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On some Extensions of Morley's Trisector Theorem

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Abstract. We establish a simple generalization for the famous theorem of Morley about trisectors in a triangle with a purely synthetic proof using only angle chasing and similar triangles. Furthermore, based on the converse construction, some other extensions of Morley's Theorem are created and proven.

Key Words: Morley's trisector theorem, Morley's triangle, equilateral triangle, perspective triangle

MSC 2020: 51M04, 51-03

1 Introduction

Over one hundred years ago, Frank Morley introduced a geometric result. This result was so classic that Alexander Bogomolny once said "it entered mathematical folklore"; see [9]. Morley's marvelous theorem (Figure 1) reads:

Theorem 1 (Morley, 1899)**.** *The three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle.*

Many mathematicians consider Morley's Theorem to be one of the most beautiful theorems in plane Euclidean geometry. Throughout history, numerous proofs have been given; see [9, 10] and [1, 2, 4, 5, 7]. There was a generalization of Morley's Theorem using projective geometry in [6]. Some extensions to this theorem have been recently analyzed by Richard Kenneth Guy in [3]. Guy's extensions are very extensive and deep research on Morley's Theorem.

In the main part of this paper, we would like to offer and prove synthetically a simple generalization of Morley's Theorem. More precisely, we prove the following theorem (Figure 2).

Theorem 2 (A generalization of Morley's trisector theorem)**.** *Let ABC be a triangle. Assume that three points X, Y, Z, and the intersections $D = BZ \cap CY$, $E = CX \cap AZ$, $F = AY \cap BX$ lie inside the triangle ABC and satisfy the following conditions*

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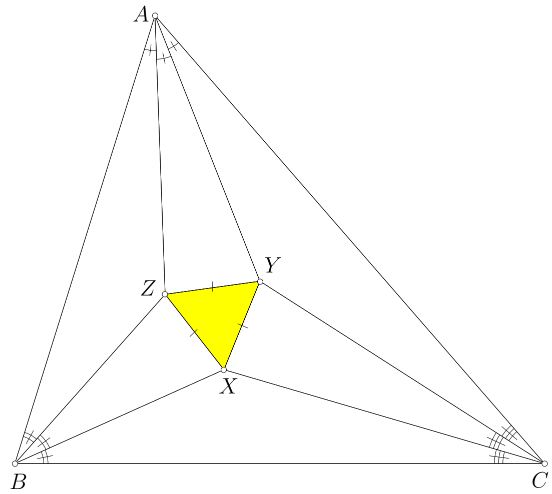

Figure 1: Morley's marvelous theorem

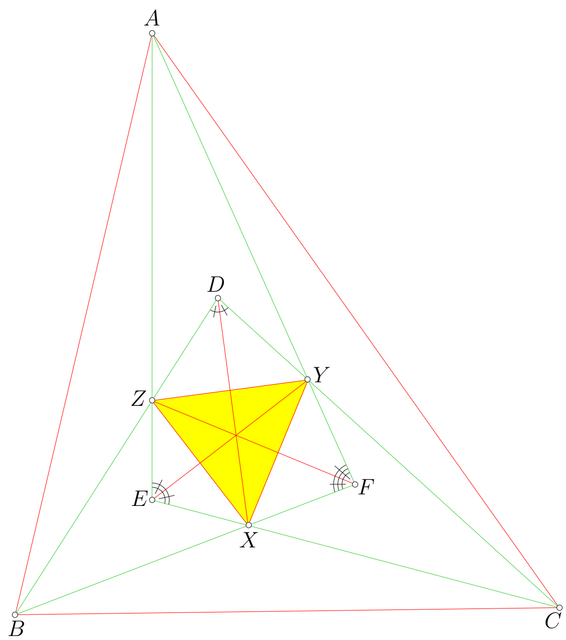

Figure 2: A generalization of Morley's trisector theorem

i) $\angle BXC = 120^\circ + \angle ZAY$, $\angle CYA = 120^\circ + \angle XBZ$, *and* $\angle AZB = 120^\circ + \angle YCX$.

ii) The points X, Y, *and* Z *lie on the interior bisectors of the angles* $\angle BDC$, $\angle CEA$, *and* $\angle AFB$, *respectively.*

Then, the triangle XYZ *is an equilateral triangle.*

Where X, Y, and Z are the intersections of the adjacent trisectors of triangle ABC, it is not hard to see that X, Y, and Z satisfy two conditions of Theorem 2. Thus, Theorem 2 is a direct generalization of Theorem 1. In addition, we prove the following theorem.

