced his perspective method in 1425 on two paintings (today lost) [8] and ALBERTI employed his expertise in optics to codify BRUNELLESCHI's findings in an easy-to-follow, step-by-step form in his book *De Pictura* (On Painting, 1434) [1]. While the former's approach was still in the spirit of art as a mirror of Nature and God's brilliance revealed by geometry, ALBERTI's 'geometric figuring' was to provide means to the representation of a wordly perfection framed by geometry.

Although the outstanding architects, artists and scientist of the Renaissance shared common intellectual grounds, the language of mathematics and science in general came to be so technical with time, that only a few giants, such as LEONARDO DA VINCI (1452 –1519) and Johann Wolfgang VON GOETHE (1749 –1832) ventured to carry out work of scientific nature — which was received with little appreciation from their respective scientific communities. Isaac NEWTON (1643 –1727) was the last scientist who attributed a role — that of the designer and the overseer — to God in the operation of the universe. Even with God on board, he concentrated rather on demonstrating a proof for the calculation of the acting forces that allow for their prediction and control. The deepening an understanding of the qualities of these forces to uncover ways to cooperate with them harmoniously was an aim pursued rather by artists.

This gradual mathematizing of the universe and an ever-increasing focus on its mechanical working was fuelled by an almost religious faith in technology's potential to bring salvation to humans already during their earthly existence. The real fascination, however, was with the powers of rational science of which technology was only a by-product. With industrialization, a completely new element in human thought occurred: the view on human labour as a subhuman drudgery. The Saint Benedictine dictum, *"laborare est orare"*, the conviction that humans may perfect themselves through work became profoundly violated. Technological progress and the resulting industrialization indeed created working conditions that degraded work to a neither pleasurable nor sacred activity.

The ills of a system that ceases to concern itself with the spirit became first apparent on the material plane, and manual labourers were the first to rebel against being robbed from their creative faculties. But, eventually they ended trading the joy and satisfaction they used to find in their work for amenities, safety and commodities. Beauty and sensibility became relegated to spare time, the only part of life when most people take pleasure in themselves as creative humans. *"... in a normal society, the artist is not a special kind of man, but every man is a special kind of artist, and there is no such hard distinction between the fine arts and others ..."* claimed the artist-workman Eric Gill (1882–1940), joining the long lineage of the English tradition of radical thought that has been addressing these issues since the dawn of industrialisation.

Along this generic and thus inexorably oversimplified introduction, present authors are equally aware of the many attempts that have been made in modern times both from the artists' and the scientists' side to establish a common platform to mingle the artistic and the scientific experience. The aim is to contribute to this ongoing process by placing side by side visually similar but in their origin strikingly different phenomena. No further speculation is intended beyond this juxtaposition. It is believed that a recognition of unity in variety inspires the interchange of ideas and in turn kindles the creative powers of all backgrounds.

In Section 2 we summarize the geometrical aspects of our investigations and in Section 3 we bring several examples from architecture and the arts where similarities to extremal tilings occur.

## 2. Mathematical background of ball packings

### 2.1. Ball packings in spaces of constant curvature

Finding the densest (not necessarily periodic) packing of balls in the 3-dimensional Euclidean space is known as the Kepler Conjecture: *No packing of spheres of the same radius has a density greater than the face-centered cubic packing.* This density can be realized by hexagonal layers (in continuum many ways). This conjecture was first published by Johannes Kepler in his monograph *The Six-Cornered Snowflake* (1611), this treatise inspired by his correspondence with Thomas Harriot (see Cannonball Problem). In 1953, László Fejes Tóth reduced the Kepler conjecture to an enormous calculation procedure that involved specific cases, and later suggested that computers might be helpful for solving the problem. In this way the above four hundred year mathematical problem has finally been solved by Thomas Hales [11]. He had proved that the guess of Kepler from 1611 was correct. Sphere packing problems usually concern arrangements of non-overlapping equal spheres (rather balls) which fill space, where space is interpreted as the usual three-dimensional Euclidean space. However, ball (sphere) packing problems can be generalized to the other 3-dimensional Thurston geometries, but a difficult problem is — similarly to the hyperbolic space — the exact definition of the packing density.

**Ball and horoball (horosphere) packings:** In an $n$-dimensional space of constant curvature $\mathbf{E}^n$, $\mathbf{H}^n$, $\mathbf{S}^n$ $(n \geq 2)$ let $d_n(r)$ be the density of $n+1$ spheres of radius $r$ mutually touching one another with respect to the simplex spanned by the centres of the spheres. L. Fejes Tóth and H.S.M. Coxeter conjectured that in an $n$-dimensional space of constant curvature the density of packing balls of radius $r$ cannot exceed $d_n(r)$. This conjecture has been proved by C.A. Rogers in the Euclidean space $\mathbf{E}^n$ [29]. The 2-dimensional spherical case was settled by L. Fejes Tóth in [9], and in [3] K. Böröczky proved the following generalization:

*In an $n$-dimensional space of constant curvature consider a packing of spheres of radius $r$. In spherical space suppose that $r < \frac{\pi}{4}$. Then the density of each sphere in its Dirichlet-Voronoi cell cannot exceed the density of $n+1$ spheres of radius $r$ mutually touching one another with respect to the simplex spanned by their centres.*

The above greatest density in $\mathbf{H}^3$ is $\approx 0.85328$ which is not realized by packing with equal balls. However, it is attained by the horoball packing (in this case $r = \infty$) of $\overline{\mathbf{H}}^3$ where the ideal centers of horoballs lie on the absolute figure of $\overline{\mathbf{H}}^3$. This ideal regular tetrahedron tiling is given with Coxeter-Schläfli symbol $\{3,3,6\}$. Ball packings of hyperbolic $n$-space and of other Thurston geometries are extensively discussed in the literature see, e.g., [3], [25] and [37], where the reader finds further references as well.

In a previous paper [13] we proved that the above known optimal horoball packing arrangement in $\mathbf{H}^3$ is not unique using the notions of horoballs in same and different types. Two horoballs in a horoball packing are of the "same type" if the local densities of the horoballs to the corresponding cell (e.g., D-V cell or ideal simplex) are equal (see [35]). We gave several new examples of horoball packing arrangements based on totally asymptotic Coxeter tilings that yield the above Böröczky–Florian packing density upper bound (see [4])

We have also found that the Böröczky–Florian type density upper bound for horoball packings of different types is no longer valid for fully asymptotic simplices in higher dimensions $n > 3$ (see [36]). For example in $\mathbf{H}^4$, the density of such optimal, locally densest horoball packing is $\approx 0.77038$ larger than the analogous Böröczky–Florian type density upper bound of $\approx 0.73046$. However, these horoball packing configurations are only locally optimal and cannot be extended to the whole hyperbolic space $\mathbf{H}^4$.

In papers [14], [16] and [15] we have continued our previous investigation in $\mathbf{H}^n$ $(n \in \{4,5\})$ allowing horoballs of different types. We gave several new examples of horoball packing configurations that yield high densities ($\approx 0.71645$ in $\mathbf{H}^4$ and $\approx 0.59421$ in $\mathbf{H}^5$) where horoballs are centered at ideal vertices of certain Coxeter simplices, and are invariant under the actions of their respective Coxeter groups.

**Hyperball (hypersphere) packings:** A hypersphere is the set of all points in $\mathbf{H}^n$, lying at a certain distance, called its *height*, from a hyperplane, on both sides of the hyperplane (cf. [48] for the planar case). In hyperbolic plane $\mathbf{H}^2$ the universal upper bound of the hypercycle packing density is $\frac{3}{\pi}$, and the universal lower bound of hypercycle covering density is $\frac{\sqrt{12}}{\pi}$, proved by I. Vermes in [47, 48, 49].

In [31] and [32] we analysed the regular prism tilings (simply truncated Coxeter orthoscheme tilings) and the corresponding optimal hyperball packings in $\mathbf{H}^n$ $(n = 3, 4)$ and we extended the method developed in [32] to 5-dimensional hyperbolic space (see [38]). In [39] we studied the $n$-dimensional hyperbolic regular prism honeycombs and the corresponding coverings by congruent hyperballs and we determined their least dense covering densities. Furthermore, we formulated conjectures for candidates of the least dense hyperball covering by congruent hyperballs in 3- and 5-dimensional hyperbolic spaces.

