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Relations Between Ceva's Theorem and the Concurrency of Midlines of Quadrilaterals in a Triangle

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Abstract. We consider two triangles, one of which is inscribed in another, and conditions for the concurrency of the midlines of quadrilaterals formed by these triangles.

Key Words: Triangle, concurrent lines, cevians

MSC 2010: 51M04, 51N10

1. Introduction

Let three arbitrary points A_1, B_1 and C_1 on the sides of any triangle ABC form a triangle $A_1B_1C_1$. Denote the midpoints of the two triangles' sides by D, F, E and D_1, F_1, E_1, respectively (see Figure 1). What is the condition for the concurrency of the midlines DD_1, FF_1 and EE_1 of the obtained quadrilaterals BC_1B_1C, BA_1B_1A and CA_1C_1A? What properties does the common point of these lines have in the case of their concurrency? The following results answer these questions.

2. Main results

Theorem 1. *Given any triangle ABC with three cevians AA_1, BB_1, CC_1. The points D, F, E are the midpoints of the sides BC, AB and AC of the triangle ABC, and D_1, F_1, E_1 are the midpoints of the sides B_1C_1, A_1B_1 and A_1C_1 of the triangle $A_1B_1C_1$ (see Figure 1). Then the lines DD_1, FF_1 and EE_1 are concurrent if and only if the cevians AA_1, BB_1 and CC_1 are concurrent.*

Proof. Without loss of generality we can prove the theorem for a right angle isosceles triangle (see Figure 2) because any other triangle can be obtained from such a triangle by an affine transformation.

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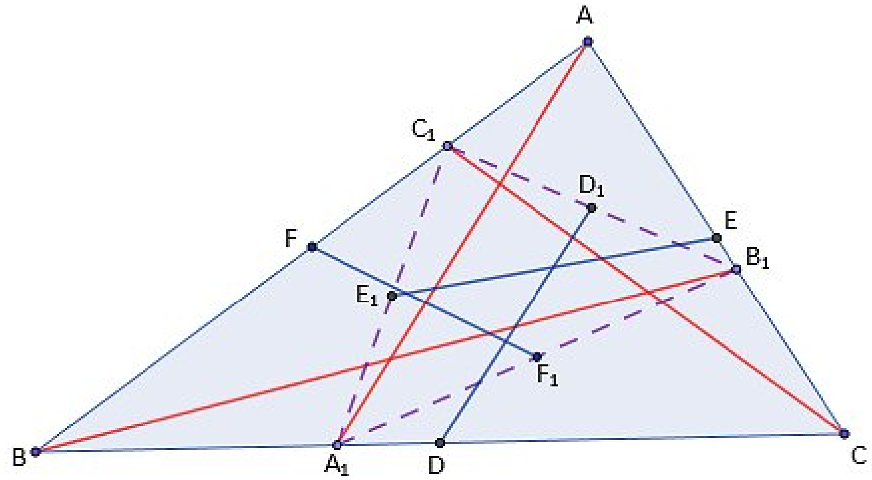

Figure 1: DD_1, FF_1, EE_1 are concurrent $\iff$ AA_1, BB_1, CC_1 are concurrent.

Suppose that coordinates of the points A, B, C are $A(0,1)$, $B(0,0)$, $C(1,0)$. Then the coordinates of D, F, E are $D(0.5,\,0)$, $F(0,\,0.5)$, $E(0.5,\,0.5)$. Denote coordinates of C_1, A_1, B_1 by $C_1(0,\alpha)$, $A_1(\beta,0)$, $B_1(\gamma,\,1-\gamma)$. Then

$$D_1\left(\frac{\gamma}{2},\,\frac{\alpha+1-\gamma}{2}\right),\quad F_1\left(\frac{\beta+\gamma}{2},\,\frac{1-\gamma}{2}\right),\quad E_1\left(\frac{\beta}{2},\,\frac{\alpha}{2}\right).$$

So the equations of lines DD_1, FF_1 and EE_1 are:

$$DD_1:\ \frac{x-0.5}{y}=\frac{\gamma-1}{\alpha+1-\gamma};\quad FF_1:\ \frac{x}{y-0.5}=\frac{\beta+\gamma}{-\gamma};\quad EE_1:\ \frac{x-0.5}{y-0.5}=\frac{\beta-1}{\alpha-1}$$

