

A Generalization of Desargues' Involution Theorem

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Abstract. This paper states and proves a generalization of the well-known Desargues' Involution Theorem from plane projective geometry.

Key Words: Desargues' Involution Theorem, generalization, quadrics, symmetric bilinear forms, projective line.

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1. Introduction

A classic theorem in projective geometry is the following remarkable result discovered by Girard DESARGUES, a French engineer and mathematician widely regarded as one of the founders of projective geometry.

Theorem (Desargues' Involution Theorem).

Consider four points in general position in the real projective plane, i.e., no three of these four points are collinear. Let F be the family of conics passing through these four points. Then for any line ℓ that does not pass through any of these four points, each conic in F will, if it intersects ℓ , do so in a pair of points that are conjugate under an involution of the line ℓ .

Here, an *involution* means a projective transformation of the line ℓ that has order two. A conic is the set of zero points of a quadratic form $\neq 0$ in projective dimension two. Specifically, if the given projective plane represents the set of lines in a vector space V of linear dimension 3, then a conic is the set of isotropic points of a symmetric bilinear form q defined on V , i.e., the projective set associated with vectors v such that $q(v, v) = 0$. Such a set of isotropic or zero points of a quadratic form is generally called a *quadric*, so a conic is the special case of a quadric in projective dimension two. In this note, we want to prove the following generalization of the Desargues' Involution Theorem.

Theorem (Generalization of Desargues' Involution Theorem).

Consider a projective space of any dimension over a field \mathbb{K} of characteristic $\neq 2$. A pencil of quadrics in that projective space will intersect a line in that space (when they do) in pairs of points that are conjugate under an involution if and only if

- (i) *the line is not part of a quadric in that pencil and*
- (ii) *the line contains no common zero point for all the quadrics in that pencil.*

Equivalently, these conditions are satisfied if and only if the pencil contains two quadrics whose intersection points with the given line over an algebraic closure of \mathbb{K} are disjoint.

As this generalization suggests, the Desargues' Involution Theorem is essentially an algebraic property of the projective line and has little to do with the ambient projective space.

The term *pencil* in this context means a linear system of projective dimension 1. In other words, if R and S are two symmetric bilinear forms that represent two different quadrics in that pencil, then all of the quadrics in the pencil can be represented by the symmetric bilinear forms $aR + bS$, where a and b are numbers running through the ground field \mathbb{K} .

In a projective plane, the family of conics passing through four given base points in general position is such a pencil. However, there may be pencils with no common base points in the projective plane, e.g., the Poncelet pencil of circles in a real projective plane generated by two non-intersecting circles. For more information about pencils of conics and the Desargues' Involution Theorem, the readers could refer to the beautiful recent book '*The Universe of Conics*' ([1] in our list of references).

The famous butterfly theorem of Euclidean plane geometry is a special case of the Desargues' Involution Theorem. Based on our generalization of the Desargues' Involution Theorem, the readers may enjoy exploring more general forms of the butterfly theorem for affine space of any dimension over a field of characteristic $\neq 2$.

2. Symmetric bilinear forms

We will work with a ground field \mathbb{K} of characteristic $\neq 2$. A *projective line* defined over \mathbb{K} is the set $\mathbf{P}(E)$ of lines through the origin in a \mathbb{K} -vector space E of linear dimension 2. If we identify such a vector space with the affine plane \mathbb{K}^2 through a suitable choice of basis, we can describe the projective line as $\mathbb{K} \cup \{\infty\}$, i.e., the affine line \mathbb{K} extended by adjoining a point at infinity denoted by the symbol ∞ , where each element x of \mathbb{K} is identified with the line in \mathbb{K}^2 passing through the point $(x, 1)$ and the origin, and the point ∞ is identified with the horizontal line consisting of all the points $(u, 0)$ (with u running through \mathbb{K}) in \mathbb{K}^2 .

A *symmetric bilinear form* on E is a mapping from the product space $E \times E$ into the ground field \mathbb{K} that is linear in each variable and symmetric. Relative to a given basis of E , such a pairing is described by a symmetric 2 by 2 matrix, and so the set B of all symmetric bilinear forms naturally has the structure of a \mathbb{K} -vector space of linear dimension 3. A symmetric bilinear form on E is *non-degenerate* if and only if the determinant of the corresponding matrix is non-zero.