In [42] we discussed congruent and non-congruent hyperball packings of the truncated regular tetrahedron tilings. These are derived from the Coxeter simplex tilings $\{p,3,3\}$ $(7 \leq p \in \mathbb{N})$ and $\{5,3,3,3,3\}$ in 3- and 5-dimensional hyperbolic space. We determined the densest hyperball packing arrangement and its density with congruent hyperballs in $\mathbf{H}^5$ and determined the smallest density upper bounds of non-congruent hyperball packings generated by the above tilings in $\mathbf{H}^n$ $(n = 3, 5)$.

In [40] we deal with the packings derived by horo- and hyperballs (briefly *hyp-hor* packings) in $n$-dimensional hyperbolic spaces $\mathbf{H}^n$ $(n = 2, 3)$ which form a new class of the classical packing problems. We constructed in the 2- and 3-dimensional hyperbolic spaces hyp-hor packings that are generated by complete Coxeter tilings of degree 1 and we determined their densest packing configurations and their densities. We proved using also numerical approximation methods that in the hyperbolic plane $(n = 2)$ the density of the above hyp-hor packings arbitrarily approximate the universal upper bound of the hypercycle or horocycle packing density $\frac{3}{\pi}$ and in $\mathbf{H}^3$ the optimal configuration belongs to the $\{7, 3, 6\}$ Coxeter tiling with density $\approx 0.83267$. Furthermore, we analyzed the hyp-hor packings in truncated orthoschemes $\{p, 3, 6\}$ $(6 < p < 7,\ p \in \mathbb{R})$ whose density function is attained its maximum for a parameter which lies in the interval $[6.05, 6.06]$ and the densities for parameters lying in this interval are larger that $\approx 0.85397$.

In [41] we proved that if the truncated tetrahedron is regular, then the density of the densest packing is $\approx 0.86338$. This is larger than the Böröczky–Florian density upper bound but our locally optimal hyperball packing configuration cannot be extended to the entirety of $\mathbf{H}^3$. However, we described a hyperball packing construction, by the regular truncated tetrahedron tiling under the extended Coxeter group $\{3, 3, 7\}$ with maximal density $\approx 0.82251$.

Recently, (to the best of author's knowledge) the candidates for the densest hyperball (hypersphere) packings in the 3-, 4- and 5-dimensional hyperbolic space $\mathbf{H}^n$ are derived by the regular prism tilings which have been studied in [31], [32] and [38].

In [43] we considered hyperball packings in 3-dimensional hyperbolic space and developed a decomposition algorithm that for each saturated hyperball packing provides a decomposition of $\mathbf{H}^3$ into truncated tetrahedra. Therefore, in order to get a density upper bound for hyperball packings, it is sufficient to determine the density upper bound of hyperball packings in truncated simplices.

In [45] we studied hyperball packings related to truncated regular octahedron and cube tilings that are derived from the Coxeter simplex tilings $\{p, 3, 4\}$ $(7 \le p \in \mathbb{N})$ and $\{p, 4, 3\}$ $(5 \le p \in \mathbb{N})$ in 3-dimensional hyperbolic space $\mathbf{H}^3$. We determined the densest hyperball packing arrangement and its density with congruent and non-congruent hyperballs related to the above tilings. Moreover, we prove that the locally densest congruent or non-congruent hyperball configuration belongs to the regular truncated cube with density $\approx 0.86145$. This is larger than the Böröczky–Florian density upper bound for balls and horoballs. We described a non-congruent hyperball packing construction, by the regular cube tiling under the extended Coxeter group $\{4, 3, 7\}$ with maximal density $\approx 0.84931$.

In [44] we examined congruent and non-congruent hyperball packings generated by doubly truncated Coxeter orthoscheme tilings in the 3-dimensional hyperbolic space. We proved that the densest congruent hyperball packing belongs to the Coxeter orthoscheme tiling of parameter $\{7, 3, 7\}$ with density $\approx 0.81335$. This density is equal — by our conjecture — with the upper bound density of the corresponding non-congruent hyperball arrangements.

### 2.2. Formulation of the problem

Let $X$ be one of the 3-dimensional geometries of constant curvature ($\mathbf{E}^3$, $\mathbf{H}^3$, $\mathbf{S}^3$) (see [37]). In the present paper we consider ball packings where their symmetry group $\Gamma$ is a fixed group of isometries of the space $X$.

In these geometries the geodesic curves are generally defined as having locally minimal arc length between any two of their points (sufficiently close to each other). The equation systems of the parametrized geodesic curves $\gamma(\tau)$ in our model can be determined by the general theory of Riemann geometry. Then a *geodesic sphere* and *ball* can be usually defined as follows below. We consider only those geodesic ball packings which are transitively generated by discrete groups of isometries of $X$ and the density of the packing is related to its Dirichlet-Voronoi cells.

**Definition.** The *distance* $d(P_1, P_2)$ between the points $P_1 \in X$ and $P_2 \in X$ is defined by the arc length of the geodesic curve from $P_1$ to $P_2$.

**Definition.** The *geodesic sphere of radius* $\rho$ (denoted by $S_{P_1}(\rho)$) with centre at the point $P_1$ is defined as the set of all points $P_2$ in the space with the condition $d(P_1, P_2) = \rho$. Moreover, we require that the geodesic sphere is a simply connected surface of the space $X$ having no self-intersection.

**Definition.** The body of the geodesic sphere of centre $P_1$ and with radius $\rho$ in space $X$ is called *geodesic ball*, denoted by $B_{P_1}(\rho)$, i.e., $Q \in B_{P_1}(\rho)$ iff $0 \leq d(P_1, Q) \leq \rho$.

In the following let $\Gamma$ be a fixed group of isometries of $X$. Denote by $d(P_1, P_2)$ the distance of two points $P_1$, $P_2$ (see Definition above).

**Definition.** We say that the point set

$$\mathcal{D}(K) = \{P \in X \,:\, d(K, P) \leq d(K^{\mathbf{g}}, P) \text{ for all } \mathbf{g} \in \Gamma\}$$

is the *Dirichlet-Voronoi cell* (D-V cell) to $\Gamma$ around the kernel point $K \in X$.

**Definition.** We say that

$$\Gamma_P = \{\mathbf{g} \in \Gamma \,:\, P^{\mathbf{g}} = P\}$$

is the *stabilizer subgroup* of $P \in X$ in $\Gamma$.

**Definition.** Assume that the stabilizer $\Gamma_K = \mathbf{I}$ is the identity, i.e., $\Gamma$ acts simply transitively on the $\Gamma$-orbit of $K \in X$. Then let $B_K$ denote the *greatest ball* with centre $K$ inside the D-V cell $\mathcal{D}(K)$. Moreover, let $\rho(K)$ denote the *radius* of $B_K$. It is easy to see that

$$\rho(K) = \min_{\mathbf{g} \in \Gamma \setminus \mathbf{I}} \frac{1}{2} d(K, K^{\mathbf{g}}).$$

**Definition.** If the stabilizer $\Gamma_K > \mathbf{I}$ then $\Gamma$ acts *multiply transitively* on the $\Gamma$-orbit of $K \in X$. Then the greatest ball radius of $\mathcal{B}_K$ is

$$\rho(K) = \min_{\mathbf{g} \in \Gamma \setminus \Gamma_K} \frac{1}{2} d(K, K^{\mathbf{g}}),$$

where $K$ belongs to a 0-, 1-, or 2-dimensional region of $X$ (vertices, axes, reflection planes).

In both cases the $\Gamma$-images of $B_K$ form a ball packing $\mathcal{B}^{\Gamma}_K$ with centre points $K^{\mathbf{G}}$.

**Definition.** The *density* of ball packing $\mathcal{B}^{\Gamma}_K$ is

$$\delta(K) = \frac{\mathrm{Vol}\,(B_K)}{\mathrm{Vol}\,\mathcal{D}(K)}.$$

It is clear that the orbit $K^\Gamma$ and the ball packing $\mathcal{B}_K^\Gamma$ have the same symmetry group. Moreover, this group contains the starting crystallographic group $\Gamma$:

$$Sym\, K^\Gamma = Sym\, \mathcal{B}_K^\Gamma \geq \Gamma.$$

**Definition.** We say that the orbit $K^\Gamma$ and the ball packing $\mathcal{B}_K^\Gamma$ is *characteristic* if $Sym\, K^\Gamma = \Gamma$, otherwise the orbit is *not characteristic.*

### 2.2.1. Simply transitive ball packings

Let $\Gamma$ be a fixed group of isometries in the space $X$. *Our problem is* to find a point $K \in X$ and the orbit $K^\Gamma$ for $\Gamma$ such that $\Gamma_K = \mathbf{I}$ and the density $\delta(K)$ of the corresponding ball packing $\mathcal{B}^\Gamma(K)$ is maximal. In this case the ball packing $\mathcal{B}^\Gamma(K)$ is said to be *optimal.*

Our aim is to determine the maximal radius $\rho(K)$ of the balls, and the maximal density $\delta(K)$. The considered space groups could have free parameters. So we have to find the densest ball packing for fixed parameters $p(\Gamma)$, and then we have to vary them to get the optimal ball packing

$$\delta(\Gamma) = \max_{K,\ p(\Gamma)} (\delta(K)).$$

We look for the optimal kernel point in a 3-dimensional region, inside of a fundamental domain of $\Gamma$. The Dirichlet-Voronoi cell belonging to the optimal kernel point is called *the optimal cell* of the considered group.