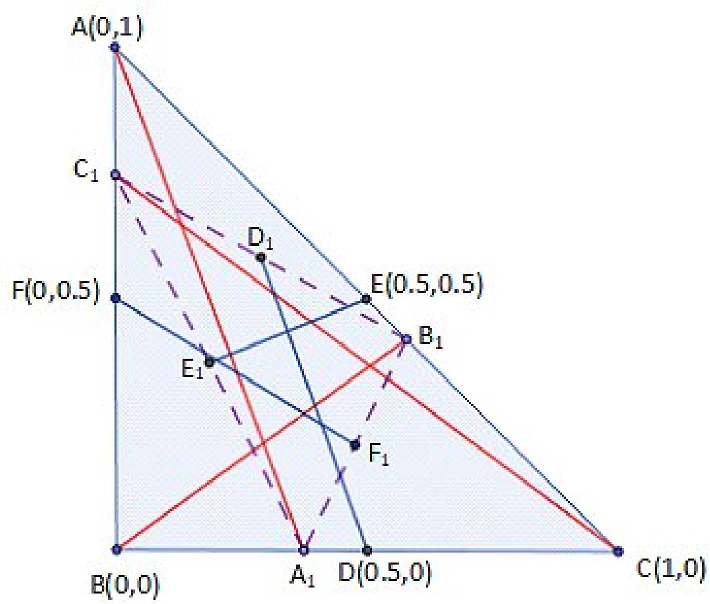

Figure 2: ABC is a right angle isosceles triangle.

or

$$\begin{array}{llcccl} DD_1: & (\gamma-1)y - & (\alpha+1-\gamma)x & + & 0.5(\alpha+1-\gamma) & = 0\,; \\ FF_1: & (\beta+\gamma)y + & \gamma x & - & 0.5(\beta+\gamma) & = 0\,; \\ EE_1: & (\beta-1)y - & (\alpha-1)x & + & 0.5(\alpha-\beta) & = 0\,. \end{array}$$

The three lines DD_1, FF_1 and EE_1 are concurrent if and only if

$$\begin{vmatrix} \gamma-1 & \beta-1 & \beta+\gamma \\ \gamma-\alpha-1 & 1-\alpha & \gamma \\ 0.5(\alpha+1-\gamma) & 0.5(\alpha-\beta) & -0.5(\beta+\gamma) \end{vmatrix} = 0.$$

By calculating the determinant we obtain

$$\beta + 2\alpha\beta\gamma - \beta\gamma - \alpha\beta - \alpha\gamma = 0,$$

i.e., three lines DD_1, FF_1 and EE_1 are concurrent if and only if this equality holds.

On the other hand according to Ceva's theorem [1, pp. 4–5], the cevians AA_1, BB_1 and CC_1 are concurrent if and only if

$$\frac{1-\alpha}{\alpha}\cdot\frac{\beta}{1-\beta}\cdot\frac{1-\gamma}{\gamma} = 1 \iff \beta + 2\alpha\beta\gamma - \beta\gamma - \alpha\beta - \alpha\gamma = 0,$$

and so we obtain the above equality. Thus the theorem is proved. □

Remark. The one-sided statement of Theorem 1, "if three cevians AA_1, BB_1 and CC_1 are concurrent, then the lines DD_1, FF_1 and EE_1 are concurrent", is given in the book [5, p. 120] as problem no. 1113. The book offers only a way to solve the problem. This is a very complicated and multi-stage way that refers to other, rather complex problems from the book.

Let the coordinates of the vertices of an arbitrary triangle ABC be $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$ and let the three cevians AA_1 , BB_1 and CC_1 be concurrent with the common point $M(x_M, y_M)$. Then three lines DD_1, FF_1 and EE_1 are concurrent. Denote their common point by $N(x_N, y_N)$. What are the coordinates of the point N? The following theorem answers this question.

Theorem 2. *If three cevians AA_1, BB_1 and CC_1 of a triangle ABC are concurrent with the common point M, then the coordinates of the common point N of the lines DD_1, FF_1 and EE_1 are the averages of the corresponding coordinates of the points A, B, C, M.*

Proof. Denote the midpoint of segment AM by K and the midpoint of segment CM by L (see Figure 3). Then in the complete quadrilateral based on AB_1MC_1 the three points D, D_1 and K are collinear [3, p. 62]. Similarly, the points F, F_1 and L are collinear. FD and KL are the midlines of triangles ABC and AMC, respectively. Then FD and KL are equal and parallel. Thus $FDLK$ is a parallelogram and N is the midpoint of DK . The coordinates of the point K are