Theorem 3. *The triangles* ABC *and* XYZ *given in the Theorem 2 are perspective (Figure 4).*

In the last section of this paper, we shall apply a converse construction to find another extension of Morley's Theorem. Some new equilateral triangles in an arbitrary triangle are

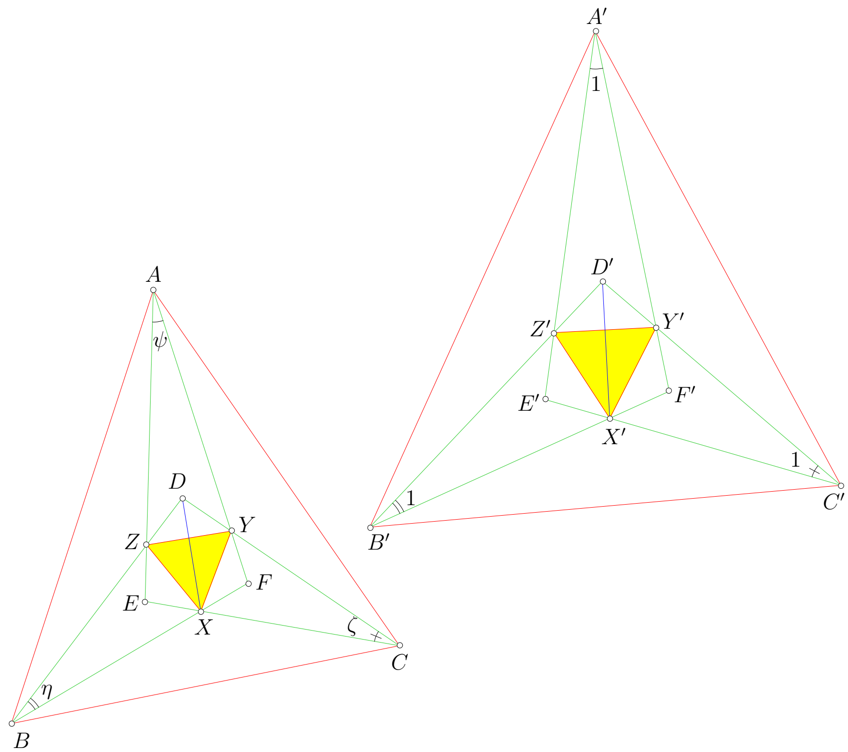

Figure 3: Proof of Theorem 2

also found. The family of these new equilateral triangles are closely related to the construction of Morley's equilateral triangle.

2 Proofs of the theorems

This section is entirely devoted to proofs of Theorem 2 and Theorem 3.

Proof of Theorem 2. The main idea of this proof comes from [10] (see Figure 3). First of all, let $\psi = \angle YAZ$, $\eta = \angle ZBX$, and $\zeta = \angle XCY$. Since X lies inside the triangle DBC (because X lies inside the triangle ABC and it lies on the interior bisector of angle $\angle BDC$, too), we have

$$\begin{aligned}\angle BDC &= \angle XDB + \angle XDC \\ &= 180^\circ - (\eta + \angle DXB) + 180^\circ - (\zeta + \angle DXC) \\ &= 360^\circ - (\angle DXB + \angle DXC) - \eta - \zeta \\ &= \angle BXC - \eta - \zeta \\ &= 120^\circ + \psi - \eta - \zeta.\end{aligned} \tag{1}$$

Similarly, we also have $\angle CEA = 120^\circ + \eta - \zeta - \psi$ and $\angle AFB = 120^\circ + \zeta - \psi - \eta$.

