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Geometric Inequalities on Parallelepipeds and Tetrahedra

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Abstract. We prove an inequality comparing the sum of areas of faces of a parallelepiped to its the volume. Then we prove an inequality on a tetrahedron analogous to Weitzenböck's Inequality on a triangle using the inequality on a parallelepiped and Yetter's Theorem. We also give a short proof of Yetter's Theorem.

Key Words: Weitzenböck's Inequality; parallelepiped; tetrahedron; Yetter's Theorem
MSC 2020: 51M16, 51M25

1 Introduction

Suppose $\vec{a}$ and $\vec{b}$ are three-dimensional vectors. Then, $\|\vec{a}\|^2 + \|\vec{b}\|^2 \geq 2\|\vec{a} \times \vec{b}\|$ since $\|\vec{a}\|^2 + \|\vec{b}\|^2 \geq 2\|\vec{a}\| \cdot \|\vec{b}\|$ and $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$. So if $ABCD$ is a parallelogram, then $AB^2 + BC^2 + CD^2 + DA^2 \geq 4\mathcal{A}_{ABCD}$, where $\mathcal{A}_{ABCD}$ is the area of the parallelogram. The equality holds when $ABCD$ is a rectangle. We will prove an analogous result to a parallelepiped in Theorem 1.

Weitzenböck's inequality states the following; if a, b, c are the side lengths of a triangle with area T, then $a^2 + b^2 + c^2 \geq 4\sqrt{3}T$, c. f. [1] and the references therein. We will obtain an inequality on a tetrahedron in Theorem 2 that is analogous to Weitzenböck's Inequality. In order to prove Theorem 2, we will use Theorem 1 and Yetter's Theorem (see Theorem 1 in [2], and [4]). We will state Yetter's theorem slightly differently from the one given in [2] and give a concise proof.

2 Parallelepiped

Let us start by giving notations and a definition.

Definition 1. 1. The area of a triangle ABC is denoted by $\mathcal{A}_{ABC}$. The area of a parallelogram $ABCD$ is denoted by $\mathcal{A}_{ABCD}$.

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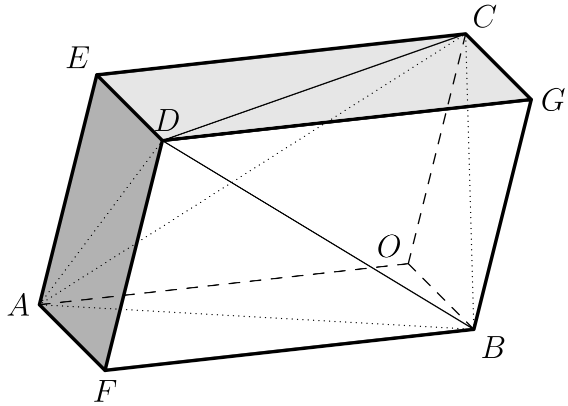

Figure 1: The parallelepiped $[OABC]$ generated by vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ that embeds a tetrahedron $ABCF$.

2. Let $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ be non-collinear vectors. We say the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ *generate* the parallelepiped with new additional vertices D, E, F, G as in Figure 1.

Note that vertices O and D of the parallelepiped in Figure 1 are diagonally opposite. We denote this parallelepiped by $[OABC]$.

Theorem 1. *Let $[OABC]$ be a parallelepiped, and W be its volume. Then $\mathcal{A}^2_{OAFB}+\mathcal{A}^2_{OBGC}+\mathcal{A}^2_{OCEA} \geq 3W^{4/3}$. The equality holds only when the parallelepiped is a cube.*

