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Entangled: From Triangles to Polygons

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Abstract. The aim of the present work is to study the configuration of two n-sided polygons with cevians where the sides and cevians of the first polygon enclose a constant angle with respective cevians and sides of the second polygon. We prove the existence of such pairs of n-gons, where sides are exchanged with cevians, and we call these polygons 'entangled'. Among the findings, there are generalizations of Miquel's theorem and Simson lines.

Key Words: orthologic triangles, entangled polygons, entanglement points, entanglement angle, Miquel point, Miquel circles, generalized Simson line
MSC 2010: 51M04

1. Introduction

In 1827, Jakob Steiner showed that for each triangle $A_1A_2A_3$ with any point N outside the sides there exists a triangle $X_1X_2X_3$ with sides X_1X_2, X_2X_3, X_3X_1 respectively orthogonal to the cevians A_3N, A_1N, A_2N where the perpendiculars through A_1, A_2, A_3 to the respective sides X_2X_3, X_3X_1, X_1X_2, meet at a point M. Steiner called such triangles *orthologic* (see [4, p. 55]). Obviously, the relation between orthologic triangles is symmetric.

In the following, we study a generalization, where $\sphericalangle PQ\,XY$ denotes the measure of the signed angle between the two lines PQ and XY, hence $\sphericalangle XY\,PQ = -\sphericalangle PQ\,XY$ modulo π. In particular, $0 < \phi := \sphericalangle PQ\,XY < \pi$ means there is an anticlockwise rotation through the angle ϕ which maps the line PQ to the line XY. Similarly, we use the symbol $\sphericalangle gh$ for the measure of the signed angle between the two lines g and h. For triangles and also for polygons we use the symbol $\widehat{A}$ for the measure of the interior angle at the vertex A.

Definition 1. Let $A_1A_2\dots A_n$ and $X_1X_2\dots X_n$ be two polygons and M, N two points where $N \neq A_1, \dots, A_n$ and $M \neq X_1, \dots, X_n$. Suppose that

$$\begin{aligned}\phi &= \sphericalangle A_1A_2\,MX_1 = \sphericalangle A_2A_3\,MX_2 = \cdots = \sphericalangle A_nA_1\,MX_n \\ &= \sphericalangle NA_1\,X_nX_1 = \sphericalangle NA_2\,X_1X_2 = \cdots = \sphericalangle NA_n\,X_{n-1}X_n,\end{aligned} \tag{1}$$

or in other words, the sides of the first polygon enclose with respective cevians through M of the second the same angle ϕ as well as the cevians through N of the first with the respective sides of the second. Then the ordered pair of polygons is called *entangled* with the points M and N as *entanglement points* and ϕ as the *entanglement angle* (see Figures 1 and 3).

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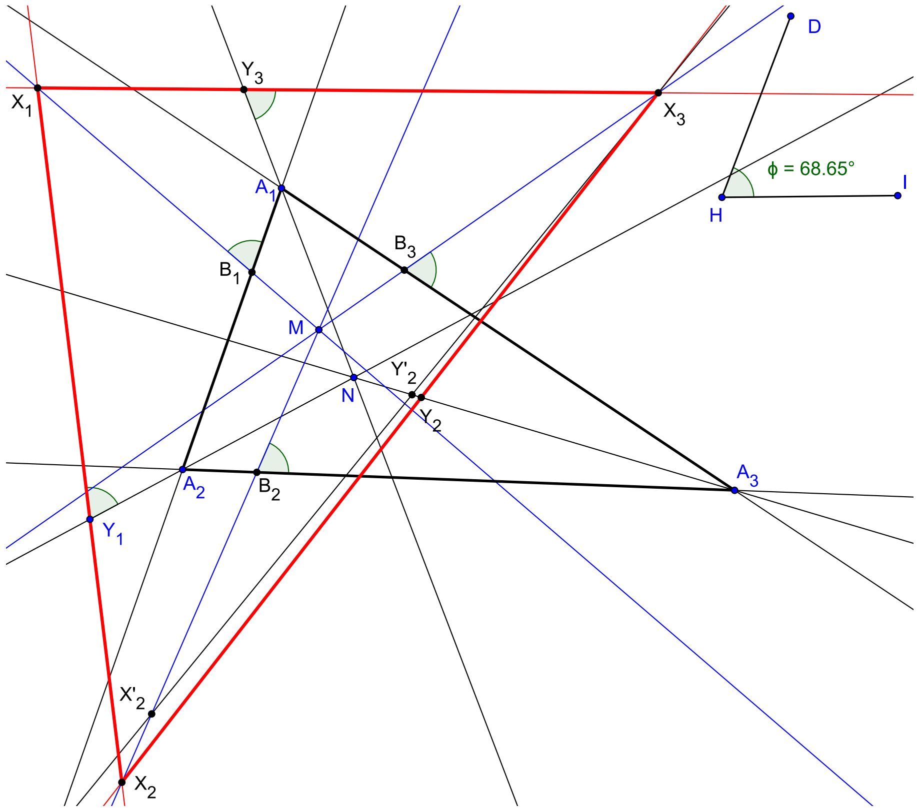

Figure 1: Entangled triangles $A_1A_2A_3$ and $X_1X_2X_3$ with entanglement points M, N and entanglement angle ϕ.

In the case $\phi = 90°$, which shows up at STEINER's orthologic triangles, the relation between the two polygons is symmetric. Otherwise the exchange of the two polygons means that the entanglement angle changes its sign.

The following lemma is a direct consequence of Definition 1.

Lemma 1. *The entanglement of polygons and the entanglement angle ϕ are preserved if the polygons with the respective points N and M are transformed independently by translations or central dilations. A rotation of the second polygon and the point M through the angle α preserves the entanglement but changes the entanglement angle to $\phi + \alpha$ modulo π.*

A non orientation-preserving motion of one polygon destroys the entanglement.

2. Main theorem

Theorem 1. *Let a polygon $A_1A_2 \ldots A_n$ together with an angle ϕ and two points M, N be given, where N lies outside the sides of the first polygon. Then there exists a polygon $X_1X_2 \ldots X_n$ such that the two polygons are entangled with entanglement points M, N and entanglement angle ϕ.*

For the proof, we recall the following lemma.

Lemma 2. *The measure of the signed angle between two lines g, h remains unchanged if the lines are replaced by two other lines g', h' where $\phi = \sphericalangle gg' = \sphericalangle hh'$.*
Conversely if two angles with $\sphericalangle gh = \sphericalangle g'h'$ are given, then $\phi = \sphericalangle gg' = \sphericalangle hh'$.

Proof. There is an orientation-preserving motion with $g \mapsto g'$ and $h \mapsto h'$ provided that g and h are not parallel. □

We split the proof and begin with triangles.

2.1. Entangled triangles

Proof. [Theorem 1 for $n = 3$] Let the triangle $A_1A_2A_3$ together with the points M, N and any angle ϕ with $0 \le \phi < \pi$ be given. The numbering of the triangle is anticlockwise. We show the existence of a second triangle $X_1X_2X_3$ by construction (see Figure 1).
We draw the lines through M which enclose with the respective sides A_1A_2, A_2A_3, A_3A_1 the given angle ϕ and denote there points of intersection with the respective sides with B_1, B_2, B_3. On MB_1 we choose a point $X_1 \neq M$. Through X_1 we draw the line which encloses with NA_2 at Y_1 the angle ϕ. It intersects MB_2 at the point X_2.
Then, we draw through X_1 the line which encloses with NA_1 at Y_3 the angle ϕ and intersect it with MB_3 at X_3. We proceed by drawing through X_3 the line which encloses with NA_3 at Y_2' the angle ϕ and denote with X_2' its point of intersection with MB_2. It remains to prove that $X_2 = X_2'$:
Now we check the angles in the triangles $A_1A_2A_3$, $X_1X_2X_3$ and $X_1X_2'X_3$ (going from Y_1 clockwise to $X_1 \to Y_3 \to X_3 \to Y_2' \to X_2'$) and conclude due to Lemma 2:

$$\begin{aligned}
\widehat{A}_{11} &:= \sphericalangle A_1A_2\, A_1N = \sphericalangle X_1M\, X_1X_3 =: \widehat{X}_{12}, & \widehat{A}_{12} &:= \sphericalangle A_1N\, A_1A_3 = \sphericalangle X_3X_1\, X_3M =: \widehat{X}_{31},\\
\widehat{A}_{21} &:= \sphericalangle A_2A_3\, A_2N = \sphericalangle X_2M\, X_2X_1 =: \widehat{X}_{22}, & \widehat{A}_{22} &:= \sphericalangle A_2N\, A_2A_1 = \sphericalangle X_1X_2\, X_1M =: \widehat{X}_{11},\\
\widehat{A}_{31} &:= \sphericalangle A_3A_1\, A_3N = \sphericalangle X_3M\, X_3X_2' =: \widehat{X}_{32}, & \widehat{A}_{32} &:= \sphericalangle A_3N\, A_3A_2 = \sphericalangle X_2'X_3\, X_2'M =: \widehat{X}_{21},
\end{aligned}$$

We notice that $A_{11} + A_{12} = \sphericalangle A_1A_2\, A_1A_3 \pmod{\pi}$, whether the cevian A_1N is disjoint to the interior of the triangle $A_3A_1A_2$ or not. Since similar equations hold for all other angles, we conclude

$$\widehat{A}_{11} + \widehat{A}_{12} + \widehat{A}_{21} + \widehat{A}_{22} + \widehat{A}_{31} + \widehat{A}_{32} = \widehat{X}_{11} + \widehat{X}_{12} + \widehat{X}_{21} + \widehat{X}_{22} + \widehat{X}_{31} + \widehat{X}_{32} = \pi \pmod{\pi}.$$

This implies

$$\sphericalangle X_2'X_3\, X_2'M = \sphericalangle X_2X_3\, X_2X_1 - \sphericalangle X_2M\, X_2X_1 = \sphericalangle X_2X_3\, X_2M \pmod{\pi},$$

and therefore $X_2' = X_2$.