### 2.2.2. Multiply transitive ball packings

Similarly to the simply transitive case we have to find a kernel point $K \in X$ and the orbit $K^\Gamma$ for $\Gamma$ such that the density $\delta(K)$ of the corresponding ball packing $\mathcal{B}^\Gamma(K)$ is maximal, but here $\Gamma_K \neq \mathbf{I}$. This ball packing $\mathcal{B}^\Gamma(K)$ is called *optimal*, too. In this multiply transitive case we look for the optimal kernel point $K$ in possible 0-, 1-, or 2-dimensional regions $\mathcal{L}$, respectively. Our aim is to determine the maximal radius $\rho(K)$ of the balls, and the maximal density $\delta(K)$.

The considered space group can have also free parameters $p(\Gamma)$. Then we have to find the densest ball packing for fixed parameters, and vary them to get the optimal ball packing. The Dirichlet-Voronoi cell belonging to the optimal kernel point is called *the optimal cell* of the considered group

$$\delta(\Gamma) = \max_{K \in \mathcal{L},\ p(\Gamma)} (\delta(K)).$$

## 3. Intriguing examples in hyperbolic space $\mathbf{H}^3$

In this section, we identify some interesting overlaps between the artistic areas and the optimal cells in hyperbolic space. Further examples can be found in other Thurston geometries, but these will be studied in a further publication.

### 3.1. Hypersphere packings – Scottish stone balls

The optimal congruent and non-congruent hyperball packing arrangements impressively resemble some of the specimens out of the around 400 stone balls found predominantly in Scotland that date back to between 3200 – 2500 BC. The size of these spheres is identical, around 7 centimetres in diameter and almost half of them have six knobs, while other designs vary from 3 to 160 knobs. There are only speculations concerning the purpose of these balls. They are obviously not by-products of a mathematical ability in our sense but are remnants of a cultural conduct described in the introduction, when peoples intuited the existence of natural forces and aimed to cooperate with them for their own benefit.

The widely diverse stone materials used for the execution and the varied quality of craftsmanship of these balls suggests that the size and the form were the dominant factors of these objects to fulfil their purpose. The time and skill required to fashion these pieces gives further hints that they were precious possessions that are unlikely to be used as weapons or games that involved throwing. They might have been instruments of fortune-telling, but it is doubtful that they were owned by individuals as they were never found in burial sites or near habitats.

The most daring explanations presume that in the preliterate societies the stone balls served to store of information about celestial motions and about how this information can be applied to the construction of calendrical monuments. According to such reckonings Neolithic societies had an awareness about the workings of magnetism and telluric earth energy currents and the stone balls formulated a network with the monolithic sites in which the stone pillars acted as antennae and the spheres functioned as distributors of energy currents to enhance crop growth among other beneficial effects. These 'charged stones' in turn might have been used as healing devices.

In geometrical sense, in the paper [45] we studied congruent and non-congruent hyperball (hypersphere) packings to the truncated regular cube and octahedron tilings. These are derived from the Coxeter truncated orthoscheme tilings $\{4,3,p\}$ $(6 < p \in \mathbb{N})$ and $\{3,4,p\}$ $(4 < p \in \mathbb{N})$, respectively, by their Coxeter reflection groups, in hyperbolic space $\mathbf{H}^3$. We determined the densest hyperball packing arrangement and its density with congruent and non-congruent hyperballs (see Figures 1 and 2).

We proved that the locally densest (non-congruent half) hyperball configuration belongs to the truncated cube with density $\approx 0.86145$, if we allow $6 < p \in \mathbb{R}$ for the dihedral angle $2\pi/p$. This local density is larger than the Böröczky–Florian density upper bound for balls and horoballs. But our locally optimal non-congruent hyperball packing configuration cannot be extended to the entire hyperbolic space $\mathbf{H}^3$. We determine the extendable densest non-congruent hyperball packing arrangement to the truncated cube tiling $\{4,3,p=7\}$ with density $\approx 0.84931$.

### 3.2. Horoball packings

In [17] we visualized the cases of optimal horoball arrangements. Four known packings of hyperbolic 3-space give the optimal packing density of approximately 0.85328. They are realized in the regular Coxeter honeycombs with Schläfli symbol $\{3,3,6\}$ and $\{4,3,6\}$. These honeycombs are totally asymptotic, and the packings involve horoballs (of different types) centered at ideal vertices. In this paper we described a method to visualize regular horoball packings of extended hyperbolic 3-space $\overline{\mathbf{H}}^3$ using the projective Cayley-Klein-Beltrami model and the Coxeter symmetry group of the packing. Using our techniques we produced images (different crowns) of the optimal horoball packings (see Figures 3, 4, 8, 9).

### 3.3. Football manifolds, fullerenes

The so-called football manifolds $\{5,6,6\}$, appearing in the Bolyai-Lobachevsky hyperbolic space $\mathbf{H}^3$, can model *"fullerenes"* very probably. Recent extremal ball packing and covering discoveries in $\mathbf{H}^3$ with systematic computations of the second author have convinced us that

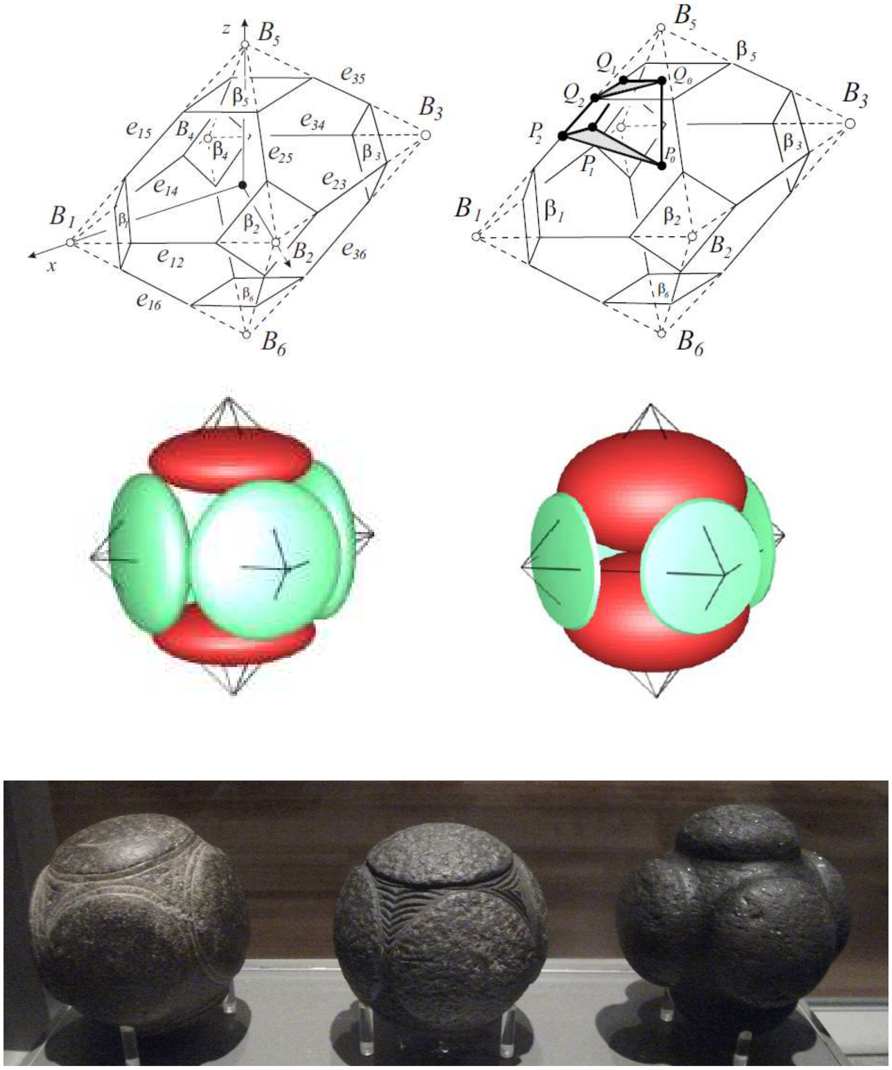


Figure 1: a,b) The optimal congruent and non-congruent hyperball packing arrangements related to truncated octahedron tilings. c) The Scottish stone balls (3200 –2500 BC) (Photocredit: Johnbod `https://commons.wikimedia.org/wiki/File:Kelvingrove_Art_Gallery_and_MuseumDSCF0239_11.JPG`, Kelvingrove Art Gallery and Museum)