Proof. Without loss of generality, we assume the parallelepiped $[OABC]$ to be generated by vectors $\vec{a}=\overrightarrow{OA}=(a,0,0)$, $\vec{b}=\overrightarrow{OB}=(x,b,0)$, $\vec{c}=\overrightarrow{OC}=(y,z,c)$, where $a,b,c>0$ and x,y,z are any real numbers. Then $W=abc$. On the other hand, we have $\vec{u}=\vec{a}\times\vec{b}=(0,0,ab)$, $\vec{v}=\vec{b}\times\vec{c}=(x,b,0)\times(y,z,c)=(bc,-xc,xz-yb)$, and $\vec{w}=\vec{c}\times\vec{a}=(y,z,c)\times(a,0,0)=(0,ac,az)$. So

$$\begin{aligned}\mathcal{A}^2_{OAFB}+\mathcal{A}^2_{OBGC}+\mathcal{A}^2_{OCEA} &= \|\vec{u}\|^2+\|\vec{v}\|^2+\|\vec{w}\|^2\\ &= (ab)^2+((bc)^2+(xc)^2+(xz-yb)^2)+((ac)^2+(az)^2)\\ &= (ab)^2+(bc)^2+(ac)^2+(xc)^2+(xz-yb)^2+(az)^2.\end{aligned}$$

By the Arithmetic Mean-Geometric Mean inequality (see Problem 49 on Page 978 of [3]), we have $(ab)^2+(bc)^2+(ac)^2 \geq 3\sqrt[3]{(ab)^2(bc)^2(ac)^2}=3(abc)^{4/3}=3W^{4/3}$. The equality holds only when $ab=bc=ac$ or, equivalently, $a=b=c$. Because of the squared terms, we have $(xc)^2+(xz-yb)^2+(az)^2 \geq 0$, and $(xc)^2+(xz-yb)^2+(az)^2=0$ only when $x=y=z=0$. Therefore, we have shown that $\mathcal{A}^2_{OAFB}+\mathcal{A}^2_{OBGC}+\mathcal{A}^2_{OCEA} \geq 3W^{4/3}$. And the equality holds only when $a=b=c$ and $x=y=z=0$, i.e., when the parallelepiped $[OABC]$ is a cube. This proves the theorem. □

3 Tetrahedron

Alsina and Nelson restated Weitzenböck's Inequality in [1] as follows; if a triangle of three sides a, b, c has the area T, and if T_s denotes the area of an equilateral triangle with side length s, then we have $T_a+T_b+T_c \geq 3T$. They proved this inequality by raising three equilateral triangles having edges a, b, and c, respectively, outside of the given triangles.

Thus, it is tempting to prove our Theorem 2 in a similar way by raising four equifacial tetrahedra outside of the given tetrahedra. An equifacial tetrahedron is a tetrahedron with four congruent triangular faces, and it is a three-dimesional analog to an equilateral triangle. However, this does not seems to work. There is no equifacial tetrahedron having obtuse triangular faces. We will prove Theorem 3 as an application of Theorem 1 and Yetter's Theorem.

Definition 2. Let $[OABC]$ be a parallelepiped with labelings as in Figure 1. The tetrahedra $ABCD$ and $OFEG$ are said to be *embedded-tetrahedra* of a parallelepiped $[OABC]$. (The tetrahedra ABD and $OFEG$ are congruent but not identical. They are mirror images.) On the other hand, if $ABCD$ is a tetrahedron, then there are two parallel planes Γ_{AB} and Γ_{CD}, Γ_{AB} containing the edge AB, and Γ_{CD} containing the edge CD. Similarly, we can construct planes (Γ_{AC} and Γ_{BF}), and (Γ_{AF} and Γ_{CB}). Then the solid enclosed by these six planes form a parallelepiped $[OABC]$ as in Figure 1. Thus, we also say that the parallelepiped $[OABC]$ *embeds* the tetrahedron $ABCD$. Note that the statement of our Yetter's theorem is slightly different from one stated in [2], but they are equivalent. We give a short proof of this theorem.

Theorem 2 (Yetter's Theorem)**.** *Suppose $[OABC]$ is the parallelepiped that embeds a tetrahedron $ABCD$ as in Figure 1. Then, $\mathcal{A}^2_{ABC}+\mathcal{A}^2_{DAB}+\mathcal{A}^2_{DBC}+\mathcal{A}^2_{DCA} = \mathcal{A}^2_{OAFB}+\mathcal{A}^2_{OBGC}+\mathcal{A}^2_{OCEA}$.*