Next, on the sides of an equilateral triangle $X'Y'Z'$, the isosceles triangles $D'Z'Y'$, $E'X'Z'$, and $F'Y'X'$ are constructed externally such that $\angle Y'D'Z' = \angle YDZ$, $\angle Z'E'X' = \angle ZEX$, and $\angle X'F'Y' = \angle XFY$. Then, take the intersections $A' = E'Z' \cap F'Y'$, $B' = F'X' \cap D'Z'$, and $C' = D'Y' \cap E'X'$. Hence, from the quadrilateral $A'E'X'F'$, we deduce that

$$\angle A_1' = 360^\circ - \angle Z'E'X' - \left[\left(90^\circ - \frac{\angle Z'E'X'}{2}\right) + 60^\circ + \left(90^\circ - \frac{\angle X'F'Y'}{2}\right)\right] - \angle X'F'Y', \tag{2}$$

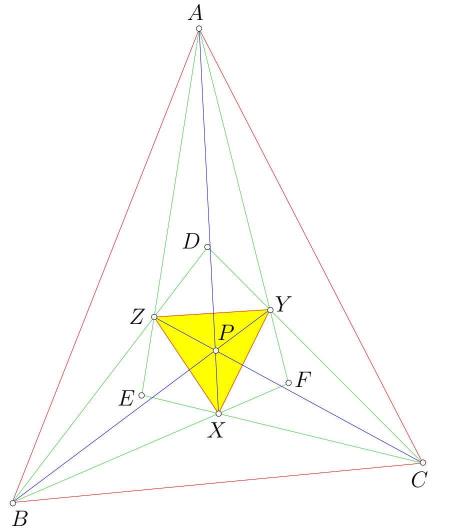

Figure 4: Perspective triangles

which implies that

$$\angle A_1' = 120^\circ - \frac{\angle Z'E'X' + \angle X'F'Y'}{2} = 120^\circ - \frac{\angle ZEX + \angle XFY}{2}$$
$$= 120^\circ - \frac{\angle AEC + \angle BFA}{2} = 120^\circ - \frac{240^\circ - 2\psi}{2} = \psi. \quad (3)$$

Similarly, one also has

$$\angle B_1' = \eta \quad \text{and} \quad \angle C_1' = \zeta. \quad (4)$$

Now, since $D'X'$ is an interior bisector of $\angle B'D'C'$ (from the constructions of the isosceles triangle $D'Z'Y'$ and the equilateral triangle $X'Y'Z'$), $\triangle DBX \sim \triangle D'B'X'$ (because they have equal angles η, $\frac{\angle BDC}{2}$), and $\triangle DCX \sim \triangle D'C'X'$ (because they have equal angles ζ, $\frac{\angle BDC}{2}$), it follows that

$$\frac{XB}{X'B'} = \frac{DX}{D'X'} = \frac{XC}{X'C'} \quad \text{or} \quad \frac{XB}{XC} = \frac{X'B'}{X'C'}, \quad (5)$$

and also

$$\angle B'X'C' = \angle B'D'C' + \angle B_1' + \angle C_1' = \angle BXC. \quad (6)$$

From (5) and (6), we deduce that $\triangle XBC \sim \triangle X'B'C'$ by SAS similarity theorem; see [8]. Analogously, $\triangle YCA \sim \triangle Y'C'A'$, and $\triangle ZAB \sim \triangle Z'A'B'$.

Finally, from these similar triangles, it follows that $\angle BAC = \angle B'A'C'$, $\angle CBA = \angle C'B'A'$, and $\angle ACB = \angle A'C'B'$. Thus, one obtains $\triangle ABC \sim \triangle A'B'C'$. This takes us to the conclusion that $\triangle XYZ \sim \triangle X'Y'Z'$, which yields XYZ is equilateral triangle, and hence the proof is complete. □

The above proof of Theorem 2 also shows that Morley's Theorem can be proven by simply using similar triangles and angle chasing in the same way. The barycentric coordinates will be used in the proof of Theorem 3.