$$x_K = \frac{x_A + x_M}{2}, \quad y_K = \frac{y_A + y_M}{2}.$$

The coordinates of the point D are

$$x_D = \frac{x_C + x_B}{2}, \quad y_D = \frac{y_C + y_B}{2}.$$

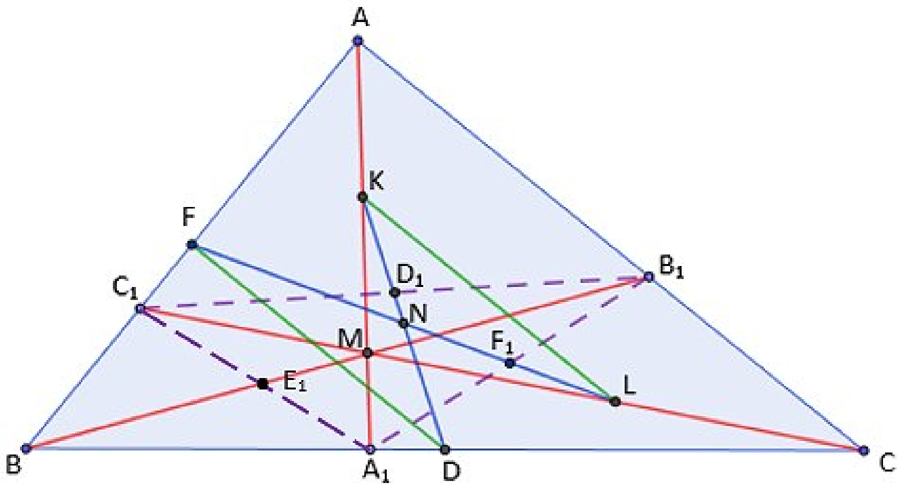

Figure 3: The coordinates of N are the averages of the corresponding coordinates of A, B, C, M.

Then the coordinates of the point N are

$$x_N = \frac{x_D + x_K}{2} = \frac{x_A + x_B + x_C + x_M}{4}, \quad y_N = \frac{y_D + y_K}{2} = \frac{y_A + y_B + y_C + y_M}{4}. \qquad \square$$

Corollary 1. *Since the coordinates of centroid G of the triangle ABC are $x_G = (x_A + x_B + x_C)/3$, $y_G = (y_A + y_B + y_C)/3$, the coordinates of the point N are $x_N = (3x_G + x_M)/4$ and $y_N = (3y_G + y_M)/4$. Thus point N lies on segment MG and divides it in the ratio 3:1, i.e., $MN : NG = 3 : 1$.*

Corollary 2. *If point M is the centroid of the triangle ABC, then M coincides with N.*

Corollary 3. *The convex hexagon (see Figure 4) with the vertices at the midpoints of the sides of the triangle ABC and at the midpoints of the segments AM, BM, CM, is a parallelogon (its opposite sides are equal and parallel). Point N is the centroid of the parallelogon's vertices. The area of the parallelogon is equal to half the area of triangle ABC. The perimeteof the parallelogon is equal to $2(AM + BM + CM)$. Therefore if all the angles of triangle ABC are less than $120°$, then the minimal value of the perimeter of the parallelogon is reached when M coincides with the Fermat-Torricelli point* [2, pp. 24–34].

Corollary 4. *If M is the circumcenter of the triangle ABC, then all sides of the parallelogon are equal to the half circumradius of the triangle ABC.*

Corollary 5. *If all the angles of the triangle ABC are less than $120°$ and M is the Fermat-Torricelli point of the triangle ABC, then all angles of the parallelogon are equal to $120°$.*

Corollary 6. *If M is the orthocenter H of the triangle ABC, then N is the nine-point center of the triangle ABC, because the nine-point center lies on the segment HG and divides it in the ratio 1:3, i.e., $HN : NG = 1 : 3$* [4, p. 153].

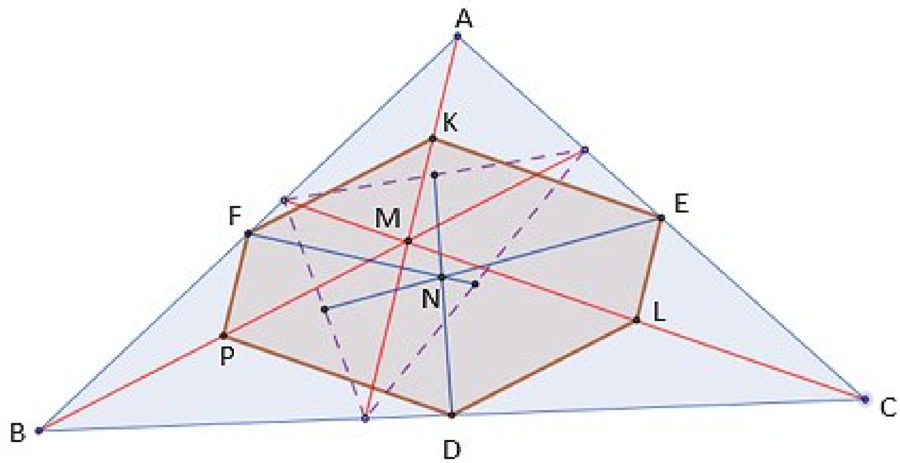

Figure 4: A parallelogon $FKELDP$.

References

[1] H.S.M. Coxeter, S.L. Greitzer: *Geometry Revisited*. Math. Assoc. Amer., Washington DC, 1967.

[2] R. Honsberger: *Mathematical Gems I*. Math. Assoc. Amer., Washington DC, 1973.

[3] R.A. Johnson: *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*. Houghton Mifflin, Boston MA, 1929.

[4] A.S. Posamentier, I. Lehmann: *The Secrets of Triangles*. Prometheus Books, 2012.

[5] G. Titeica: *Culegere de Probleme de Geometrie* [Romanian]. Editura Technica, Bucuresti 1965.

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