H. Stachel proposed an alternative proof: Due to Lemma 1, it means no restriction of generality to specify $\phi = 0$. Then the cevians of one triangle are parallel to the sides of the other (Figure 2). For given $A_1A_2A_3$, M and N, the construction presented above yields, on the respective cevians through M, for a chosen X_1 the points X_2 and X_3. Since the points A_1, A_2, A_3, N are supposed as a quadrangle, we can apply Desargues' involution theorem: the points at infinity of the sides and the cevians yield three pairs of an involution. According to the construction of the second triangle so far, the corresponding involution is already fixed by the ideal points of the pairs $(MX_3,\, X_1X_2)$ and $(MX_2,\, X_1X_3)$ and thus identical with the first one. Therefore, also the ideal points of MX_1 and X_2X_3 belong to this involution, which proves that X_2X_3 must be parallel to the cevian NA_1. □

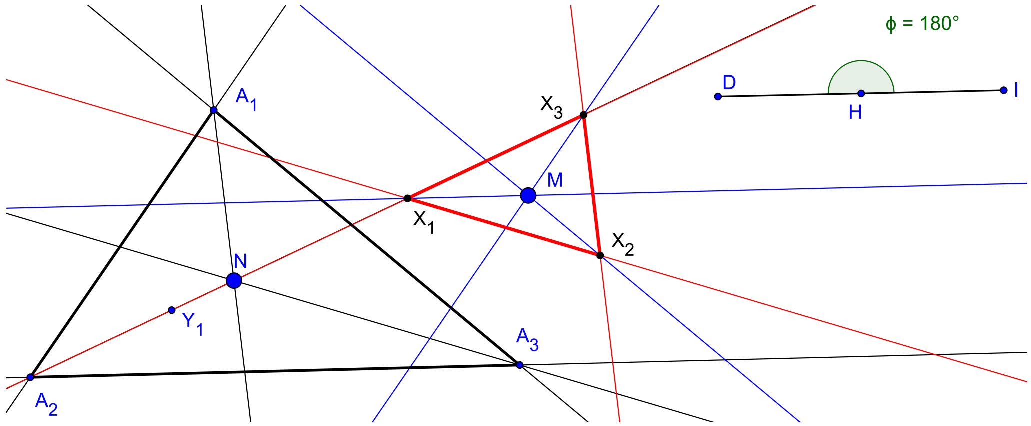

Figure 2: Entangled triangles for $\phi = 0 \pmod{\pi}$.

2.2. Entangled polygons

Now we try to apply the previous construction in a case where an n-gon $A_1A_2\ldots A_n$ is given together with two points M, N and an angle ϕ (Figure 3). The point N lies outside the sides of $A_1A_2\ldots A_n$.

Proof. [Theorem 1 for $n > 3$] The sides A_1A_2, A_2A_3, ... define the cevians MX_1, MX_2, ... of the requested n-gon $X_1X_2\ldots X_n$ satisfying (1). In the same way as for triangles, we specify any point X_1 on the respective cevian and draw X_1X_2 enclosing ϕ with NA_2. Then we go the other way round and draw X_nX_1 with $\phi = \measuredangle NA_1\,X_nX_1$, then $X_{n-1}X_n$ with $\phi = \measuredangle NA_n\,X_{n-1}X_n$ and so on, until side $X_2'X_3$ with $\phi = \measuredangle NA_3\,X_2'X_3$. This yields the following equalities of angles:

$$
\begin{array}{lclclcl}
\widehat{A}_{11} & := & \measuredangle A_1A_2\,A_1N & = & \measuredangle X_1M\,X_1X_n & =: & \widehat{X}_{12}, \\
\widehat{A}_{12} & := & \measuredangle A_1N\,A_1A_n & = & \measuredangle X_nX_1\,X_nM & =: & \widehat{X}_{n1}, \\
\widehat{A}_{21} & := & \measuredangle A_2A_3\,A_2N & = & \measuredangle X_2M\,X_2X_1 & =: & \widehat{X}_{22}, \\
\widehat{A}_{22} & := & \measuredangle A_2N\,A_2A_1 & = & \measuredangle X_1X_2\,X_1M & =: & \widehat{X}_{11}, \\
\widehat{A}_{31} & := & \measuredangle A_3A_2\,A_3N & = & \measuredangle X_3M\,X_3X_2 & =: & \widehat{X}_{32}, \\
\widehat{A}_{32} & := & \measuredangle A_3N\,A_3A_2 & = & \measuredangle X_2'X_3\,X_2'M & =: & \widehat{X}_{21}, \\
 & & \cdots & & \cdots & & \\
\widehat{A}_{n-1\,1} & := & \measuredangle A_{n-1}A_n\,A_{n-1}N & = & \measuredangle X_{n-1}M\,X_{n-1}X_{n-2} & =: & \widehat{X}_{n-1\,2}, \\
\widehat{A}_{n-1\,2} & := & \measuredangle A_{n-1}N\,A_{n-1}A_{n-2} & = & \measuredangle X_{n-2}X_{n-1}\,X_{n-1}M & =: & \widehat{X}_{n-2\,1}, \\
\widehat{A}_{n1} & := & \measuredangle A_nA_1\,A_nN & = & \measuredangle X_nM\,X_nX_{n-1} & =: & \widehat{X}_{n2}, \\
\widehat{A}_{n2} & := & \measuredangle A_nN\,A_nA_{n-1} & = & \measuredangle X_{n-1}X_n\,X_{n-1}M & =: & \widehat{X}_{n-1\,1}.
\end{array}
$$

We proceed like in the proof for triangles: In the system of equations above, the sum of the angles on the left-hand side equals $(n-2)\pi \pmod{\pi}$ and also the sum of the angles on the right-hand side, which leads to the conclusion $X_2' = X_2$, and thus, to the existence on the n-gon $X_1X_2\ldots X_n$. □

Figure 3 shows the case of entangled pentagons, with the special characteristics that $A_1A_2A_3A_4A_5$ is cyclic and M is on the circumcircle, in order to illustrate Theorem 2. In the coming section, some of the properties of entangled polygons are presented. In Section 3,