non-Euclidean crystallography is and remain to be a timely research topic in the near future. As an important principle, we mention that our football manifold seems to be minimal (left for later publication), i.e., it does not cover a smaller manifold. But it is covered by a hyperbolic dodecahedron manifold $M_1 = F_D$ in Figure 5, described also by I. Prok [28]. The classification of minimal compact 3-dimensional manifolds seems to be a very hard and timely open problem. The 10 Euclidean 3-space forms are well-known from crystallography.

The inscribed ball into $\widetilde{\mathbf{F}}_{\mathbf{G}}$, and so the ball packing by the tiling under group $\mathbf{G}$, symbolizes the atomic (molecule) structure with the best known top density $0,77147\ldots$. Similarly, the circumscribed ball of $\widetilde{\mathbf{F}}_{\mathbf{G}}$ serves the best known loosest ball covering for hyperbolic space $\mathbf{H}^3$ [25] with density $1.36893\ldots$. To this we need the generalization of the volume formula of N.I. Lobachevsky for complete orthoscheme as we mention only by [25]. For other analogous ball packing and covering problems, we refer to [25, 31, 32, 35, 39, 45], Figures 5, 8 and Subsection 3.5.

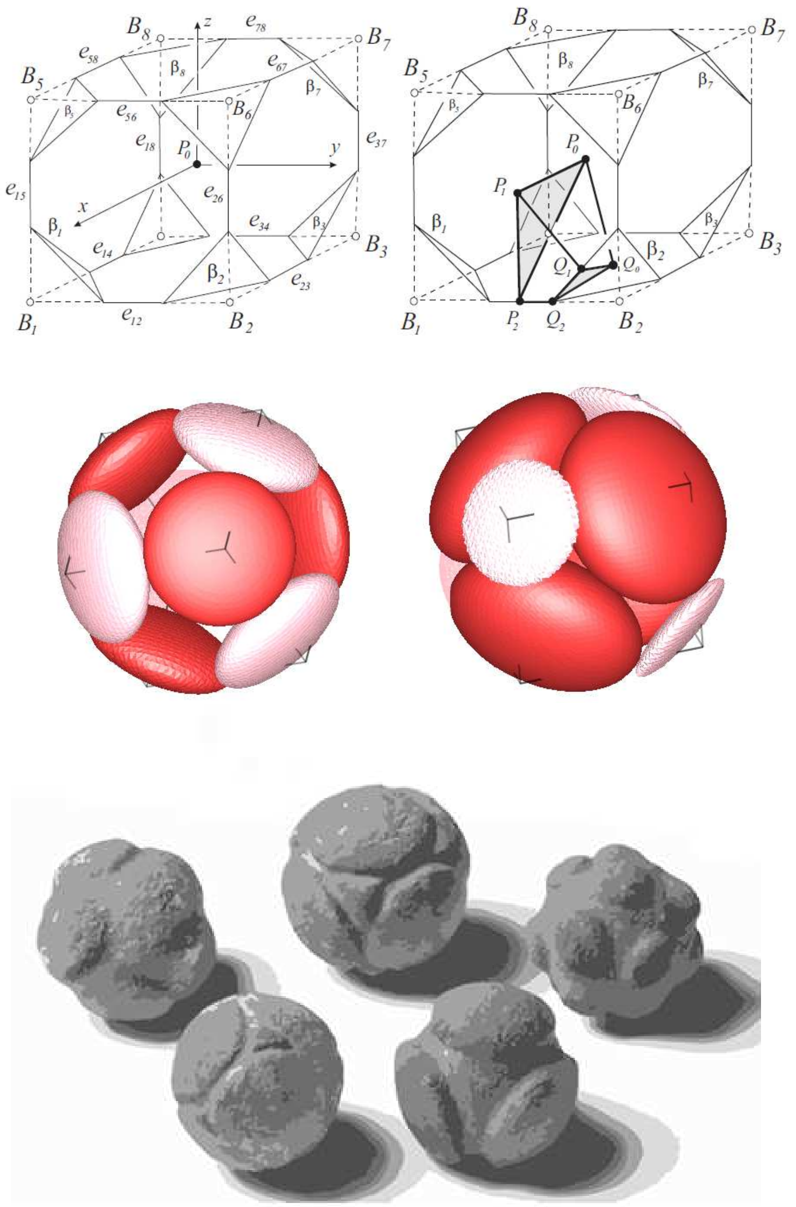


Figure 2: a,b) The optimal congruent and non-congruent hyperball packing arrangements related to truncated cube tilings. c) The Scottish stone balls (authors' vectorgraphic reproduction of a photograph taken from https://www.ancient-origins.net/artifacts-ancient-technology/geometric-stone-spheres-scotland-part-2-explanations-platonic-solids-021577)

### 3.4. Cobweb manifolds, nanotubes – the "Crystal Chainers" work

In [24] and [26] we have constructed a fixed point free group acting in hyperbolic space $\mathbf{H}^3$ with the given compact fundamental domain. In Figure 6 we have described the extended reflection group $\mathbf{G}(6,6,6) = \mathbf{G}(6)$ with fundamental domain $W(6)$, as a half of the complete Coxeter orthoscheme $\mathcal{O}(6)$, and glued together to the cobweb polyhedron $Cw(6,6,6){=}Cw(6)$ as Dirichlet-Voronoi (in short D-V) cell of the kernel point $Q$ by its orbit under the group $\mathbf{G}(6)$. Now by Figure 6 we shall give the face identification of $Cw(6)$, so that it will be