Proof of Theorem 3. We assume that barycentric coordinates of points X, Y, and Z are as follows

$$X = (1:0:0)\,,\ Y = (0:1:0)\,,\ \text{and}\ Z = (0:0:1)\,. \quad (7)$$

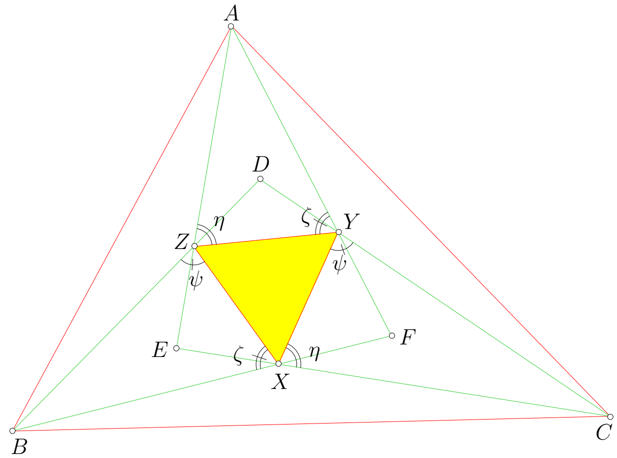

Figure 5: Inverse of Theorem 2

Because the points D, E, and F lie on the perpendicular bisector of sides BC, CA, and AB, respectively, we may assume that coordinates of D, E, and F are as follows

$$D = (-p : 1 : 1)\,, \;\; E = (1 : -q : 1)\,, \text{ and } F = (1 : 1 : -r)\,. \tag{8}$$

Now by using the equations of lines in barycentric coordinates, we obtain

$$A = (-1 : q : r)\,, \;\; B = (p : -1 : r)\,, \text{ and } C = (p : q : -1)\,. \tag{9}$$

Obviously, the lines AX, BY, and CZ concur at the point $P = (p : q : r)$, which finishes the proof. □

3 Some other extensions

In this section, some newly discovered equilateral triangles based on a given arbitrary triangle are found.

To describe more clearly, the equilateral triangle XYZ which is mentioned in Theorem 2, we will give an inverse of Theorem 2 (Figure 5).

Theorem 4 (Inverse of Theorem 2)**.** *Let XYZ be an equilateral triangle. Arbitrary isosceles triangles DYZ, EZX, and FXY are constructed externally of triangle XYZ with their bases on the sides of XYZ. Assume that the pairs of lines (EZ, FY), (FX, DZ), and (DY, EX) meet at A, B, and C, respectively, and the pairs of points (A, D), (B, E), and (C, F) are the same side with respect to the sides YZ, ZX, and XY, respectively. Then,*

$$\angle BXC = 120^\circ + \angle ZAY, \;\; \angle CYA = 120^\circ + \angle XBZ, \;\; \textit{and } \angle AZB = 120^\circ + \angle YCX. \tag{10}$$

Proof. Since X lies on the bisector of $\angle D$, and the sides XY, XZ of the equilateral triangle XYZ are isogonal with respect to sides DB, DC and so we denote by