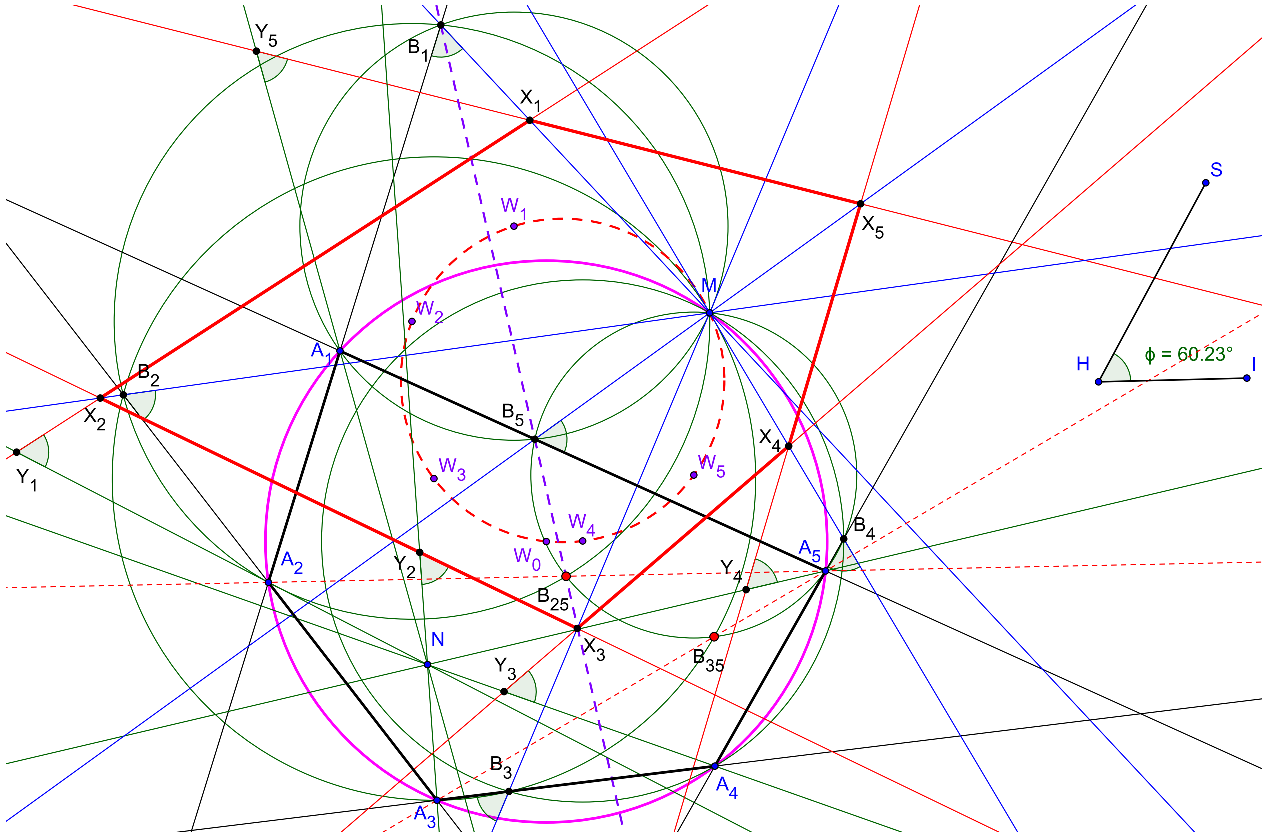

Figure 3: Entangled pentagons $A_1A_2\dots A_5$ and $X_1X_2\dots X_5$ with entanglement angle ϕ.

we examine the extension of the Simson line related to a combination of concave and cyclic entangled polygons and $M = N$ and on the relevant polygon circle.

3. Some properties of entangled polygons

In Figure 1 two entangles triangles are displayed with an entanglement angle $\phi \neq 0 \pmod{\pi}$. Here, M is the *Miquel point* of the triangle $A_1A_2A_3$ with respect to the points B_1, B_2, B_3 on the sides, and N is the Miquel point of $X_1X_2X_3$ with respect to Y_1, Y_2, Y_3 (see [1, 5]).

If we place M on the circumcircle of $A_1A_2A_3$, as stated in [5], then the centers W_0, W_1, W_2, W_3 of the four circumcircles (the first related to the triangle and the 3 others, called Miquel circles, are related to the three subtriangles) lie on a circle which passes through M as well. The points B_1, B_2, B_3 are collinear and form the *Simson line* of $A_1A_2A_3$ in a generalized sense, since the lines from point M form congruent angles equal to ϕ with the sides of the triangle (see [6]).

The whole concept of a Miquel point on the circumcircle and the relevant Miquel circumcircles centres of a given triangle, as stated in [1, 5], can be extended to cyclic n-gons. A method for the first part of this extension can be found in [7] where it is stated that the author could not find any reference for such a generalization.

For the second part (Miquel circumcircles), the proof is as follows: In Figure 3, we have a cyclic pentagon $A_1A_2A_3A_4A_5$ with centre W_0. The point M is the Miquel point with the property that

$$\phi = \sphericalangle A_1A_2\, MB_1 = \sphericalangle A_2A_3\, MB_2 = \cdots = \sphericalangle A_5A_1\, MB_5.$$

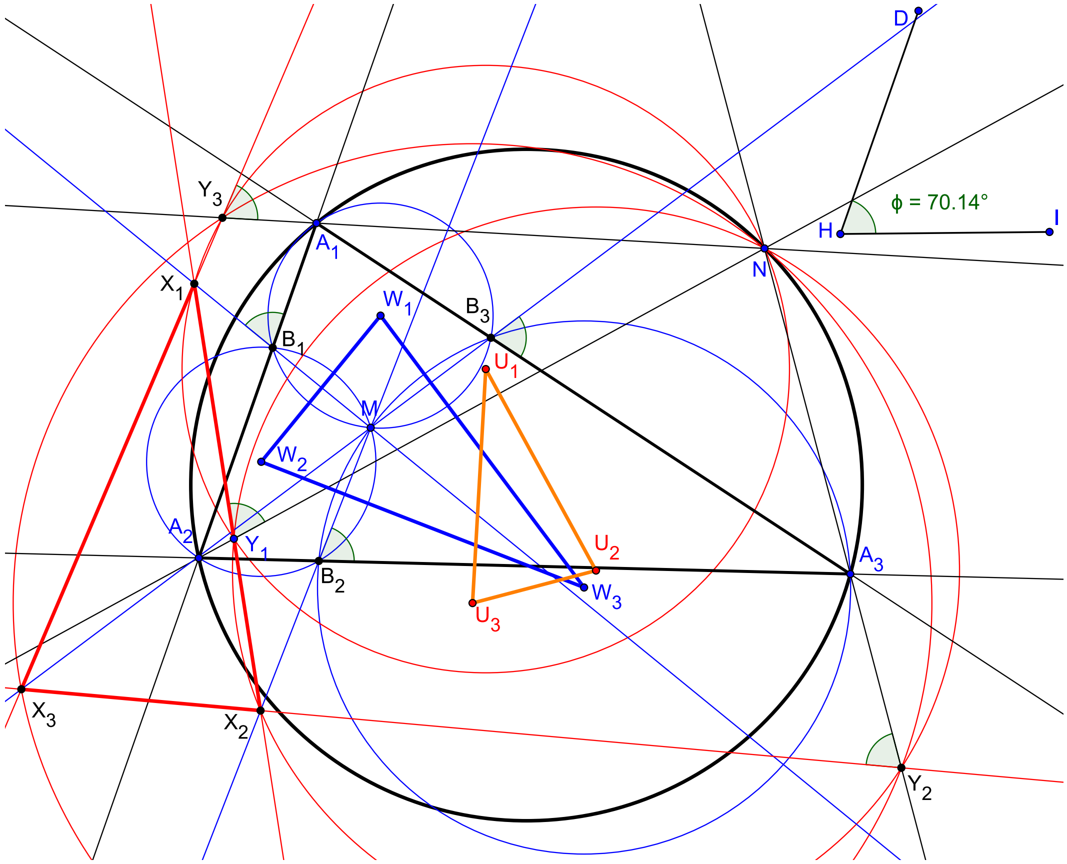

Figure 4: For the given triangle $A_1A_2A_3$, the point M is the Miquel point w.r.t. B_1, B_2, B_3, and W_1, W_2, W_3 are the centers of the Miquel circles. The points N and $U_0, \dots, U_3$ play the analogue role for $X_1X_2X_3$.

The points W_1, W_2, $\dots$, W_5 are the centers of the five Miquel circles related to the five pentagon's vertices.

The case of the triangle $A_1A_2A_5$ with point M is equivalent to that of the triangle $A_1A_2A_3$ and M on its circumcircle (Figure 4), as mentioned in the previous paragraph. Therefore the points A_2, B_{25} and A_5 are collinear (where B_{25} is the intersection of the circumcircles with respective centres W_2 and W_5) and W_0, W_1, W_2, W_5, and M are cyclic. The line passing through B_1, B_5 and B_{25} is the Simson line of the triangle $A_1A_2A_5$, and M, B_5, B_{25}, B_{35}, A_5, and B_4 are cyclic for obvious reasons based on $A_2A_3A_5$ and $A_3A_4A_5$. Similarly follows from the triangle $A_2A_3A_5$ that M, A_5, B_{35} and A_3 are collinear, and W_3 belongs to the circle mentioned before. From $A_3A_4A_5$ and point M follows that also W_4 lies on this circle.

Additionally, $A_1A_2 \dots A_5 \sim W_1W_2 \dots W_5$ holds also when $A_1A_2 \dots A_n$ is not cyclic (Figure 5). If angle ϕ becomes $\pi/2$, then $\sphericalangle A_1B_1\,B_1M$ becomes $\pi/2$, too, and the distance $\overline{W_1W_2} = 0.5 \cdot \overline{A_1A_2}$. Similarly we handle the remaining sides W_iW_{i+1} and A_iA_{i+1} in order to prove that $A_1A_2 \dots A_5 \sim W_1W_2 \dots W_5$ for any value of ϕ, since the shape is preserved when ϕ varies.

Following the same approach, a chain of proofs can be easily made for n-sided polygons, starting with the first triangle consisting of the first two vertices A_1, A_2 and the last vertex A_n, moving on to a series of triangles keeping the last vertex constant and changing one vertex, such as $A_2A_3A_n$, $A_3A_4A_n$, $\dots$, $A_{n-2}A_{n-1}A_n$ and applying the method described above for each triangle of the chain.