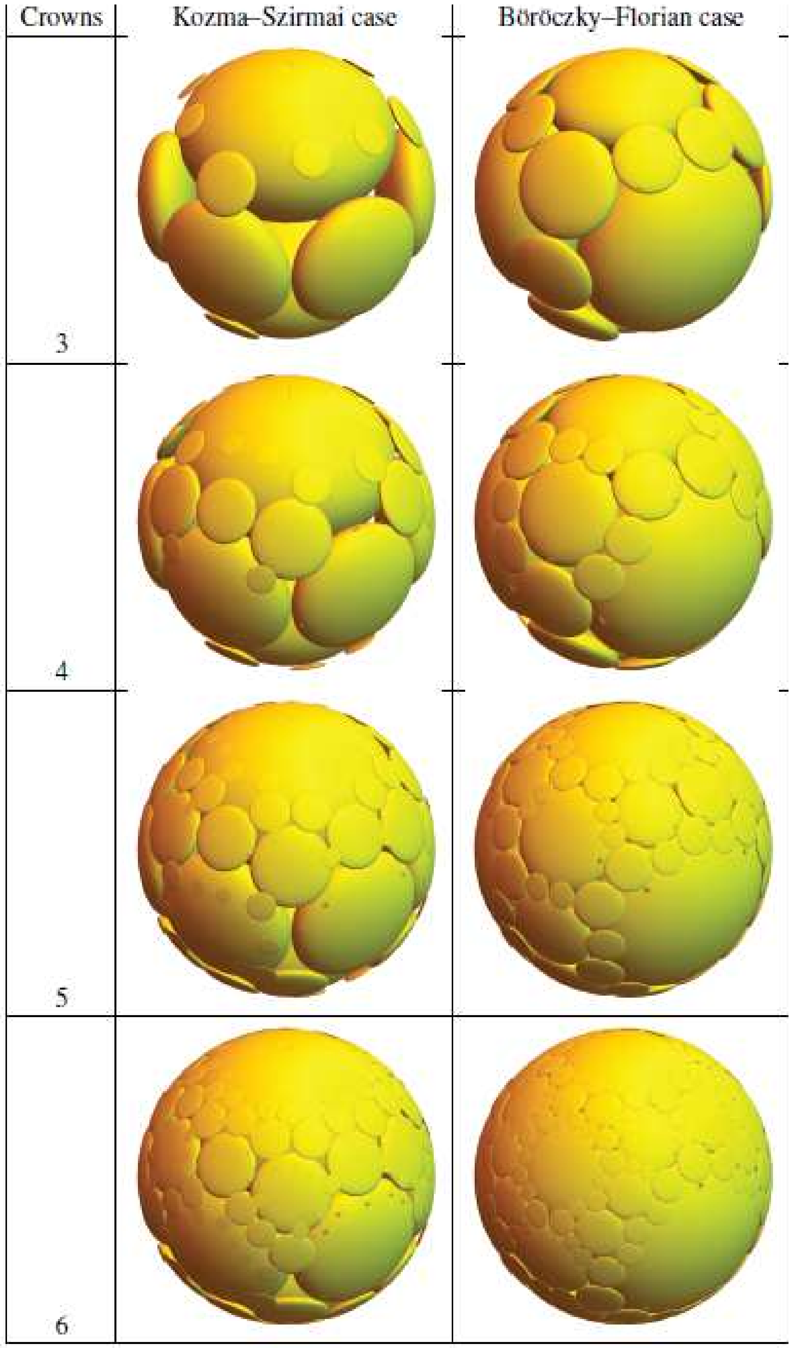


Figure 3: The optimal horoball packing related to regular ideal tetrahedron tiling in Beltrami-Cayley-Klein model of $\mathbf{H}^3$

fundamental polyhedron of the fixed-point-free group, denoted also by $\mathbf{Cw}(6)$, generated just by the face identifying isometries (as hyperbolic screw motions).

The densities of packing and covering are

$$\begin{aligned}\delta(6,6,6) &= \mathrm{Vol}\,(B(r))/\mathrm{Vol}\,(Cw) = 0.10503,\\ \Delta(6,6,6) &= \mathrm{Vol}\,(B(R))/\mathrm{Vol}\,(Cw) = 6.05670,\end{aligned}$$

respectively, and play relevant roles for our manifold.

The cobweb manifold as pictured in the Beltrami-Cayley-Klein model recalls the fantastic

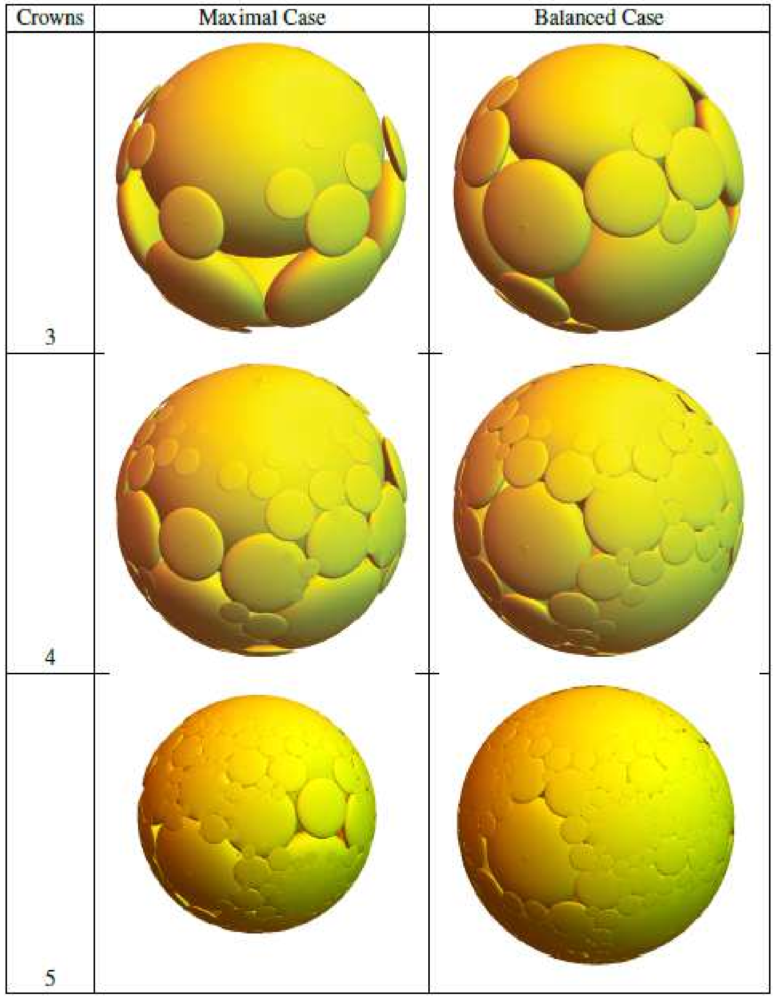


Figure 4: The optimal horoball packing related to regular ideal cube tiling in Beltrami-Cayley-Klein model of $\mathbf{H}^3$

crystal worlds depicted by early 20th century artist, in our examples those that united for a short period in 1919–1920 in an intellectual exchange in Germany that is known to architectural history as the *Crystal Chain Correspondence*. The turn of the 19th and 20th century witnessed a triumphant march of technology, breaking through boundaries that were formerly believed to be beyond the reach of humanity. The railway penetrated into hitherto hard to access territories, facilitating an unprecedentedly intense removal of their natural resources while linking them into the global current. Aviation technology made its most significant conquests of the sky and with the development of electric power transmission and distribution technology arc lights replaced gas lamps in the city and streetcars stepped in the place of horse-powered carriages.

Artists dedicated themselves to conceptualise a society that harmoniously coexisted with these new unleashed forces for its own betterment. The ills of the industrialized society were clearly seen and the awareness was there that the magnified power thus gained was equally capable for the destruction as well as for the redemption of humanity. Writers, painters, sculptors, architects all took up an almost apostolic mission to put forth ideas and visions that demonstrated a spiritual wakening of humanity as a whole, aided by technology. With our present mindset, it is almost impossible to interpret this activity as anything realistic.