Given a symmetric 2 by 2 matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, its determinant $ac - b^2$ is a quadratic form of the matrix entries, so we can associate a symmetric bilinear form to the determinant. Specifically, we can define the following pairing $B \times B \rightarrow \mathbb{K}$ for any two symmetric 2 by 2 matrices:

$$\left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix} \right\rangle = \frac{1}{2}(ac' + ca') - bb'.$$

Such a pairing is well-defined because the ground field \mathbb{K} has characteristic $\neq 2$, and it is clearly bilinear and symmetric. With this pairing, B is a quadratic space isometric to the orthogonal sum of an Artinian plane (also known as a hyperbolic plane) and the one-dimensional space

\mathbb{K} (with multiplication). Accordingly, this pairing on B is regular, meaning that $\langle x, y \rangle = 0$ for all $y \in B$ if and only if $x = 0$. The non-isotropic elements of B are the symmetric matrices z such that $\langle z, z \rangle = \det z \neq 0$, i.e., the matrices that represent non-degenerate symmetric bilinear forms on E .

Because the determinant pairing is regular, any non-degenerate symmetric bilinear form q gives us an orthogonal decomposition $B = \mathbb{K}q \oplus$ (the orthogonal complement of $\mathbb{K}q$). The orthogonal complement of $\mathbb{K}q$ has linear dimension 2, so whenever there is a subspace W of B of dimension at least 2 such that W is orthogonal to a non-degenerate element q of B , then W must in fact be the 2-dimensional orthogonal complement of $\mathbb{K}q$, and B must be the direct sum of $\mathbb{K}q$ and W .

Implicit in the determinant pairing is the choice of a basis for E to allow us to identify the space B with the space of 2 by 2 symmetric matrices. If we change the coordinates for E by a general linear transformation $x = S(x')$, then the matrix M of a symmetric bilinear form in the old coordinate x will become tSMS (where tS is the transpose of S) in the new coordinates x' . If we take the norm $\langle M, M \rangle$ of M under the determinant pairing relative to the old basis, we get $\det(M)$. On the other hand, the norm of M in the determinant pairing relative to the new basis becomes

$$\langle {}^tSMS, {}^tSMS \rangle = \det({}^tSMS) = \det(S)^2 \det(M).$$

Therefore the simple linear transformation $q \mapsto (\det S)q$ gives us an isometry between the vector space B with the determinant pairing in the new coordinate x' and the vector space B with the determinant pairing in the old coordinate x . Accordingly, orthogonal properties in the space B under the determinant pairing are independent of any basis chosen for the coordinates.

A symmetric bilinear form $q \neq 0$ on E falls into one of 3 types:

- The form q is non-degenerate and anisotropic, meaning there is no non-zero vector v in E such that $q(v, v) = 0$. The form q has no isotropic point on the projective line $\mathbf{P}(E)$.
- The form q is non-degenerate and isotropic. In that case, the quadratic space (E, q) is isometric to an Artinian plane, and the isotropic vectors v such that $q(v, v) = 0$ consist of exactly two different lines in E . The form q has two distinct isotropic points on the projective line $\mathbf{P}(E)$. Any symmetric bilinear form that has the same isotropic vectors as q must be a scalar multiple of q . Indeed, relative to the basis consisting of those two isotropic vectors the matrices of these two bilinear forms both have zeros in the diagonal and a non-zero number in the cross diagonal.
- The form q is degenerate. In that case, its radical is a one-dimensional subspace of E , and the form q has exactly one isotropic point on the projective line $\mathbf{P}(E)$. Any symmetric bilinear form that has the same isotropic vectors as q must be a scalar multiple of q . Indeed, relative to any basis that includes an isotropic vector, the matrices of these two bilinear forms both have just one non-zero number in the same diagonal position.

The above three situations tell us how a projective line in an arbitrary projective space can intersect a quadric. Let such a quadric be the projective set of isotropic points of a symmetric bilinear form $q \neq 0$ defined on a vector space V , and let the given projective line be $\mathbf{P}(E)$, where E is a subspace of dimension 2 of V . If the restriction of q to E is a non-zero symmetric bilinear form on E , then:

- (i) such a restriction being anisotropic means that the line $\mathbf{P}(E)$ does not intersect the given quadric;

- (ii) such a restriction being non-degenerate and isotropic means that the line $\mathbf{P}(E)$ intersects the given quadric in two distinct points; and
- (iii) such a restriction being degenerate means that the line $\mathbf{P}(E)$ intersects the given quadric in one tangent point.