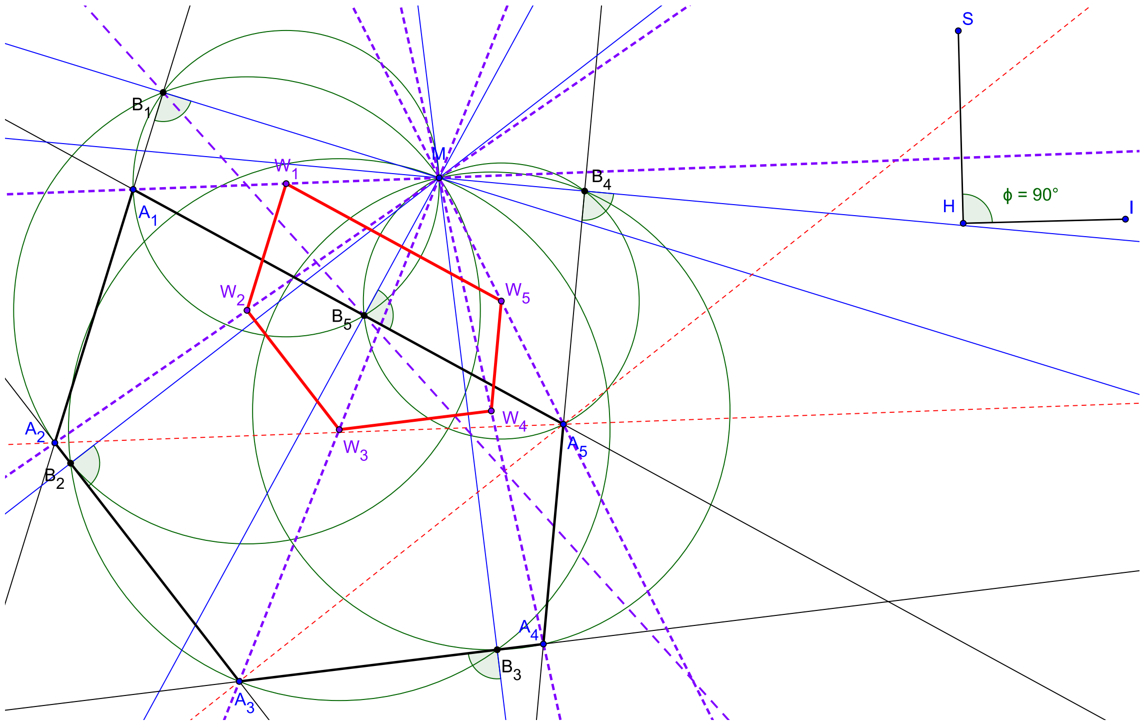

Figure 5: Also for a non-cyclic pentagon $A_1A_2 \dots A_5$ we obtain a similarity $A_1A_2 \dots A_5 \sim W_1W_2 \dots W_5$.

From all the above, we can obtain the following theorem.

Theorem 2. *Let $A_1A_2 \dots A_n$ be a cyclic n-gon and M a relevant Miquel point M on the circumcircle, such that the lines MB_1, MB_2, ..., MB_n enclose a given angle $\phi \neq 0 \pmod{\pi}$ with the n sides A_1A_2, A_2A_3, ..., A_5A_1 of the given polygon at B_1, B_2, ..., B_n. The point M and the centres of the circumcircle and the n Miquel circles (= circumcircles of the quadrangles $A_1B_1B_nM$, $A_2B_2B_1M$, ..., $A_nB_nB_{n-1}M$) lie on a circle, while the n centres of the Miquel circles form a polygon similar to the given one. The similarity of the two polygons is also valid when the polygon $A_1A_2 \dots A_n$ is not cyclic.*

In entangled triangles and n-gons, there are four particular cases related to the point N:

Case 1. Point N on the circumcircle of $A_1A_2A_3$ (Figure 4). From Lemma 2 we get:

$$\begin{aligned} \measuredangle X_1X_3\,X_1X_2 &= \measuredangle NY_3\,NY_1 = \measuredangle NA_1\,NA_2 = \measuredangle A_3A_1\,A_3A_2, \\ \measuredangle X_2X_1\,X_2X_3 &= \measuredangle NY_1\,NY_2 = \measuredangle NA_2\,NA_3 = \measuredangle A_1A_2\,A_1A_3, \end{aligned}$$

and therefore $X_1X_2X_3 \sim A_1A_2A_3$. The numbering of the triangle $X_1X_2X_3$ is clockwise (note Lemma 5, which is valid only for $n = 3$).

Case 2. In the case $n = 3$ and $N = M$, the points where the sides enclose the fixed angle $\phi \neq 0 \pmod{\pi}$ with the respective cevians, form similiar triangles $B_1B_2B_3 \sim Y_2Y_3Y_1$ (Figure 6). This holds only for $n = 3$. For a proof, see the Appendix.

Case 3. If N is the orthocentre of $A_1A_2A_3$ (see Figure 6), whether $N = M$ or $N \neq M$, we have the following: Since $A_1A_3 \perp NA_2$ and $A_2A_3 \perp NA_1$ we have

$$\widehat{A}_3 = \measuredangle A_3A_1\,A_3A_2 = \measuredangle NA_2\,NA_1 = \measuredangle X_1X_2\,X_1X_3 = \widehat{X}_1.$$

Similarly, we get $\widehat{A}_1 = \measuredangle A_1A_2\,A_1A_3 = \measuredangle X_2X_3\,X_2X_1 = \widehat{X}_2$, so that $A_1A_2A_3 \sim X_2X_3X_1$. Since by (1) $\measuredangle A_2A_3\,MX_2 = \phi = \measuredangle NA_1\,X_3X_1$, the orthogonality $NA_1 \perp A_2A_3$ implies $X_3X_1 \perp MX_2$ etc., so that M is the orthocentre of $X_1X_2X_3$ as well (only for $n = 3$).[1]

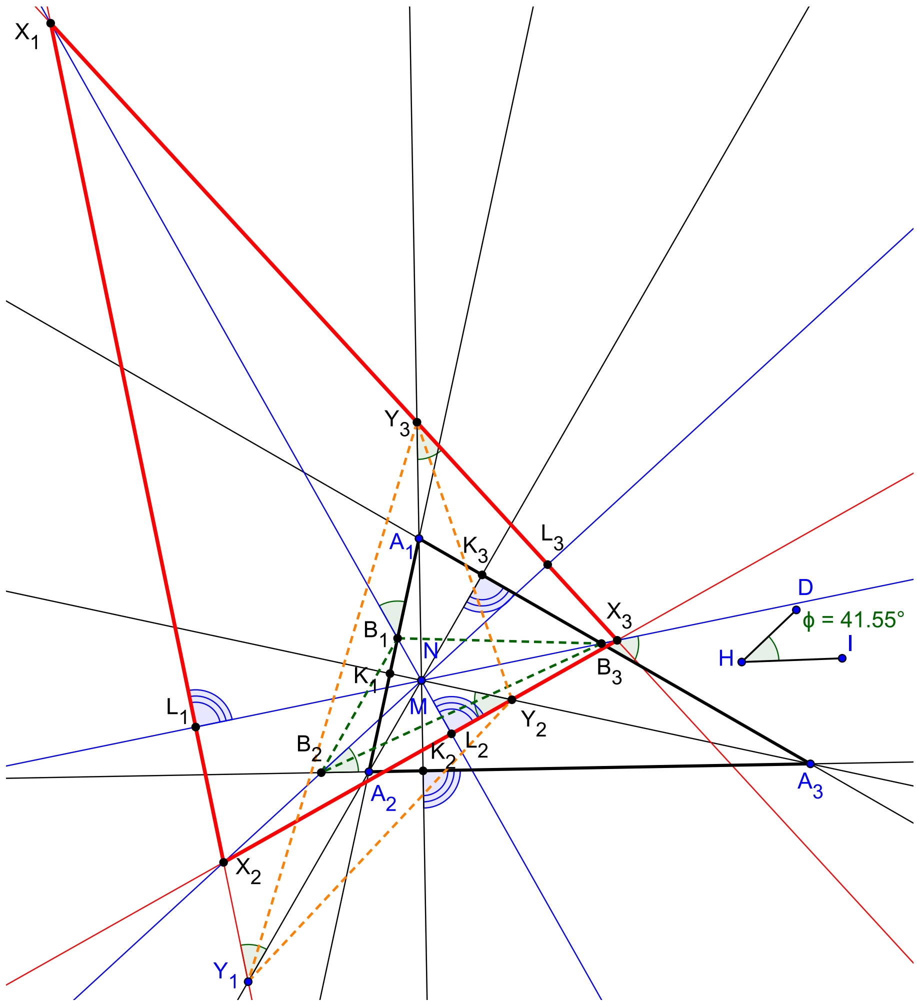

Figure 6: Case 2: If N is the orthocenter of $A_1A_2A_3$, then the triangles $A_1A_2A_3$ and $X_2X_3X_1$ are similar and M is the orthocenter $X_1X_2X_3$.