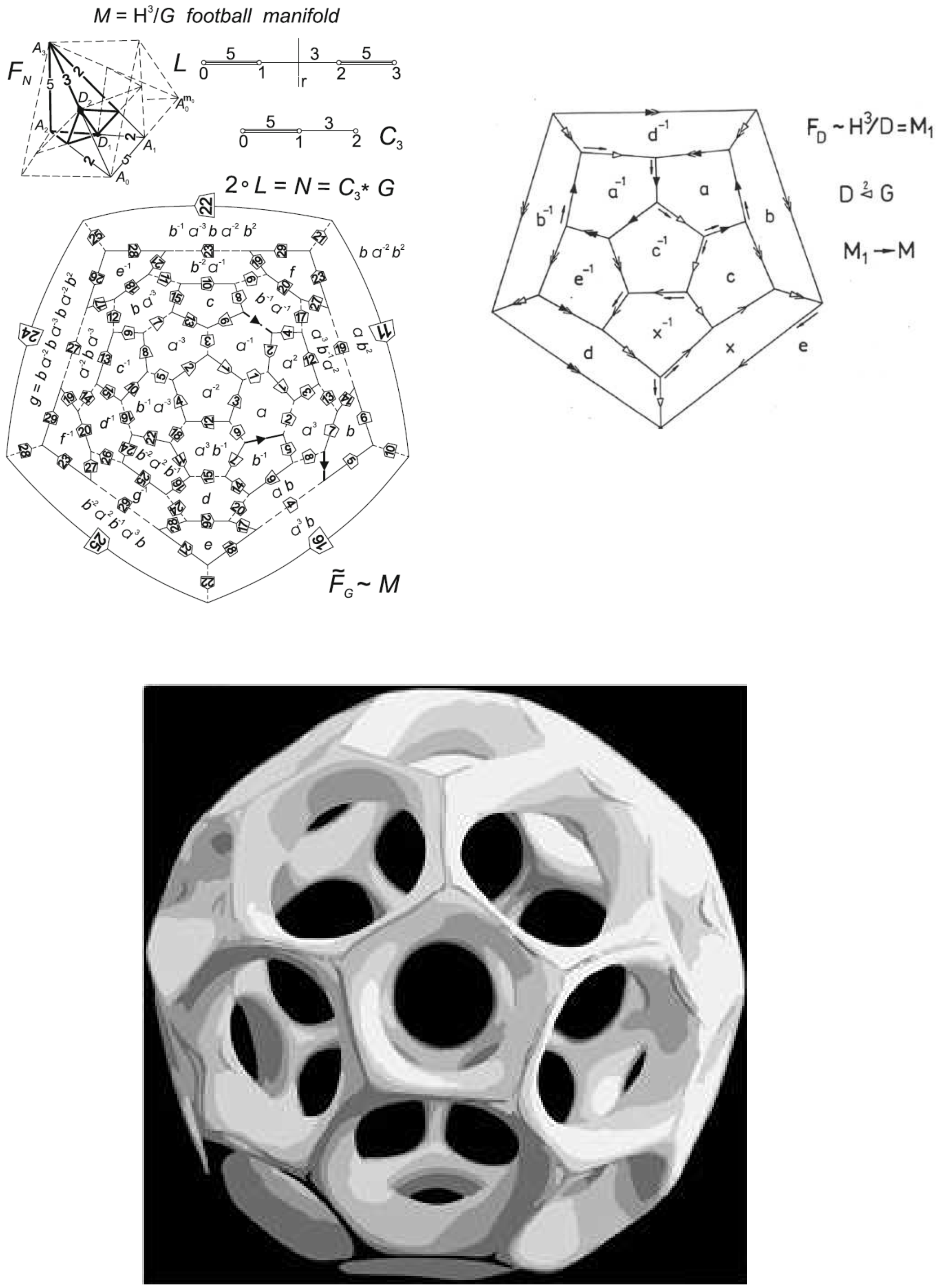


Figure 5: a,b) The hyperbolic football manifold (fullerene), realized by face pairing isometries of the Archimedean solid $\{5, 6, 6\}$. c) Bobby JABER's porcelain buckminsterfullerene (see description of artwork under Subsection 3.6) (authors' vectorgraphic reproduction of image taken from documentary 'Porcelainia', by Dave MAZE `https://vimeo.com/71474416`)

Today the literary output of this period is considered to be the forerunner of science fiction literature and the architectural projects are seen as mere fantasies, of which only the cinematic arts may gain inspiration.

This revelation was very prompt, the initiator of the *Crystal Chain Correspondence* and

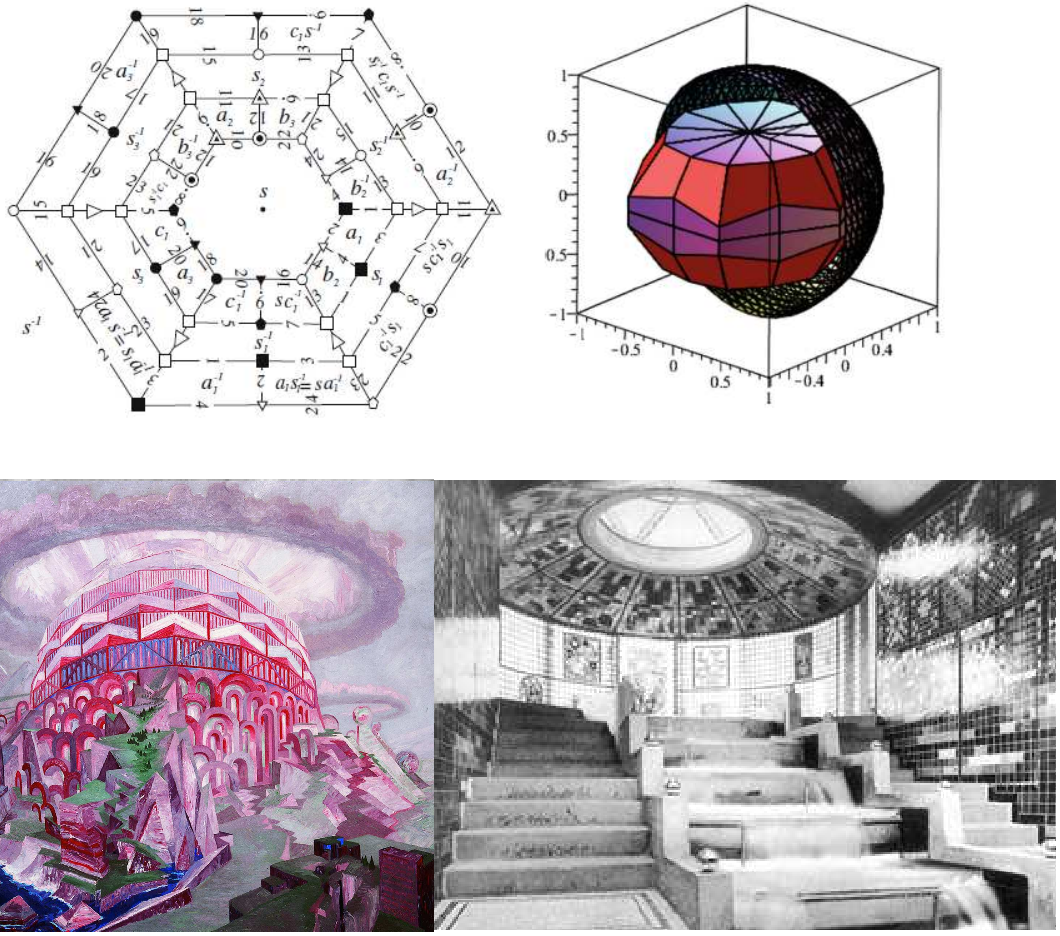

Figure 6: a,b) The cobweb (tube) manifold $Cw(6)$ with its symbolic face pairing isometries. Edge and vertex equivalence classes are indicated. Any point has a ball-like neighbourhood (nanotube). A picture of its animation in Beltrami-Cayley-Klein model. c) Wenzel Hablik: *Freitragende Kuppel mit fünf Bergspitzen als Basis* (1918/23/24), © Wenzel-Hablik-Foundation, Itzehoe. d) Bruno Taut: *Glass Pavilion interior* (1914) `https://hu.wikipedia.org/wiki/Fájl:Taut_Glass_Pavilion_interior_1914.jpg`, public domain because of age

the head of the Arbeitsrat für Kunst, the architect Bruno Taut (1880–1938), and the Bohemian painter and craftsman Wenzel Hablik (1881–1934) — whose works are shown in Figure 6 — recognized during their correspondence the unfeasibility of their visions and concluded that for the time being they record them in movie scripts and approach their acquaintances in the motion picture industries for prospects of realization. This soon led to the dissolution of the *Crystal Chain Correspondence* by the end of 1920 with Taut's declaring: *"I am now finished with intuitive works, I almost hope forever"* [23, p. 354] and went on to take commissions for the design of low cost housing estates, accepted a city architect position in Magdeburg before migrating to Japan and from there to Istanbul.

### 3.5. Hyperbolic dodecahedra – Roman dodecahedra and icosahedra

In [28] I. Prok investigated the dodecahedron tilings and proved that the dodecahedron has 12 essentially different face identifications which generate fixed point free orientation preserving transformation groups acting simply transitively on the $\{5,3,5\}$ tiling of the hyperbolic space. These face identifications determine compact orientable hyperbolic manifolds. Moreover, he proved, considering their first homology groups that there are at least 7 different ones among them. His algorithm also shows that compact non-orientable hyperbolic dodecahedron manifolds do not exist. The first compact hyperbolic dodecahedron manifold was discovered by C. Weber and H. Seifert. Then L.A. Best listed 7 more ones, however 3 of them were not different. If we consider the hyperbolic dodecahedron tiling $\{5,3,6\}$ then its vertices are ideal points. In this case in [13] we investigated the horoball packings and we obtained: The density of the optimally dense horoball arrangement for the dodecahedral Coxeter tiling $\{5,3,6\}$ is $\delta(5,3,6) \approx 0.787251$ (see Figure 7).

Our artistic examples for the hyperbolic dodecahedron, so-called *Roman dodecahedra* (see Figure 7c) induce very similar speculations to those of the Scottish stones. About a hundred have been found on sites across Rome's northern provinces (today France, Germany, Wales, and Hungary) and a single icosahedron, all dating back to 200–300 AD. The size of these artefacts is similar to, although not as uniform in dimensions as that of the stone balls. Each face of these hollow copper alloy polyhedra have different sized circular holes, in some cases surrounded by incised pentagonal lines, and the vertices are accentuated with large spherical projections. Measuring devices for calculating the trajectories of projectiles or that of celestial motions to identify the ideal day for sowing, surveying tools are all candidates for the possible uses of a Roman dodecahedron, without plausible demonstrations of its exact functioning.

