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On CR-Structures and the General Quadratic Structure

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Abstract. The object of the present paper is to determine the relationship between CR-structure and the general quadratic structure and find some basic results. We discuss integrability conditions and prove certain theorems on CR-structure and the general quadratic structure.

Key Words: CR-structure, general quadratic structure, almost complex structure, Nijenhuis tensor, integrability

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1 Introduction

The theory of the tangent bundle of submanifolds is an interesting topic in the differential geometry. There are three types of submanifolds concerning the almost complex structure of the ambient manifold, namely, given as holomorphic submanifolds, totally real submanifolds, and CR-(Cauchy-Riemannian) submanifolds. Bejancu [2] has studied CR-submanifold of a Kählerian manifold in which a new class of submanifolds of the complex manifold was initiated. Bejancu has introduced the concept of CR-submanifold and gave its basic properties. Numerous investigators made valuable contributions to CR-submanifolds including Bejancu [1], Blair and Chen [3], Chen [4], Dragomir at el [6] and Yao and Kon [9]. In this paper, we study the integrability conditions and Nijenhuis tensor on CR-structures and the general quadratic structure.

Let us consider the general quadratic equation $x^2 + \alpha x + \beta = 0$, α, β are integers. The set of solutions denoted by $x = \frac{1}{2}(-\alpha \pm \sqrt{\alpha^2 - 4\beta})$. In n-dimensional manifold M, suppose a tensor field $F(\neq 0)$ of the type $(1,1)$ and of class C^∞ on M such that

$$F^2 + \alpha F + \beta I = 0 \tag{1}$$

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such structure on M is called the general quadratic structure of rank r. If the rank of F is constant and $r = r(F)$, then M is called the general quadratic manifold.

Let us introduce the operators as follows

$$l = -\frac{F^2 + \alpha F}{\beta}, \quad m = I + \frac{F^2 + \alpha F}{\beta} \tag{2}$$

where I denotes the identity operator on M.

Proposition 1.1. *Let M be the general quadratic manifold. Then*

$$l + m = I, \quad l^2 = l, \quad \text{and} \quad m^2 = m. \tag{3}$$

Proof. In the view of Equation (2), the proof is trivial. □

For $F \neq 0$ satisfying Equation (1), there exist complementary distributions D_l and D_m corresponding to the projection operators l and m respectively. If the rank(F) = constant and $r = r(F)$ on M, then $\dim D_l = r$ and $\dim D_m = n - r$ [5, 7].

Proposition 1.2. *Let M be the general quadratic manifold. Then*

$$Fl = lF = F, \quad Fm = mF = 0 \tag{4}$$

$$\frac{F^2 + \alpha F}{\beta} = -l, \quad \frac{F^2 + \alpha F}{\beta} l = -l, \quad \frac{F^2 + \alpha F}{\beta} m = 0. \tag{5}$$

Thus $\left(\frac{F^2+\alpha F}{\beta}\right)^{\frac{1}{2}}$ acts on D_l as an almost complex structure and on D_m as a null operator.

Proof. In the view of Equation (1), the proof is trivial. □

2 Nijenhuis tensor

Definition 2.1. *If X, Y are two vector fields in M, then their Lie bracket $[X, Y]$ is given by [5]*

$$[X, Y] = XY - YX. \tag{6}$$

The Nijenhuis tensor $N(X, Y)$ of F satisfying Equation (1) in M is expressed as follows

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y], \tag{7}$$

for every vector field X, Y on M.

We state the following proposition [8]:

Proposition 2.2. *A necessary and sufficient condition for the general quadratic structure F to be integrable is that $N(X, Y) = 0$ for any two vector fields X and Y on M.*

3 CR-structure

Let M be a differentiable manifold and T_cM its complexified tangent bundle. A CR-structure on M is a complex subbundle H of T_cM such that $H_p \cap \bar{H}_p = 0$ and H is involutive, i. e., for complex vector fields X and Y in $H, [X,Y]$ is in H. In this case we say M is a CR-manifold. Let F be the general quadratic integrable structure satisfying Equation (1) of rank $r = 2m$ on M. We define complex subbundle H of T_cM by $H_p = \{X - \sqrt{-1}FX, X \in \chi(D_l)\}$, where $\chi(D_l)$ is the $\wp(D_m)$ module of all differentiable sections of D_l. Then $\mathrm{Re}(H) = D_l$ and $H_p \cap \bar{H}_p = 0$, where $\bar{H}_p$ denotes the complex conjugate of H [5].