Case 4. If the triangle $A_1A_2A_3$ is isosceles with $\widehat{A}_1 = \widehat{A}_2$, then the height through A_3 is an axis of symmetry. Suppose that N is placed on this axis. Then we have

$$\widehat{A}_{12} := \measuredangle A_1N\,A_1A_3 = -\measuredangle A_2N\,A_2A_3 = \widehat{A}_{21} \text{ and } \widehat{A}_{31} = -\widehat{A}_{32}.$$

Since $\widehat{A}_{31} + \widehat{A}_{21} = \widehat{X}_2$ and $\widehat{A}_{12} + \widehat{A}_{31} = \widehat{X}_3$, we get $\widehat{X}_2 = \widehat{X}_3$. Therefore, also the triangle $X_1X_2X_3$ is isosceles.
This property can be generalized to entangled n-gons: When the given polygon has an axis of symmetry on which N is placed, then the other polygon is also symmetric with respect to an axis passing through M. For the proof choose $\phi = 0°$.

Case 5. If Y_1 is kept fixed while the angle ϕ varies, then we notice in the case $N = M$, that the triangle $X_1X_2X_3$ rotates about $N = M$ (Figure 6) and either reduces or increases

[1] The latter follows also from the fact that the Desargues involution induced by the quadrangle $A_1A_2A_3N$ on the line at infinity is the right-angle involution.

its size (it coincides with M when $\phi = 0$), while also the vertices Y_2 and Y_3 are fixed (the angles $\sphericalangle Y_1 N Y_1 Y_i$ and $Y_i N Y_i Y_1$ remain constant, as in the Appendix) and all the above can be extended in exactly the same way for n-gons. If $N \neq M$, then all angles $\sphericalangle Y_1 N Y_1 Y_i$ and $\sphericalangle Y_i N Y_i Y_1$ vary (contrary to the case of the Appendix), so apart from Y_1 all Y_i change their positions along NA_i, while the n-gon $X_1 X_2 \ldots X_n$ keeps its shape and reduces or increases its sidelengths (it coincides with M when $\sphericalangle Y_1 M Y_1 N = \phi$).

From the above, we obtain the following two Lemmas.

Lemma 3. *For two entangled triangles $A_1A_2A_3$ and $X_1X_2X_3$ with $\phi \neq 0 \ (mod\,\pi)$, the following holds:*

- *If N is on the circumcircle of $A_1A_2A_3$ (Figure 4), we get similar triangles $A_1A_2A_3 \sim X_2X_3X_1$ and a clockwise numbering of $X_1X_2X_3$ (Lemma 5).*
- *If $N = M$, the triangles formed by pairs of lines which enclose the entanglement angle ϕ are similar (Figure 6), i.e., $B_1B_2B_3 \sim Y_2Y_3Y_1$).*
- *If N is the orthocentre of $A_1A_2A_3$, then M is the orthocentre of $X_1X_2X_3$ as well and $A_1A_2A_3 \sim X_2X_3X_1$.*

Lemma 4. *For two entangled n-gons $A_1A_2 \ldots A_n$ and $X_1X_2 \ldots X_n$, $n \geq 3$ with $\phi \neq 0 \ (mod\,\pi)$, the following holds:*

- *If we vary ϕ, the second polygon performs a stretch-rotation about M. If we fix Y_1, then in the case $N = M$ all points $Y_1, Y_2, \ldots, Y_n$ can remain fixed, while under $N \neq M$ all other Y_i change their positions on the cevians NA_i. They can even coincide with M when $\sphericalangle Y_1 M Y_1 N = \phi$.*
- *If one polygon has an axis of symmetry on which N is placed, then the other polygon is symmetric as well with respect to an axis passing through M.*

4. Generalized Simson line at entangled polygons

Theorem 3. *Let two entangled polygons $A_1A_2 \ldots A_n$ and $X_1X_2 \ldots X_n$ be given, where the first is cyclic and $M = N$ lies on its circumference. Then,*

a) *the polygon $X_1X_2 \ldots X_n$ is concave at one of its vertices,*

b) *the points $Y_1, Y_2, \ldots, Y_n$ are collinear, forming the Entangled Polygons Simson Line (EPSL),*

c) *the lines spanned by the sides of $X_1X_2 \ldots X_n$ define triangles which have collinear orthocentres.*

Proof. [Theorem 3a)] If the first polygon is convex, then each side A_iA_{i+1} defines two half-planes. The one which contains the remaining vertices is called *in-half-plane* of A_iA_{i+1} and the other the *ex-half-plane.* If N is inside the convex polygon, then it belongs to all in-half-planes of the sides of the polygon. Taking the example of entangled pentagons in Figure 3, as stated in Section 1.2, we notice that

$$\widehat{X}_5 = \widehat{X}_{51} + \widehat{X}_{52} = \widehat{A}_{52} + \widehat{A}_{12} = \sphericalangle A_5 N A_5 A_1 + \sphericalangle A_1 N A_5 A_1.$$

Therefore, as the interior point N approaches the segment A_5A_1, we obtain $\widehat{X}_5 \to 0$. If N moves out of the polygon, as in Figure 7, and more specifically to the ex-half-plane[2] of A_5A_1,

[2] since N lies on the circumcircle of the first polygon, it can belong only to the ex-half-plane of one side.

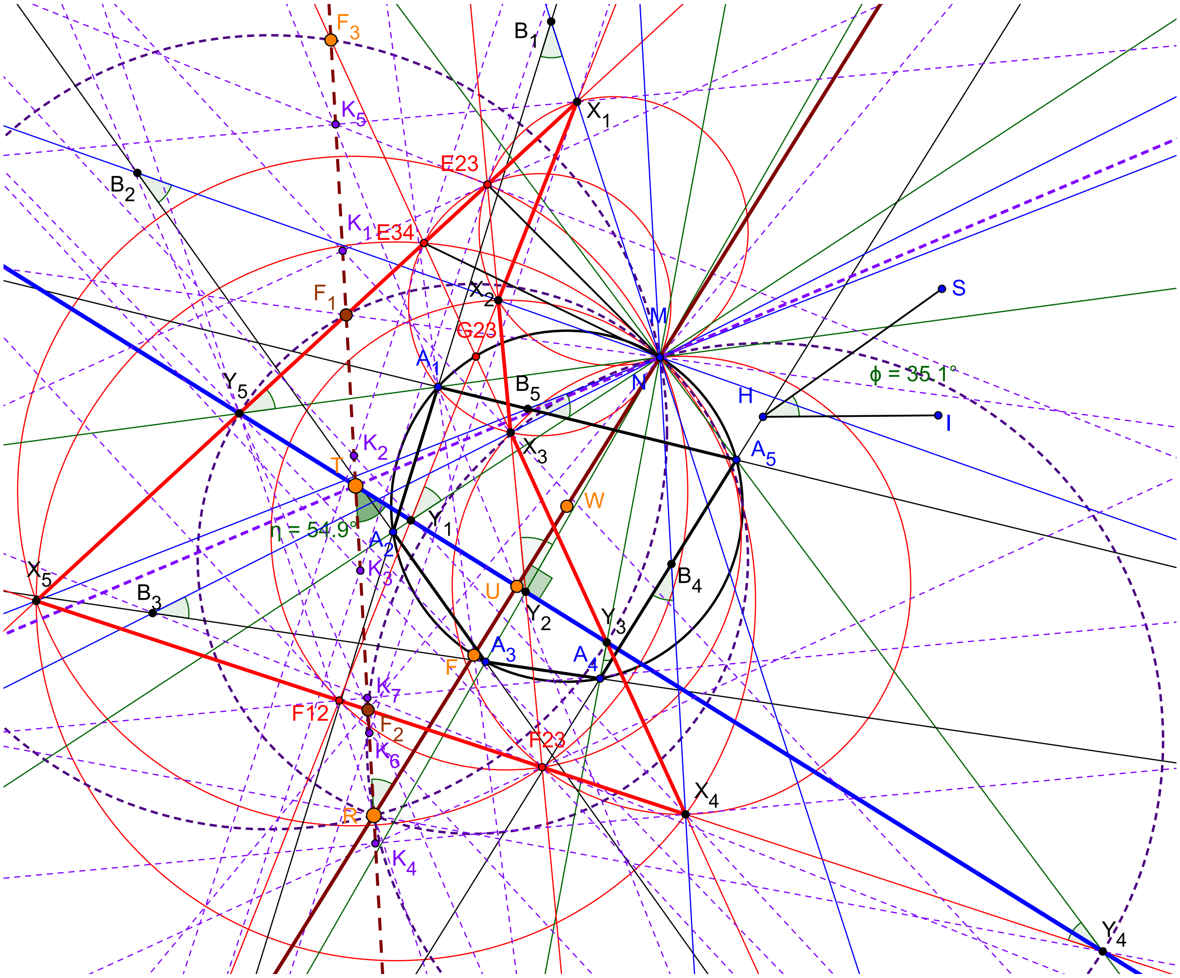

Figure 7: Two entangled pentagons where $A_1A_2\dots A_5$ is cyclic with $M = N$ placed on the circumcircle. The points $Y_1,\dots,Y_5$ are located on the generalized Simson line.

the angles $\widehat{A}_{52}$ and $\widehat{A}_{12}$ acquire a clockwise rotation and, as a result, the angles $\widehat{X}_4$, $\widehat{X}_5$, $\widehat{X}_1$, and vertex X_5 have clockwise rotation too. Due to the vertex X_5, we get $X_3 \to X_4 \to X_5$, $X_4 \to X_5 \to X_1$, and $X_5 \to X_1 \to X_2$ have a clockwise rotations. The other angles $\widehat{X}_2$ and $\widehat{X}_3$ remain anticlockwise, so $X_1X_2X_3X_4X_5$ becomes concave due to vertex X_5. If $n = 3$ (Figure 4), the numbering of the vertices in the second triangle becomes clockwise. □

From the above we can generalize and obtain the following Lemma.