The dodecahedral arrangement of these objects hints at their relatedness to the celestial realm, a connection Plato alluded in his *Timaeus*. Here, drawing on Empedocles (∼494–∼434 BC), he equated the four elements and their variously proportioned mixtures as the basis of all material manifestation in the cosmos. But Plato went further to develop his own abstract-geometric version by linking each element to a regular solid. The fifth regular solid, the dodecahedron was identified by him with the world as a whole. Many attributes the declaration of the dodecahedron as the symbol of heaven to Plato, but it came from Luca Pacioli (1445–1517) in his *Divina Proportione* (1509), a short treatise on the sacred meanings of the Golden Mean, that he assembled during his service in the Sforzas' court, where at the same time Leonardo was employed to realize military and artistic projects.

Leonardo was seeking help in order to master linear perspective for the design of the *Last Supper* to the convent of Santa Maria della Grazie. In exchange for tuition, he supplied Pacioli's book with exquisite illustrations. Centuries later, the Romanian prince, mathematician and philosopher Matila Ghyka (1881–1965) was a prime proponent of sacred geometry, a cryptic body of knowledge whose mysteries are believed to have transmitted from Pythagorean brotherhoods through the medieval guilds to the secret societies of 18th century Europe. Ghyka claimed that old masters, especially Leonardo, had a mathematical key for achieving beauty in art and included layout diagrams as an aid for the successful realization of similar artistic goals. Salvador Dalí (1904–1989) fell under the spell of these teachings, went on to read Pacioli's book and paid tribute to Leonardo with his *The Sacrament of the Last Supper* (1955) (Figure 7d). In the composition, he applied Ghyka's diagrams to position his figures and painted a dodecahedron in the background to symbolize the cosmos.

Richard Buckminster Fuller (1895–1983) was one of the most charismatic figures in the 20th century who ardently campaigned for a unified intellectual basis of the entire human experience. His theoretic work consisted of a compilation of universal laws primarily of geometric nature that he called Synergetic Energetic Geometry, or *Synergetics*. His philosophy was partially Platonic, postulating that the universe or space had an underlying geometric

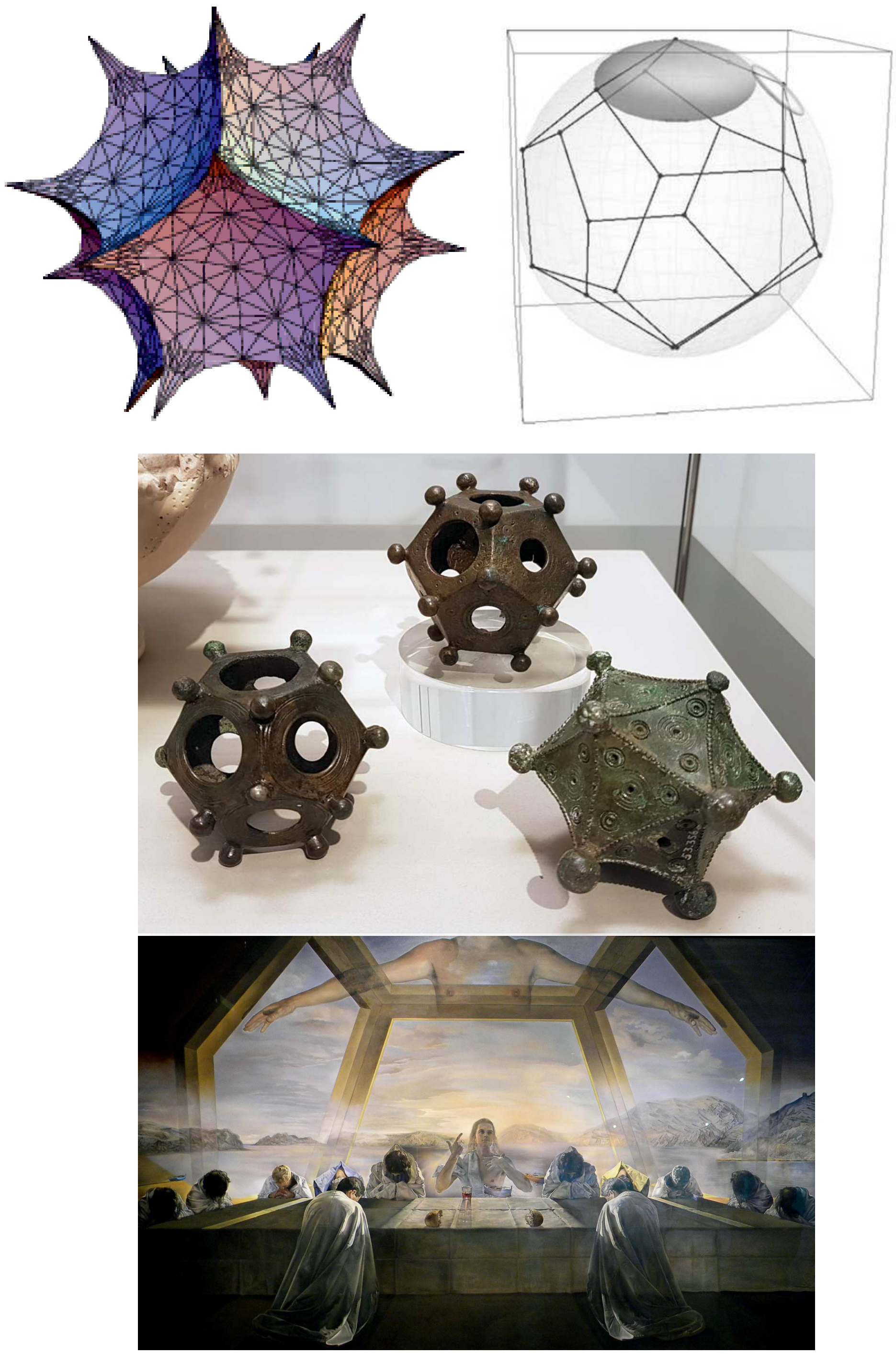

Figure 7: a,b) Hyperbolic dodecahedra in different models and the optimal horoball related to $\{5,3,6\}$ tiling. c) Roman dodecahedra and icosahedron (3rd century AD). Two ancient Roman bronze dodecahedrons and an icosahedron (3rd century AD) in the Rheinisches Landesmuseum in Bonn, Germany. The dodecahedrons were excavated in Bonn and Frechen-Bachem; the icosahedron in Arloff. 6 May 2018, Kleon3 (Wikimedia Commons user) `https://commons.wikimedia.org/wiki/File:2018RheinischesLandesmuseumBonn`, Dodekaeder. d) Salvador Dalí: *The Sacrament of the Last Supper*, 1955, ©Fundació Gala-Salvador Dalí / Bildrecht, Wien 2020

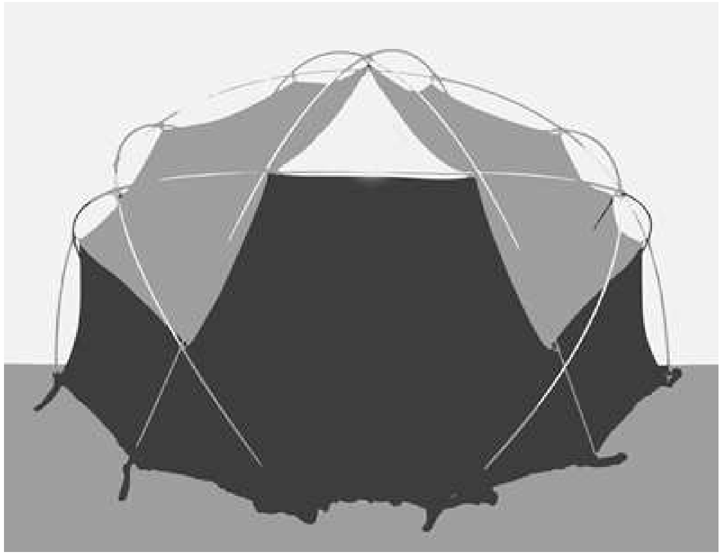

Figure 8: Richard BUCKMINSTER FULLER's *Oval Intention* tent (authors' vectorgraphic reproduction of photographic image)

structure, but he condemned the Platonic and the generic mathematical approach that followed from it, that the senses lead to a misperception of the world and the realm of pure forms can be understood only through abstract rational cognition. Instead, he gave primacy to a hands-on approach and advised model building exercises as the golden way to truly get acquainted with the "shape of space".