Of course, if the restriction of q to E is the zero symmetric bilinear form on E , then that simply means the entire line $\mathbf{P}(E)$ is part of the quadric.

When the form q has one or two isotropic points we can tell what the symmetric matrices that represent q (relative to a given basis of E) would look like. Suppose that relative to a given basis of E , the form q has isotropic vectors (s, t) and (u, v) which are linearly independent in the non-degenerate case and proportional in the degenerate case. The following quadratic form has the same isotropic vectors, and therefore must be the same quadratic form associated with q , up to a scalar factor:

$$(tX - sY)(vX - uY) = tvX^2 - (tu + sv)XY + suY^2.$$

Accordingly, the 2 by 2 symmetric matrix associated with q would look like the following, up to a scalar factor:

$$\begin{pmatrix} tv & w \\ w & su \end{pmatrix}, \text{ where } w = -\frac{1}{2}(tu + sv).$$

3. Involutions

A projective transformation of a projective line $\mathbf{P}(E)$ is induced by an invertible linear transformation of E . So relative to a given basis of E , a projective transformation can be described by an invertible 2 by 2 matrix with coefficients in the ground field \mathbb{K} , up to a scalar factor. A major focus of the classical geometry of projective line is the study of involutions, defined as projective transformations of order two. For a projective transformation to be an involution, it is necessary and sufficient that any 2 by 2 matrix T corresponding to such an involution satisfies a minimal equation $T^2 - s Id = 0$ with $s \neq 0$ (Id stands for the identity matrix). At the same time, we know from the Hamilton-Cayley theorem that

$$T^2 - \text{tr}(T)T + \det(T) Id = 0.$$

So involutions are given by invertible 2 by 2 matrices with zero trace.

Given an involution on $\mathbf{P}(E)$ represented by $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, we can ask whether a point represented by the vector (x, y) is a fixed point of this involution. The vector (x, y) is transformed into the vector $(u, v) = (ax + by, cx - ay)$ under the action of the involution. The vector (x, y) represents a fixed point of the involution if and only if (u, v) is proportional to (x, y) , or equivalently if and only if $uy = vx$. That means

$$(ax + by)y = (cx - ay)x$$

or that

$$byy + 2axy - cxx = 0,$$

i.e., (x, y) is an isotropic vector of the symmetric bilinear form $\begin{pmatrix} -c & a \\ a & b \end{pmatrix}$.

Accordingly, to any involution on $\mathbf{P}(E)$ represented by $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, let us associate the element of $\mathbf{P}(B)$ represented by $\begin{pmatrix} -c & a \\ a & b \end{pmatrix}$, which we will call the *Desargues bilinear form*

associated to the involution. Note that the determinant of $\begin{pmatrix} -c & a \\ a & b \end{pmatrix}$ is equal to the determinant of $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, so that the symmetric matrix $\begin{pmatrix} -c & a \\ a & b \end{pmatrix}$ represents a non-degenerate symmetric bilinear form. Moreover, the fixed points of the involution (if any) are exactly the isotropic points (if any) of the associated Desargues bilinear form.

This correspondence is a bijection between involutions on $\mathbf{P}(E)$ and non-degenerate elements of $\mathbf{P}(B)$, with the inverse correspondence mapping a non-degenerate element of $\mathbf{P}(B)$ represented by $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ to the involution on $\mathbf{P}(E)$ represented by $\begin{pmatrix} b & c \\ -a & -b \end{pmatrix}$. While this bijective correspondence between involutions and non-degenerate elements of $\mathbf{P}(B)$ is defined based on the matrix representation of involutions and symmetric bilinear forms relative to a specific basis of the vector space E , the correspondence is really independent of any basis, based on the following proposition.

Proposition 1. *Each pair of conjugate points in an involution are the isotropic points of a symmetric bilinear form that is orthogonal (under the determinant pairing) to the Desargues bilinear form associated to the given involution.*

Proof. Consider an involution on $\mathbf{P}(E)$ represented by $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, and let (x, y) and $(ax + by, cx - ay)$ be a pair of conjugate points. These points can be proportional if (x, y) happens to be a fixed point of the involution. Note that if we take any two conjugate points of the given involution and treat them as isotropic points of a symmetric bilinear form, then that form is uniquely determined up to a scalar factor, per our discussion of symmetric bilinear forms above. A symmetric bilinear form with these two isotropic points is represented by the matrix

$$\begin{pmatrix} y(cx - ay) & z \\ z & x(ax + by) \end{pmatrix}, \text{ where } z = -\frac{1}{2}y(ax + by) - \frac{1}{2}x(cx - ay).$$