Lemma 5. *Given two entangled n-gons $A_1A_2\dots A_n$ and $X_1X_2\dots X_n$. If the first is convex with N in the interior, then the second is convex as well with M in the interior. If N moves out of the first polygon into the ex-half-plane of A_iA_{i+1}, then $X_1X_2\dots X_n$ becomes concave (for $n > 3$) at the vertex $X_i \in MB_i$ with a clockwise rotation.*

In Figure 7 with $M = N$ on the circumcircle of the first pentagon, the pentagon $X_1X_2\dots X_5$ becomes concave at the vertex X_5. So, the sides X_1X_2 and X_2X_3 intersect the side X_5X_4 at points $F_{1,2}$ and $F_{2,3}$, respectively, and the sides X_2X_3 and X_3X_4 intersect the side X_5X_1 at $E_{2,3}$ and $E_{3,4}$. When we have two entangled hexagons and, similarly to Figure 7, the hexagon $X_1X_2\dots X_6$ becomes concave at the vertex X_5, we have on the side X_5X_4 the points $F_{2,3}$, $F_{1,2}$ and $F_{6,1}$, and on the side X_5X_6 the points $E_{1,2}$, $E_{2,3}$ and $E_{3,4}$.

Table 1: Types of triangles in the case of the pentagon in Figure 7.

Triangle	*Type*	*Simson line*	*Triangle*	*Type*	*Simson line*
1. $\triangle E_{2,3}X_1X_2$	A	Y_5, Y_2, Y_1	2. $\triangle F_{2,3}X_4X_3$	A	Y_4, Y_3, Y_2
3. $\triangle E_{2,3}E_{3,4}X_3$	C	Y_5, Y_3, Y_2	4. $\triangle F_{2,3}F_{1,2}X_2$	C	Y_4, Y_1, Y_2
5. $\triangle E_{3,4}X_5X_4$	B	Y_5, Y_4, Y_3	6. $\triangle F_{1,2}X_1X_5$	B	Y_1, Y_5, Y_4
7. $\triangle E_{2,3}X_5F_{2,3}$	D	Y_5, Y_4, Y_2			

Going from pentagons to hexagons increases the points on the two sides connected to the concave vertex by one per side. Following this reasoning, in the concave quadrangle case we get just one point per side connected with the concave vertex. The above gives rise to the following Remark:

Remark 1. Given two entangled n-gons, where due to Theorem 3 $X_1X_2 \dots X_n$ is concave at X_i, then we have $n-3$ intersection points on each of the sides $X_{i-1}X_i$ and X_iX_{i+1} with the remaining ones, as shown in Figure 7.

In Figure 7, the number of triangles formed by points of intersection between sides of $X_1X_2 \dots X_5$ is seven, following the rule that in each triangle at least one of the sides X_5X_4 and X_5X_1 must be involved. This allows to choose pairs of triangles with two of their sides on two common lines, thus sharing the Simson line. There are always four types of triangles obeying the above rule for entangled n-gons:

- *Type A,* with one point of intersection on one of the two sides connecting the concave vertex of the polygon but not the concave vertex ($\triangle E_{2,3}X_1X_2$ and $\triangle F_{2,3}X_4X_3$)
- *Type B,* with one point of intersection on one of the two sides connecting the concave vertex of the polygon and the concave vertex ($\triangle E_{3,4}X_5X_4$ and $\triangle F_{1,2}X_1X_5$)
- *Type C,* with two points of intersection on one of the two sides connecting the concave vertex but not the concave vertex ($\triangle E_{2,3}E_{3,4}X_3$ and $\triangle F_{2,3}F_{1,2}X_2$)
- *Type D,* with two points of intersection, one on each of the two sides connecting the concave vertex and the concave vertex ($\triangle E_{2,3}X_5F_{2,3}$)

Due to this generalization, only $n-2$ triangles are needed at any application of the chain of proofs for entangled n polygons and the extended Simson line. It is simpler to use only triangles of type A, B, C, D (Table 1 and Figure 7), because of use of angles equalities related to quadrangles which have been proven cyclic in previous steps of the chain of proof (case g) and also because of exactly similar approach in the proof of equivalent types of triangles (cases a,b and c,d).

Let the number of these triangles be N_t, depending on the number of sides of the entangled n-gon $X_1X_2 \dots X_n$, which is concave at X_i, according to the rule that all sides of each triangle belongs to a side or extension of a side of the polygon from which one or two are X_iX_{i+1} or $X_{i-1}X_i$ line, with or without the concave vertex X_i. In the case of pentagons (Figure 7) we have $N_t = 7$ (2 Type A, 2 Type B, 2 Type C, 1 Type D), in the case of hexagons N_t is equal to 10 (2 A, 2 B, 4 C, 2 D) and it is obvious that for quadrangles holds $N_t = 4$ (2 A, 2 B); for heptagons holds $N_t = 13$ (2 A, 2 B, 6 C, 3 D). Since we have $n-3$ intersection points on each of the two sides connecting the concave vertex with the other sides (Remark 1), we deduce for the numbers of triangles belonging to types A, B, C, D that

$$N_t = 4 + 3(n-4), \quad N_A = 2,\ N_B = 2,\ N_C = 2(n-4),\ N_D = n-4. \tag{2}$$

Proof. [Theorem 3b)] We could study all the triangles formed by the sides or extensions of sides of the concave pentagon, but this selection of triangles makes the proof of the extension of the Simson theorem. The collinearity of the points Y_1, Y_2, Y_3, Y_4, Y_5 can be proven as follows:

Case a. If we prove that the triangle $E_{34}X_5X_4$ (Type B triangle 5 in Table 1) together with $M = N$ form a cyclic quadrangle, then Y_5, Y_4, Y_3 are collinear because $Y_5Y_4Y_3$ is the Simson line of the triangle, so :

• $E_{3,4}Y_5Y_3M$ is cyclic since $\measuredangle E_{34}X_5\, E_{34}X_4 = \measuredangle NY_5\, NY_3$ by Lemma 2
• $\widehat{A}_5 = \measuredangle NY_5\, NY_3$ since $A_1A_2 \ldots A_5$ is cyclic and N is on the circumcircle,
• $\widehat{A}_5 = \measuredangle MX_5\, MX_4$ by Lemma 2,

Thus $\measuredangle E_{34}X_5\, E_{34}X_4 = \measuredangle MX_5\, MX_4$, so the quadrangle $E_{34}X_5X_4M$ is cyclic and the points Y_5, Y_4, and Y_3 collinear.

Case b. In exactly the same way, we can prove that the quadrangle $F_{12}X_1X_5M$ (Type B triangle 6 of Table 1) is cyclic and Y_1, Y_5, Y_4 are collinear. The triangles $E_{34}X_5X_4$ and $F_{12}X_1X_5$ are equivalent in view of the proof of collinearity of their Simson lines.

Case c. If we prove that the quadrangle $F_{23}X_4MX_3$ (Type A triangle 2 of Table 1) is cyclic, then Y_4, Y_3, and Y_2 are collinear. From angle relations of entangled n-gons in Section 1.2 and Figure 7 we have:

• $\measuredangle X_3M\, X_3E_{23} = \measuredangle X_3M\, X_3X_2 = \widehat{X}_{32} = \widehat{A}_{31} = \measuredangle A_3A_4\, A_3N$,
• $\measuredangle X_4M\, X_4X_5 = \widehat{X}_{41} = \pi - \widehat{A}_{52} = \pi - \measuredangle A_5N\, A_5A_4$ ($\widehat{X}_4$ has acquired clockwise orientation, the quadrangle $A_5B_4X_4Y$ is cyclic, since $\measuredangle A_4B_4\, B_4X_4 = \measuredangle A_5Y_4\, Y_4X_4$),
• the quadrangle $A_3A_4A_5N$ is cyclic,

thus $\measuredangle X_3M\, X_3E_{23} = \measuredangle X_4M\, X_4X_5$, so the quadrangle $F_{23}X_4MX_3$ is cyclic and Y_4, Y_3, and Y_2 are collinear.