Besides his broad publishing activity, he travelled around the world in responding to invitations from teaching institutions that organized public lectures and construction courses under his guidance. It is hard to overestimate the continuing impact BUCKY had on generations of architects, designers and artists. Among them was Bruce HAMILTON, a product developer at *The North Face* outdoor gear company between 1970 and 1989 who invited BUCKMINSTER FULLER as an advisor to the design of a tent that exploited the sturdiness of BUCKY's tensegrity structures and geodesic domes. The result of this collaboration, the *'Oval Intention'* tent became available to public in 1975. The structural considerations live on in all of *The North Face*'s "lightweight, back-packable environment controlling devices" as FULLER termed tents.

Another distant disciple of BUCKY is Bobby JABER, a retired high school science teacher, who after 35 years of educational practice decided to base his work on his passion for the capturing of nature's symmetries in porcelain vessels, inspired both by his Arabic heritage and by his fascination with FULLER's geometric system, most of all with the buckminsterfullerene $C60$ (Figure 5c), which he calls *"supposedly the most beautiful molecule in existence"*. Over the decades JABER refined a sculptural method to accomplish his pieces, starting from a solid sphere that is gradually truncated with the carving of holes into it to arrive at the final hollow shape. This painstaking method takes about a year to complete due to the slow evaporation of water when aiming to maintain a state of equilibrium of moisture in the clay at every stage of sculpting. JABER has been employing a scientific rigour throughout the development and perfection of this process. He can be regarded as a contemporary paraphrase of the Neolithic craftsman who fashioned the stone balls or the Roman age coppersmiths who created the Roman dodecahedra.

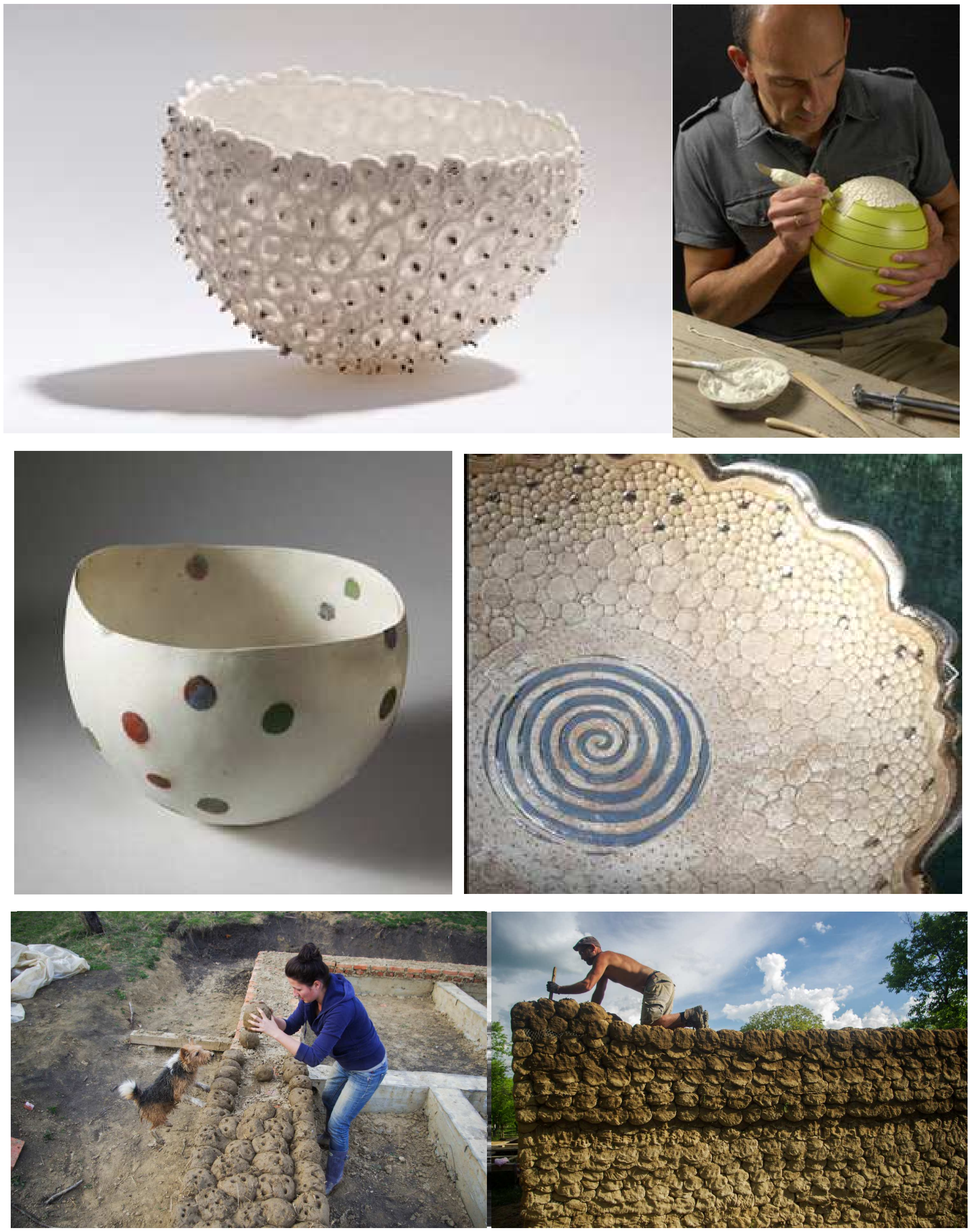

Figure 9: a) *Pineapple*, porcelain bowl by Guy van Leemput, 2013, photography: Dirk Theys. b) Guy van Leemput using an inflated balloon base to sculpt a vessel, photography: Dirk Theys. c) *Solar system*, porcelain bowl by Guy van Leemput, 2016, photography: Dirk Theys. d) Clay bubble bowl, Edit Bukrán, photography: Edit Bukrán. e,f) Construction of a cob building by stacking mud balls in contemporary Transylvania, photography: Egyed Ufo Zoltán.

### 3.6. Clay "ball packings" in art and architecture resembling some locally optimal ball and horoball packings

For the same problem of how nature builds a sturdy structure with as little material as possible that resulted in the *'Oval Intention' tent* from the collaboration of Buckminster Fuller and *The North Face*, mathematician and ceramist Guy van Leemput produces sculpted porcelain bowls as answers, in his own words in a *"search for the origin of all things"*. This quest takes place both in intuited and in more mathematized forms (Leemput's research interest while a mathematician was 'the monohedral and isohedral tilings of the plane'). For his *Pineapple*-series (Figures 9a, 3, 4), he uses inflated balloon bases onto which he adds minute porcelain slip balls which he flattens to achieve a continuous surface (Figure 9b). In the context where spatial structures are to be expressed in materials that have their own specific physical behaviour, in this case clay, spatial considerations are adapted to these constraints and the visual outcome is less obvious of the concept itself — although strongly sensible — to which Leemput's other shown work, the *Solar System* (Figures 9c, 3, 4) is an outstanding example.

Leemput's technique for the making of the bowls is a refined version of a coarser archaic way of creating pots of clay balls (which in most cases is mixed with coils). Formerly working on large plane murals, in recent decades ceramist Edit Bukrán bends her tilings into curved, mostly hyperbolic surfaces (Figure 9d). The reason for the bending is to allow the clayballs to organise themselves into a self-supporting structure, where the individual spherical elements are only glued with the final glazing of the plates. In a larger scale the similarly old earthen building technique of cob wall construction employs the same principle to achieve a structure in which the even division of clay particles and moisture, thus a balanced drying and weight distribution is ensured (Figure 9e, f).

### 3.7. Concluding remarks

Space is a curious phenomenon that imposes challenges upon human cognition. The ability to create an internal spatial map in order to navigate in it is an attribute in the animal kingdom already due to the evolutionary pressure of survival. But when it comes to humans, and their unique quality of a participatory approach in their own evolution, the question of the nature of space is more of a dialogue with it rather than a passive response to its constraints. Although our quest in finding parallels of visualizations of space — or records of spatial information — in the worlds of arts and sciences is confined to superficial appearances, we find it a valuable path to be followed. Optimal sphere packings in other homogeneous Thurston geometries represent another huge class of open mathematical problems. For these non-Euclidean geometries only very few results are known (e.g., [37]). Therefore, we find in the analysis of the optimal cells of other Thurston geometries and their relationship to the arts a challenge worth to respond to.

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Received January 27, 2020; final form April 3, 2020