Proof. Let $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, $\vec{c} = \overrightarrow{OC}$, and $\vec{u} = \vec{a}\times\vec{b}$, $\vec{v} = \vec{b}\times\vec{c}$, $\vec{w} = \vec{c}\times\vec{a}$. Then $\mathcal{A}^2_{OAFB} + \mathcal{A}^2_{OBGC} + \mathcal{A}^2_{OCEA} = \|\vec{u}\|^2 + \|\vec{v}\|^2 + \|\vec{w}\|^2$. On the other hand, we have

$$\begin{aligned}
\mathcal{A}^2_{ABC} &= \frac{1}{4}\|(\vec{b}-\vec{a})\times(-\vec{a}+\vec{c})\|^2\\
&= \frac{1}{4}\|\vec{u}+\vec{v}+\vec{w}\|^2 = \frac{1}{4}(\|\vec{u}\|^2+\|\vec{v}\|^2+\|\vec{w}\|^2+2(\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w})),\\
\mathcal{A}^2_{DAB} &= \frac{1}{4}\|(\vec{b}-\vec{a})\times(\vec{b}+\vec{c})\|^2\\
&= \frac{1}{4}\|\vec{v}-\vec{u}+\vec{w}\|^2 = \frac{1}{4}(\|\vec{u}\|^2+\|\vec{v}\|^2+\|\vec{w}\|^2+2(-\vec{u}\cdot\vec{v}-\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w})),\\
\mathcal{A}^2_{DBC} &= \frac{1}{4}\|(\vec{a}+\vec{b})\times(\vec{b}-\vec{c})\|^2\\
&= \frac{1}{4}\|\vec{u}-\vec{v}+\vec{w}\|^2 = \frac{1}{4}(\|\vec{u}\|^2+\|\vec{v}\|^2+\|\vec{w}\|^2+2(-\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{w}+\vec{w}\cdot\vec{u})),\\
\mathcal{A}^2_{DCA} &= \frac{1}{4}\|(\vec{a}+\vec{b})\times(\vec{a}-\vec{c})\|^2\\
&= \frac{1}{4}\|-\vec{u}-\vec{v}+\vec{w}\|^2 = \frac{1}{4}(\|\vec{u}\|^2+\|\vec{v}\|^2+\|\vec{w}\|^2+2(\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{w}-\vec{w}\cdot\vec{u})).
\end{aligned}$$

Hence, we have $\mathcal{A}^2_{ABC}+\mathcal{A}^2_{DAB}+\mathcal{A}^2_{DBC}+\mathcal{A}^2_{DCA} = \|\vec{u}\|^2+\|\vec{v}\|^2+\|\vec{w}\|^2$. Therefore, we have shown that $\mathcal{A}^2_{ABC}+\mathcal{A}^2_{DAB}+\mathcal{A}^2_{DBC}+\mathcal{A}^2_{DCA} = \mathcal{A}^2_{OAFB}+\mathcal{A}^2_{OBGC}+\mathcal{A}^2_{OCEA}$. □

Now we are ready to state and prove Theorem 3.

Theorem 3. *Suppose the volume of a tetrahedron $ABCD$ is V. Then $\mathcal{A}^2_{ABC}+\mathcal{A}^2_{DAB}+\mathcal{A}^2_{DBC}+\mathcal{A}^2_{DCA} \geq 9\sqrt[3]{3}V^{4/3}$. The equality holds only when the tetrahedron $ABCD$ is regular.*

Proof. Let $[OABC]$ be the parallelepiped that embeds the tetrahedron $ABCD$. Let W be the volume of $[OABC]$. Then, since we know that $W = 3V$, we have $\mathcal{A}^2_{OADB}+\mathcal{A}^2_{OBGC}+\mathcal{A}^2_{OCEA} \geq$

$3W^{4/3} = 3(3V)^{4/3} = 9\sqrt[3]{3}V^{4/3}$ by Theorem 1. Therefore, $\mathcal{A}^2_{ABC} + \mathcal{A}^2_{DAB} + \mathcal{A}^2_{DBC} + \mathcal{A}^2_{DCA} = \mathcal{A}^2_{OADB} + \mathcal{A}^2_{OBGC} + \mathcal{A}^2_{OCEA} \geq 9\sqrt[3]{3}V^{4/3}$ by Yetter's Theorem. Note that a regular tetrahedron is the only tetrahedron that can be embedded in a cube. Hence, the equality holds only when the tetrahedron is regular. □

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