From the three cases above we deduce that the points Y_1, Y_2, Y_3, Y_4, and Y_5 are collinear. Also other combinations of triangles can be used for this proof, such as:

Case d. As in case c, since the quadrangle $E_{23}X_2MX_1$ (Type A triangle 1 of Table 1) is cyclic, Y_5, Y_2, and Y_1 are collinear (the triangles $E_{23}X_2X_1$ and $F_{23}X_3X4$ are equivalent).

Case e. In order to prove that the quadrangle $E_{23}E_{34}X_3M$ (Type C triangle 3 of Table 1) is cyclic, we have: the quadrangle $E_{34}X_5X_4M$ is cyclic (case a), so $\measuredangle X_4M\, X_4E_{34} = \measuredangle X_5M\, X_5E_{34}$ and $\measuredangle E_{34}M\, E_{34}E_{23} = \measuredangle X_4M\, X_4X_5$. Since the quadrangle $F_{23}X_4MX_3$ is cyclic (case c), we get: $\measuredangle E_{34}M\, E_{34}E_{23} = \measuredangle X_3M\, X_3E_{23}$. Thus, the quadrangle $E_{23}E_{34}X_3M$ is cyclic and Y_5, Y_3, Y_2 are collinear.

Case f. As in case e, since the quadrangle $F_{12}F_{23}MX_2$ (Type C triangle 4 of Table 1) is cyclic, Y_4, Y_1, and Y_2 are collinear (the triangles $E_{23}E_{34}X_3$ and $F_{12}F_{23}X_2$ are equivalent).

Case g. In order to prove that the quadrangle $F_{23}X_5E_{23}M$ (Type D triangle 7 of Table 1) is cyclic, we have: $\measuredangle F_{23}E_{23}\, E_{23}M = \measuredangle X_3E_{23}\, E_{23}M = \measuredangle X_3E_{34}\, E_{34}M = \measuredangle X_4E_{34}\, E_{34}M = \measuredangle X_4X_5\, X_5M = \measuredangle F_{23}X_5\, X_5M$ (as shown previously, the quadrangles $E_{23}E_{34}X_3M$ and $E_{34}X_5X_4M$ are cyclic), therefore the quadrangle $F_{23}X_5E_{23}M$ is cyclic and Y_5, Y_4, and Y_2 are collinear.

In order to prove that the five points $Y_1, Y_2, \ldots, Y_5$ of the concave pentagon are collinear, three triangles out of seven of Table 1 are needed. Also, in the case of entangled hexagons, exactly the same process can be used by proving that four triangles of types A, B, C, and D form cyclic quadrangles with the points $M = N$, thus showing that this process holds for any number of vertices.

The above method of proof can be extended using exactly the same process of chain proofs for two entangled polygons $A_1A_2 \dots A_n$ and $X_1X_2 \dots A_n$, where the first is cyclic, and $M = N$ on the circumcircle of the first polygon in order to prove that the points $Y_1, Y_2, \dots, Y_n$ are always collinear. This is because the proof that each triangle belonging to one of the four types of triangles stated above and the point $M = N$ form a cyclic quadrangle does not depend on the number of vertices of the entangled polygons, but on the following types of relations, valid for entangled n-gons:

- Equality of angles $\widehat{A}_{kl} = \widehat{X}_{ij}$ such as those in Section 1.3
- Equality of angles $\sphericalangle A_{i-1}A_i\, A_iA_{i+1} = \pi - \sphericalangle A_{i+1}M\, MA_{i-1} = \pi - \sphericalangle A_{i+1}N\, NA_{i-1}$, since $A_1A_2 \dots A_n$ is cyclic and the point $M = N$ is placed on the circumcircle.
- Angle equalities of cyclic quadrangles as found in previous steps of the proofs.
- Equivalence of types of triangles (triangles 1 and 2, 3 and 4, 5 and 6 of Table 1). □

Proof. [Theorem 3c)] As can be seen in Figure 7, $M = N$ is Miquel point ([3, Theorem 3.34, p. 62]), for the circumcircles of the triangles $E_{23}E_{34}X_3$, $E_{34}X_5X_4$, $X_3F_{23}X_4$, $F_{23}X_5E_{23}$ created by the intersections of four lines: $E_{23}X_5$, $E_{23}F_{23}$, $E_{34}X_4$, and X_5X_4, forming a complete quadrilateral. Also the triangles $E_{23}E_{34}X_3$, $E_{34}X_5X_4$, etc. have common Simson line passing through the points Y_2, Y_3, Y_4, Y_5 which are collinear (taking two appropriate triangles which have two sides on the same lines, hance the two triangles have the same Simson line).

Let us assume that just the above mentioned four lines and angle ϕ were initially given and N was defined as above, creating a concave quadrangle (such as $E_{23}X_3X_4X_5$ of Figure 7, ignoring the displayed pentagon $A_1A_2A_3A_4A_5$). We construct the Simson line $Y_2Y_3Y_4Y_5$ for ϕ, of the triangles formed by the intersections of the given lines. Let us construct a fifth line which creates triangles by intersecting the other lines. We choose X_1 on $E_{23}X_5$ and F_{12} on X_5X_4. So, the line X_1F_{12} intersects the Simson line at the point Y_1 (it also intersects $E_{23}F_{23}$ at the point X_2). If we keep X_1 fixed while F_{12} moves along X_5X_4, then the angle $\sphericalangle Y_1N\, Y_1X_1$ can vary from 0 to π until $\sphericalangle Y_1N\, Y_1X_1 = \phi$ in order to allow that the triangles $E_{23}X_1X_2$, $F_{23}F_{12}X_2$ and $F_{12}X_1X_5$ have the Simson line $Y_1Y_2Y_3Y_4Y_5$.

All seven triangles mentioned above are those of Table 1. The first four triangles ($E_{23}E_{34}X_3$, $E_{34}X_5X_4$, $X_3F_{23}X_4$, and $F_{23}X_5E_{23}$) have collinear orthocentres (line $K_1K_2K_3K_4$) as stated in [1, 5], based on the property of the Simson line of a complete quadrilateral according to which, the segments joining the Miquel point with the orthocentres of the four triangles are bisected by their common Simson line $Y_2Y_3Y_4Y_5$ which is parallel to the line of orthocentres when $\phi = \pi/2$. This construction of the fifth line X_1F_{12} allows the other three triangles mentioned above to have the same Simson line which becomes $Y_1Y_2Y_3Y_4Y_5$. So, taking the quadrilateral formed by the lines $E_{23}X_5$, $E_{23}F_{23}$, $E_{34}X_4$ and X_1F_{12}, we get that the Simson line bisects the segments joining K_1 and K_5 with the Miquel point when $\phi = \pi/2$, so K_5 is collinear with K_1, K_2, etc. Similarly, we can use any other combinations forming quadrilaterals of the five lines in order to deduce that all K_i are collinear, and this is valid for any value of ϕ when it varies, since the shape of the pentagon does not change. Another property is related to the sum of the angles formed by the line of orthocentres and the Simson line plus the angle ϕ and more specifically,

$$\sphericalangle K_1K_2\, Y_1Y_2 + \phi = \pi/2, \tag{3}$$

because the line of orthocentres and the Simson line are parallel when $\phi = \pi/2$ and, since the shape of the pentagon does not change and also $M = N$ remains constant when ϕ changes, this sum of angles remains constant.

In Figure 7, we draw the line MU orthogonal to Y_1Y_2 and we have $\sphericalangle RM\,RF_1 = \phi$ because of (3). Moreover, when ϕ changes, R remains constant, since $\sphericalangle F_1R\,F_1Y_5 = \sphericalangle MR\,RY_5$. If $\phi = \pi/2$, then K_5M is bisected by $Y_1Y_2\ldots$ (as shown in [5], since M is the Miquel point of the triangle $X_1E_{23}X_2$) and also all segments K_iM are bisected, so the line $Y_1Y_2\ldots$ is the perpendicular bisector of RM ($MU = UR$) (when ϕ varies, point R remains constant and the line $K_1K_2\ldots$ rotates about R).

We have $\sphericalangle A_1A_5\,A_1M = \sphericalangle X_5M\,X_5F_1$ due to isogonal sides, $\sphericalangle F_1R\,F_1X_5 = \sphericalangle F_1F_2\,F_1X_5 = \sphericalangle MR\,MY_5 = \sphericalangle MR\,MA_1$, and $\sphericalangle MY_4\,MR = \sphericalangle F_2Y_4\,F_2R = \sphericalangle F_2F_1\,F_2X_5$ (sides isogonal), so from the triangles $X_5F_1F_2$ and A_1MA_5 we get respectively:

$$\begin{aligned}\sphericalangle X_5M\,X_5F_1 + \sphericalangle X_5F_2\,X_5M + \sphericalangle F_1F_2\,F_1X_5 + \sphericalangle F_2F_1\,F_2X_5 &= \pi \pmod{\pi} \quad \text{and}\\ \sphericalangle A_1A_5\,A_1M + \sphericalangle MR\,MA_1 + \sphericalangle MY_4\,MR + \sphericalangle A_5A_1\,A_5M &= \pi \pmod{\pi},\end{aligned}$$

therefore $\sphericalangle X_5F_2\,X_5M = \sphericalangle A_5M\,A_5A_1$.