The determinant pairing between that symmetric bilinear form and the Desargues bilinear form $\begin{pmatrix} -c & a \\ a & b \end{pmatrix}$ gives us the number

$$\begin{aligned} & \frac{1}{2}(-cx(ax + by) + by(cx - ay)) - az \\ &= \frac{1}{2}(-cx(ax + by) + by(cx - ay)) + \frac{1}{2}(ay(ax + by) + ax(cx - ay)) \\ &= \frac{1}{2}(-caxx - bayy) + \frac{1}{2}(abyy + acxx) = 0. \quad \square \end{aligned}$$

Recall that orthogonal relationships under the determinant pairing is independent of any basis. Hence, regardless of any basis we choose for E , the Desargues bilinear form corresponding to a given involution is orthogonal to all the elements of $\mathbf{P}(B)$ associated with pairs of conjugate points of the involution.

Because we can find at least two different pairs of such conjugate points, these elements of $\mathbf{P}(B)$ generate a subspace of projective dimension at least one. Since a Desargues bilinear form is non-degenerate by construction, the above proposition implies that all the symmetric bilinear forms associated with pairs of conjugate points of the involution generate a projective subspace of dimension 1 (a pencil) in $\mathbf{P}(B)$, and the Desargues bilinear form is the uniquely determined bilinear form up to a scalar factor that is orthogonal to that pencil.

Suppose we have an involution T , and by extending the ground field to its algebraic closure if necessary, let us assume that the involution T has two fixed points A and B on the

projective line. Let us call t the Desargues bilinear form associated to T . What Proposition 1 shows is that if C and D are two conjugate points under T , i.e., if T transforms C to D and D to C , then the bilinear form t is orthogonal (under the determinant pairing) to the bilinear form s that has C and D as isotropic points (such a bilinear form s is determined up to a scalar factor, as we previously noted). If S is the involution associated to the bilinear form s , what can we say about T and S ? We know that C and D are the fixed points of S . Moreover, the cross ratio $[A, B, C, D] = [T(A), T(B), T(C), T(D)] = [A, B, D, C]$, and so this cross ratio must be equal to -1 , i.e., the points A, B, C, D form a harmonic range. It is well-known that two involutions commute with each other if and only if their fixed points (over an algebraic closure of the ground field) form a harmonic range, i.e., $ST = TS$, and this product is naturally also an involution. So two Desargues bilinear forms are orthogonal under the determinant pairing if and only if the corresponding involutions commute with each other.

4. Generalization of Desargues' Involution Theorem

Based on Proposition 1, there is a natural bijection (independent of any basis or coordinates) between involutions of a projective line and non-degenerate bilinear forms on that line or, equivalently, the orthogonal complements of such non-degenerate bilinear forms. A necessary and sufficient condition for a pencil of bilinear forms on a projective line to be the orthogonal complement of a non-degenerate bilinear form is that the pencil must be regular under the determinant pairing. In other words, that pencil must correspond to a two-dimensional subspace of B which is a regular quadratic space under the determinant pairing.

In general, it is straight-forward to check whether a bilinear space of dimension 2 is regular under a given symmetric pairing. We can just write down the matrix of that pairing relative to a suitable basis of the space and determine if the matrix has non-zero determinant. In our particular case, we also have another geometric criterion.

Proposition 2. *A 2-dimensional subspace of the space of all symmetric bilinear forms on a vector space of dimension 2 is regular with respect to the determinant pairing if and only if there is no common isotropic vector for all the forms in that subspace, or equivalently, if the subspace can be generated by two forms with no common isotropic vector.*

Proof. As before, we denote by B the space of all symmetric bilinear forms on a 2-dimensional \mathbb{K} -vector space E . With a choice of basis for E , B can be identified with the space of symmetric 2 by 2 matrices with coefficients in \mathbb{K} and therefore has linear dimension 3. Let G be a 2-dimensional subspace of B . Recall that B is regular under the determinant pairing. Because B is regular, the subspace H of B orthogonal to G is therefore a 1-dimensional subspace, say generated by a bilinear form h .