Proposition 3.1. *Let $P, Q \in H$, then we have*

$$[P,Q] = [X,Y] - [FX,FY] - \sqrt{-1}([X,FY] + [FX,Y]), \tag{8}$$

for every vector field X, Y on M.

Proof. Consider $P = X - \sqrt{-1}FX$ and $Q = Y - \sqrt{-1}FY$. Then we have

$$\begin{aligned}[P,Q] &= [X - \sqrt{-1}FX, Y - \sqrt{-1}FY] \\ &= [X,Y] - [FX,FY] - \sqrt{-1}([X,FY] + [FX,Y]).\end{aligned}$$ □

Proposition 3.2. *If the general quadratic structure satisfying Equation* (1) *is integrable, then we have*

$$(F + \alpha I)([FX,FY] + F^2[X,Y]) = -\beta l([FX,Y] + [X,FY]), \tag{9}$$

for every vector field X, Y on M.

Proof. Since $N(X,Y) = 0$, from (7) we obtain

$$[FX,FY] + F^2[X,Y] = F([FX,Y] + [X,FY]). \tag{10}$$

Operating Equation (10) by $\frac{F+\alpha I}{\beta}$, we get

$$\frac{(F + \alpha I)}{\beta}([FX,FY] + F^2[X,Y]) = \frac{(F^2 + \alpha F)}{\beta}([FX,Y] + [X,FY]). \tag{11}$$

Making use of Equation (2), we obtain

$$(F + \alpha I)([FX,FY] + F^2[X,Y]) = -\beta l([FX,Y] + [X,FY]). \tag{12}$$

This completes the proof. □

Theorem 3.3. *The following conditions are equivalent*

$$\begin{aligned}
&(i) && mN(X,Y) = 0, \\
&(ii) && m[FX,FY] = 0, \\
&(iii) && mN\left(\frac{F^2 + \alpha F}{\beta}X, Y\right) = 0, \\
&(iv) && m\left[\frac{F^2 + \alpha F}{\beta}FX, FY\right] = 0, \\
&(v) && m\left[\frac{F^2 + \alpha F}{\beta}lFX, FY\right] = 0,
\end{aligned}$$

where X and Y are vector fields.

Proof. We have to show that $(i) \iff (ii)$. Let $mN(X,Y)=0$. Then in the view of Equation (7), we have

$$mN(X,Y)=m[FX,FY]-mF[FX,Y]-mF[X,FY]+mF^2[X,Y].$$

Since $mF=0$, we have $mN(X,Y)=m[FX,FY]$. Since $mN(X,Y)=0$, then $m[FX,FY]=0$. Thus, $mN(X,Y)=0$ if and only if $m[FX,FY]=0$.

$(ii) \iff (iii)$: Let $m[FX,FY]=0$. Then in the view of Equation (7), we have

$$mN\Big(\frac{F^2+\alpha F}{\beta}X,Y\Big)=m\Big[\frac{F^2+\alpha F}{\beta}FX,FY\Big]-mF\Big[\frac{F^2+\alpha F}{\beta}FX,Y\Big]-mF\Big[\frac{F^2+\alpha F}{\beta}X,FY\Big]+mF^2\Big[\frac{F^2+\alpha F}{\beta}X,Y\Big].$$

Since, $\frac{F^2+\alpha F}{\beta}=-l$, $mF=0$, then above equation becomes

$$mN\Big(\frac{F^2+\alpha F}{\beta}X,Y\Big)=m[-lFX,FY]=-m[FX,FY] \quad \text{as} \quad lF=F.$$

Since $m[FX,FY]=0$, then we have

$$mN\Big(\frac{F^2+\alpha F}{\beta}X,Y\Big)=0.$$

Thus, $m[FX,FY]=0$ if and only if $mN(\frac{F^2+\alpha F}{\beta}X,Y)=0$.