$$\psi = \angle XZB = \angle XYC. \tag{11}$$

Analogously, since the equal angles, we denote by

$$\eta = \angle YXC = \angle YZA, \tag{12}$$

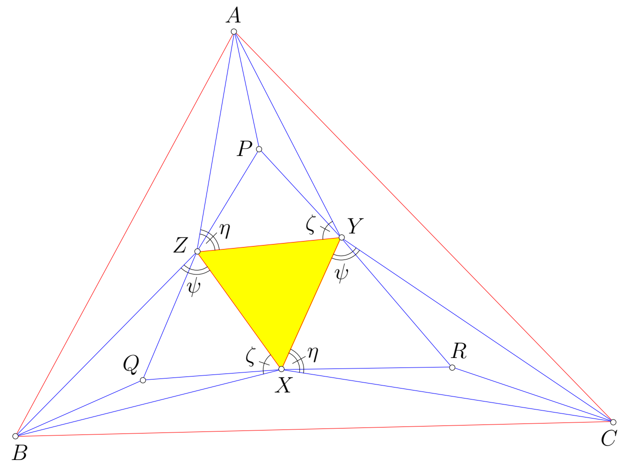

Figure 6: Addition to Theorem 4

and

$$\zeta = \angle ZYA = \angle ZXB. \tag{13}$$

We have

$$\angle BXC = 360^\circ - \eta - 60^\circ - \zeta = 120^\circ + (180^\circ - \eta - \zeta) = 120^\circ + \angle ZAY. \tag{14}$$

Similarly, $\angle CYA = 120^\circ + \angle XBZ$ and $\angle AZB = 120^\circ + \angle YCX$, which finishes the proof. □

On the configuration of Theorem 4, we see a following property (Figure 6):

Theorem 5 (Addition to Theorem 4)**.** *Using the hypothesis as Theorem 4, let P, Q, and R be the circumcenters of triangles AYZ, BZX, and CXY, respectively. Assume that P, Q, and R lie inside triangle AYZ, BZX, and CXY, respectively. Then,*

$$\angle BXR = \angle CXQ = \angle CYP = \angle AYR = \angle AZQ = \angle BZP. \tag{15}$$

Proof. As in the proof of Theorem 4, let

$$\psi = \angle XZB = \angle XYC. \tag{16}$$

Analogously, since the equal angles, let

$$\eta = \angle YXC = \angle YZA, \tag{17}$$

and

$$\zeta = \angle ZYA = \angle ZXB. \tag{18}$$

From these,

$$\angle CXR = \angle BXQ = 90^\circ - \psi, \tag{19}$$

so

$$\angle BXR = \angle QXC = 360^\circ - \eta - \zeta - 60^\circ + 90^\circ - \psi = 390^\circ - \psi - \eta - \zeta. \tag{20}$$

This means that there are six equal angles

$$\angle BXR = \angle CXQ = \angle CYP = \angle AYR = \angle AZQ = \angle BZP. \tag{21}$$

This completes proof. □

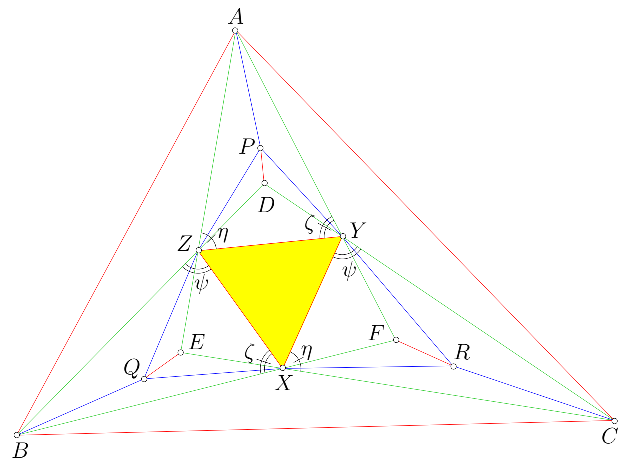

Figure 7: Another extension of Morley's Theorem

At this point, if we use the conclusion of Theorem 5 as hypothesis, we propose another extension of Morley's Theorem as follows (Figure 7):

Theorem 6 (Another extension of Morley's Theorem)**.** *Let ABC be a triangle. Assume that three points X, Y, and Z lie inside triangle ABC such that*

$$\angle BXR = \angle CXQ = \angle CYP = \angle AYR = \angle AZQ = \angle BZP, \tag{22}$$

where P, Q, and R are circumcenters of triangles AYZ, BZX, and CXY, respectively. Assume that P, Q, and R lie inside triangle AYZ, BZX, and CXY, respectively. Then, triangle XYZ is an equilateral triangle.