G is a regular subspace of B if and only if h does not belong to G , i.e., if and only if h is a non-isotropic vector under the determinant pairing. If h is isotropic, then it is a degenerate bilinear form orthogonal to all the forms in G . Moreover, h has a one-dimensional radical. Choose a basis of E where the first vector of that basis lies in the radical of h . Relative to that basis, the matrix of h has the form $\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$ where $x \neq 0$. Any symmetric bilinear form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ orthogonal to h under the determinant pairing must satisfy the equation $ax = 0$, i.e., $a = 0$. That means the first vector of the basis is also an isotropic vector for any form orthogonal to h .

Conversely, if there is a common isotropic vector for all forms in the 2-dimensional subspace G , i.e., if all forms in G can be expressed as matrices $\begin{pmatrix} 0 & b \\ b & c \end{pmatrix}$ relative to a suitable basis of E , then G also contains the degenerate form represented by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$. That form is orthogonal to all forms in G , and therefore G is not regular under the determinant pairing. \square

We can now prove the generalization of Desargues' Involution Theorem stated at the beginning of this note.

Theorem (Generalization of Desargues' Involution Theorem).

Consider a projective space of any dimension over a field \mathbb{K} of characteristic $\neq 2$. A pencil of quadrics in that projective space will intersect a line in that space (when they do) in pairs of points that are conjugate under an involution if and only if

- (i) *the line is not part of a quadric in that pencil and*
- (ii) *the line contains no common zero point for all the quadrics in that pencil.*

Equivalently, these conditions are satisfied if and only if the pencil contains two quadrics whose intersection points with the given line over an algebraic closure of \mathbb{K} are disjoint.

Proof. Consider a pencil of quadrics in the given projective space $\mathbf{P}(V)$ defined by a family of symmetric bilinear forms on V of linear dimension 2. Let ℓ be a line in that projective space with $\ell = \mathbf{P}(E)$ for some 2-dimensional \mathbb{K} -vector subspace E of V . The family of symmetric bilinear forms when restricted to E will give us a linear system of symmetric bilinear forms on E of dimension at most 2.

For the linear system of bilinear forms to be 2-dimensional when restricted to E , it is necessary and sufficient that there is no non-zero form in that system that becomes zero when restricted to E , i.e., if and only if for any non-zero form q in that system, we do not have $q(u, v) = 0$ for all $u, v \in E$, or equivalently, that we do not have $q(v, v) = 0$ for all $v \in E$. That is the case if and only if the line ℓ is not part of any quadric in the pencil.

Assuming that is the case, the quadrics in the given pencil intersect the line ℓ (when they do) in pairs of points conjugate under an involution of the line precisely when the linear system of symmetric bilinear forms as restricted to E is regular under the determinant pairing (in whatever coordinates of the line). In light of Proposition 2, that is equivalent to the lack of a common isotropic point on the line for all the bilinear forms in that linear system, i.e., the line ℓ contains no common zero point for all the quadrics in the pencil.

Equivalently, note that the given pencil of quadrics gives us a regular 2-dimensional space of symmetric bilinear forms on E , as a vector space over \mathbb{K} , if and only if that pencil gives us a regular 2-dimensional space of symmetric bilinear forms on $E \otimes \mathbb{L}$ as a vector space over an algebraic closure \mathbb{L} of \mathbb{K} . That is because the bilinear forms and their determinant pairing vary naturally with the extension in scalars.

If the quadrics give us a regular 2-dimensional space of symmetric bilinear forms on $E \otimes \mathbb{L}$, then it is clear that we must have two different quadrics whose intersection points with $\mathbf{P}(E \otimes \mathbb{L})$ are disjoint, in light of Proposition 2. Conversely, suppose we have two different quadrics whose intersection points with $\mathbf{P}(E \otimes \mathbb{L})$ are disjoint. Note that over an algebraic closure \mathbb{L} of \mathbb{K} any bilinear form on $E \otimes \mathbb{L}$ has isotropic vectors, i.e., any quadric intersects a line in the same projective space. Let R and S be two symmetric bilinear forms that represent these two different quadrics. By the disjoint intersection condition, we can

choose two different points M and N of $\mathbf{P}(E \otimes \mathbb{L})$, where M is an isotropic point of R but not of S , and N is an isotropic point of S but not of R . In that case, M and N cannot both be isotropic points for a linear combination $aR + bS$ unless both a and b are zero. In other words, the line $\mathbf{P}(E \otimes \mathbb{L})$ cannot be part of any quadric in the given pencil. According to Proposition 2, these two quadrics then give us a regular 2-dimensional space of symmetric bilinear forms on $E \otimes \mathbb{L}$. \square

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