Given that F is the intersection of MU and the circumcircle of $A_1A_2\ldots A_n$, we have $\sphericalangle FM\,FA_1 = \sphericalangle A_5M\,A_5A_1$, therefore $\sphericalangle FM\,FA_1 = \sphericalangle X_5F_2\,X_5M = \sphericalangle Y_5Y_4\,Y_5M$ and since $\sphericalangle Y_5Y_4\,Y_5M + \sphericalangle MY_5\,MF = \pi/2$, we get $\sphericalangle A_1F\,A_1M = \pi/2$. From all the above we deduce that MR passes through W, the circumcentre of $A_1A_2\ldots A_5$.

This construction gives us the concave pentagon $X_1X_2X_3X_4X_5$. It has always one solution and can be repeated for $m = n - 4$ lines in order to give us an n-gon. □

The above process for a given concave quadrangle with a Simson line for the ϕ, enables us to construct a family of entangled n-sided polygons following Theorem 3 and based on Theorem 1. Therefore:

Lemma 6. *Given four lines forming a complete quadrilateral, any three of them form four triangles whose circumcircles pass through a common point N and define a common Simson line for these triangles at the angle ϕ. Therefore this line has four collinear points. The four lines define a concave quadrangle (such as $E_{23}X_3X_4X_5$ of Figure 7).*
We can define $m = n - 4$ additional lines which form, together with the previous four lines, triangles having a common Simson line (EPSL) with n collinear points $Y_1, Y_2, \ldots$, also having their orthocentres collinear (K_1, K_2, $\ldots$) and forming a concave n-gon related with its entangled convex and cyclic polygon, following Theorems 1 and 3.
Moreover, the equation (3) holds, and the line $Y_1Y_2\ldots$ is the perpendicular bisector of MR. When ϕ varies, the line $K_1K_2\ldots$ rotates about R, and the line MR always passes through the circumcenter W of the entangled cyclic polygon.

The case of Figure 7 with two pentagons following Theorem 3 can be extended to include hexagons which are convex as follows: Z_0 is placed on the line X_5X_4 and Z_1 on the line X_5X_1. If we keep one of the two points constant and we move the other along its line, then we can find a position where we have $\sphericalangle Y_ZZ_0\,Y_ZM = \phi$ (within the limits of these parameters) where Y_Z is the intersection point of the two lines going through points Z_0, Z_1 and Y_1, Y_2 (the remaining points Y_i are collinear). This gives us a new convex hexagon $X_1X_2X_3X_4Z_0Z_1$ for which the extended Simson line has the collinear points Y_1, Y_2, Y_3, Y_4, Y_5, and Y_Z (Figure 8). The above process can be generalized for n-gons following Theorem 3.

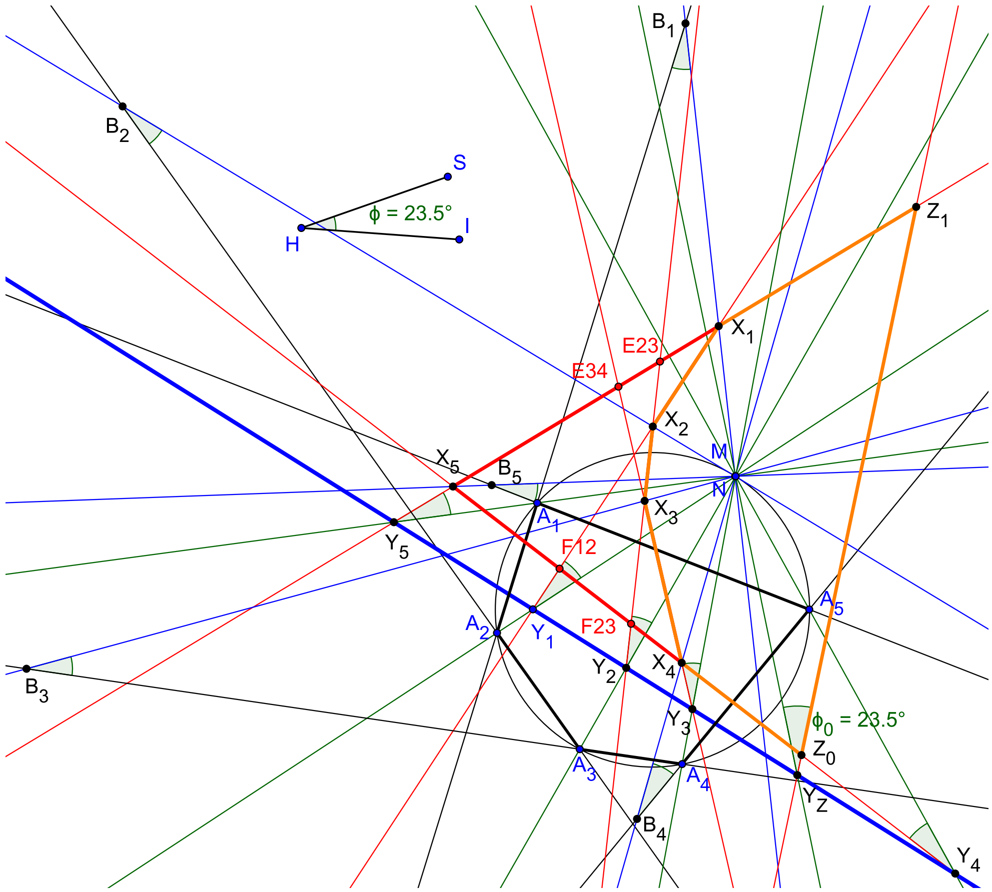

Figure 8: A convex hexagon $X_1X_2X_3X_4Z_0Z_1$ with generalized Simson line passing through Y_1, Y_2, Y_3, Y_4, Y_5, and Y_Z .

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A. Appendix

From Figure 6 (special case of Figure 1 with $N = M$) and the triangles $A_1A_2A_3$, $X_1X_2X_3$, $B_1B_2B_3$, $Y_1Y_2Y_3$, based on the Miquel circles applied to the entangled triangles (the quadrangles $A_1B_1MB_3$, ..., $X_1Y_1NY_3$, ... are cyclic), we have the following relations (see also the construction of the $X_1X_2X_3$ of Figure 1 in Section 1.1). Below we use the symbol $\measuredangle ABC$ for the measure of the interior angle at B in the triangle ABC.

$$\begin{aligned}
\widehat{A}_{11} &= \sphericalangle A_3A_1N = \sphericalangle MX_3X_1 = \widehat{X}_{32} = \sphericalangle B_3B_1M = \widehat{B}_{11} = \sphericalangle NY_2Y_3 = \widehat{Y}_{22},\\
\widehat{A}_{12} &= \sphericalangle NA_1A_2 = \sphericalangle X_3X_1M = \widehat{X}_{11} = \sphericalangle MB_3B_1 = \widehat{B}_{32} = \sphericalangle Y_3Y_1N = \widehat{Y}_{11},\\
\widehat{A}_{21} &= \sphericalangle A_1A_2N = \sphericalangle MX_1X_2 = \widehat{X}_{12} = \sphericalangle B_1B_2M = \widehat{B}_{21} = \sphericalangle NY_3Y_1 = \widehat{Y}_{32},\\
\widehat{A}_{22} &= \sphericalangle NA_2A_3 = \sphericalangle X_1X_2M = \widehat{X}_{21} = \sphericalangle MB_1B_2 = \widehat{B}_{12} = \sphericalangle Y_1Y_2N = \widehat{Y}_{21},\\
\widehat{A}_{31} &= \sphericalangle A_2A_3N = \sphericalangle MX_2X_3 = \widehat{X}_{22} = \sphericalangle B_2B_3M = \widehat{B}_{31} = \sphericalangle NY_1Y_2 = \widehat{Y}_{12},\\
\widehat{A}_{32} &= \sphericalangle NA_3A_1 = \sphericalangle X_2X_3M = \widehat{X}_{31} = \sphericalangle MB_2B_3 = \widehat{B}_{22} = \sphericalangle Y_2Y_3N = \widehat{Y}_{31}.
\end{aligned}$$

From the above follows

$$\widehat{B}_{11} = \widehat{Y}_{22},\ \ \widehat{B}_{12} = \widehat{Y}_{21},\ \ \widehat{B}_{21} = \widehat{Y}_{32},\ \ \widehat{B}_{22} = \widehat{Y}_{31},\ \ \widehat{B}_{31} = \widehat{Y}_{12},\ \ \widehat{B}_{32} = \widehat{Y}_{11},$$

therefore we obtain similar triangles $B_1B_2B_3 \sim Y_1Y_2Y_3$.

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