Proof. By the hypothesis, we conclude that $\angle CXR = \angle QXB$, which leads to

$$90^\circ - \angle XYC = 90^\circ - \angle BZX, \tag{23}$$

or

$$\angle XYC = \angle BZX, \tag{24}$$

so let

$$\psi = \angle XZB = \angle XYC \tag{25}$$

Analogously, since the equal angles, let

$$\eta = \angle YXC = \angle YZA, \tag{26}$$

and

$$\zeta = \angle ZYA = \angle ZXB. \tag{27}$$

Therefore, one has

$$\angle BXR = \angle CXQ = 360^\circ - \eta - \zeta - \angle ZXY + 90^\circ - \psi = 450^\circ - \psi - \eta - \zeta - \angle ZXY. \tag{28}$$

Hence, we get

$$\angle CYP = \angle AYR = 450^\circ - \psi - \eta - \zeta - \angle XYZ, \tag{29}$$

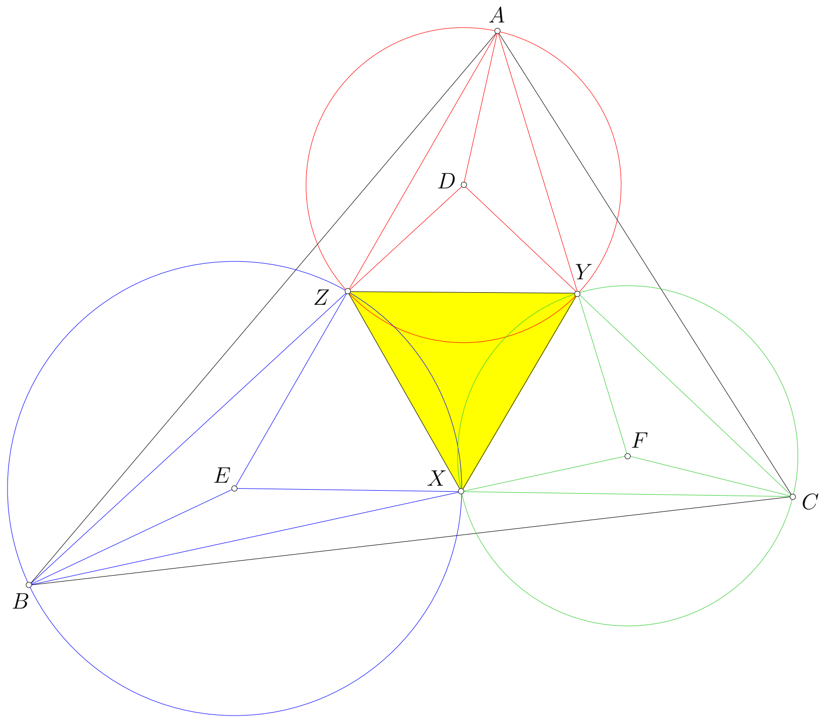

Figure 8: A consequence of Theorem 6

and

$$\angle AZQ = \angle BZP = 450^\circ - \psi - \eta - \zeta - \angle YZX. \tag{30}$$

Combining (28), (29), and (30) with six equal angles of the hypothesis (22), we conclude that

$$\angle ZXY = \angle XYZ = \angle YZX. \tag{31}$$

Thus the triangle XYZ is equilateral. □

Note that the equilateral triangle XYZ in Theorem 6 will become Morley triangle if we add the following conditions

$$\angle BXR = \angle CXQ = \angle CYP = \angle AYR = \angle AZQ = \angle BZP = 150^\circ. \tag{32}$$

We end this section by an interesting consequence of Theorem 6 in which all six equal angles (in Theorem 6) are 180° (see Figure 8).

Theorem 7 (A consequence of Theorem 6)**.** *Assume that three points X, Y, and Z lie inside a given triangle ABC and satisfying the following conditions*

- *BZ and CY meet at D lying inside triangle AYZ and D is the circumcenter of triangle AYZ.*
- *CX and AZ meet at E lying inside triangle BZX and E is the circumcenter of triangle BZX.*
- *AY and BX meet at F lying inside triangle CXY and F is the circumcenter of triangle CXY.*

Then, the triangle XYZ is an equilateral triangle.

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