$(iii) \iff (iv)$: Let $mN(\frac{F^2+\alpha F}{\beta}X,Y)=0$. Using $mF=0$, then we have

$$mN\Big(\frac{F^2+\alpha F}{\beta}X,Y\Big)=m\Big[\frac{F^2+\alpha F}{\beta}FX,FY\Big]=0,$$

whence $mN(\frac{F^2+\alpha F}{\beta}X,Y)=0$ indeed implies $m[\frac{F^2+\alpha F}{\beta}FX,FY]=0$.

$(iv) \iff (v)$: Let $m[\frac{F^2+\alpha F}{\beta}FX,FY]=0$. Using $lF=F$, $Fm=0$, then we have

$$mN\Big(\frac{F^2+\alpha F}{\beta}lX,Y\Big)=m\Big[\frac{F^2+\alpha F}{\beta}lFX,FY\Big]=m\Big[\frac{F^2+\alpha F}{\beta}FX,FY\Big], \quad \text{as} \quad lF=F.$$

So

$$m\Big[\frac{F^2+\alpha F}{\beta}lFX,FY\Big]=m\Big[\frac{F^2+\alpha F}{\beta}FX,FY\Big], \quad \text{as} \quad m\Big[\frac{F^2+\alpha F}{\beta}FX,FY\Big]=0$$

and

$$m\Big[\frac{F^2+\alpha F}{\beta}FX,FY\Big]=0 \quad \text{implies} \quad m\Big[\frac{F^2+\alpha F}{\beta}lFX,FY\Big]=0.$$

$(v) \iff (i)$: Let $m[\frac{F^2+\alpha F}{\beta}lFX,FY]=0$. Making use of (2) and (3), we obtain

$$m[-l^2FX,FY]=0, \quad -m[lFX,FY]=0', \quad m[FX,FY]=0, \quad \text{as} \quad lF=F.$$

Since $mN(X,Y)=m[FX,FY]=0$, we have $mN(X,Y)=0$. Thus, $m[\frac{F^2+\alpha F}{\beta}lFX,FY]=0$ implies $mN(X,Y)=0$. This completes the proof. $\square$

Proposition 3.4. *If* $\left(\frac{F^2+\alpha F}{\beta}\right)^{\frac{1}{2}}$ *acts on* D_l *as an almost complex structure, then*

$$m\left[\frac{F^2+\alpha F}{\beta}lX, FY\right] = m[-X, FY] = 0 \tag{13}$$

for every vector field X, Y *on* M.

Proof. From Equation (5), we know that $(\frac{F^2+\alpha F}{\beta})^{\frac{1}{2}}$ acts on D_l as an almost complex structure then Equation (13) follows in an obvious manner. To prove that $m[\frac{F^2+\alpha F}{\beta}lX, FY] = 0$, by using the formula $[X,Y] = XY - YX$ where X, Y are C^∞ vector fields and Equation (5), we obtain Equation (13). □

Proposition 3.5. *For* $X, Y \in \chi(D_l)$, *we have*

$$l([X, FY] + [FX, Y]) = [X, FY] + [FX, Y].$$

Proof. By Definition 2.1, we have

$$[X, Y] = XY - YX.$$

Now,

$$\begin{aligned} l[X, FY] &= l(XFY - FYX) = X(lFY) - (lF)YX \\ &= XFY - FYX, \quad \text{as} \quad lF = F, \quad \text{using Equation (4)}, \\ &= [X, FY]. \end{aligned}$$

Similarly, $l[FX, Y] = [FX, Y]$. Hence,

$$l([X, FY] + [FX, Y]) = [X, FY] + [FX, Y].$$

□

Theorem 3.6. *The integrable general quadratic structure satisfying Equation* (1) *on* M *defines a CR-structure* H *on it such that* $\operatorname{Re} H \equiv D_l$.

Proof. To prove the general quadratic equation satisfies Equation (1) defines CR-structure on M it suffices to prove $[P, Q] \in \chi(D_l)$.

From Equation (8), we have

$$[P, Q] = [X, Y] - [FX, FY] - \sqrt{-1}([X, FY] + [FX, Y]). \tag{14}$$

Now,

$$\begin{aligned} [P, Q] - \sqrt{-1}F[P, Q] &= [X, Y] - [FX, FY] - \sqrt{-1}([X, FY] + [FX, Y]) \\ &\quad - \sqrt{-1}F([X, Y] - [FX, FY]) - F([X, FY] + [FX, Y]). \end{aligned}$$

Making use of Theorem (3.5) and Equation (9), we obtain

$$\begin{aligned} [P, Q] - \sqrt{-1}F[P, Q] &= [X, Y] - [FX, FY] - \sqrt{-1}\left(\frac{F+\alpha I}{-\beta}\right)([FX, FY] + F^2[X, Y]) \\ &\quad - \sqrt{-1}F([X, Y] - [FX, FY] - \sqrt{-1}\left(\frac{F+\alpha I}{-\beta}\right)([FX, FY] + F^2[X, Y])). \end{aligned}$$

By definition of CR-structure, we have $[P, Q] \in \chi(D_l)$. This completes the proof. □

Definition 3.7. *Let $\tilde{K}$ be the complementary distribution of* $\mathrm{Re}(H)$ *to* TM*. We define a morphism of vector bundles* $F : TM \to TM$ *given by* $F(X) = 0$ *for all* $X \in \chi(\tilde{K})$*, such that*

$$F(X) = \frac{1}{2}\sqrt{-1}(P - \bar{P}) \tag{15}$$

where $P = X + \sqrt{-1}Y \in \chi(H_p)$ *and* $\bar{P}$ *is a complex conjugate of* P *[5].*

Corollary 3.8. *If* $P = X+\sqrt{-1}Y$ *and* $\bar{P} = X-\sqrt{-1}Y$ *belong to* H_p *and* $F(X) = \frac{1}{2}\sqrt{-1}(P-\bar{P}), F(Y) = \frac{1}{2}(P + \bar{P})$ *and* $F(-Y) = -\frac{1}{2}(P + \bar{P})$*, then* $F(X) = -Y, F^2(X) = -X$ *and* $F(-Y) = -X$*.*

Proof. On using Definition 3.7, we have

$$\begin{aligned} F(X) &= \frac{1}{2}\sqrt{-1}(X + \sqrt{-1}Y - (X - \sqrt{-1}Y)) = \frac{1}{2}\sqrt{-1}(2\sqrt{-1}Y), \\ F(X) &= -Y. \end{aligned} \tag{16}$$

Applying F to both sides of Equation (16), we obtain

$$F(F(X)) = F(-Y). \tag{17}$$

But

$$F(Y) = \frac{1}{2}(X + \sqrt{-1}Y + X - \sqrt{-1}Y),$$

which on simplifying gives $F(Y) = X$. Also,

$$F(-Y) = -\frac{1}{2}(X - \sqrt{-1}Y + X + \sqrt{-1}Y) = -X. \tag{18}$$

Combining Equations (17) and (18), we get $F^2(X) = -X$. □

Theorem 3.9. *If* M *has a CR-structure* H*, then we have* $F^2+\alpha F+\beta I = 0$ *and consequently the general quadratic structure is defined on* M *such that the distributions* D_l *and* D_m *coincide with* $\mathrm{Re}(H)$ *and* $\tilde{K}$ *respectively.*

Proof. Suppose M has a CR-structure on M. Then in view of Definition 3.7 and Corollary 3.8 we can write [5]

$$F(X) = -Y. \tag{19}$$

Operating Equation (19) by $\frac{F+\alpha I}{\beta}$, we get

$$\begin{aligned} \frac{F+\alpha I}{\beta}F(X) &= \frac{F+\alpha I}{\beta}(-Y) = \frac{F(-Y)+\alpha(-Y)}{\beta} \\ &= \frac{-X-\alpha Y}{\beta} = \frac{-X+\alpha F(X)}{\beta}, \quad \text{as} \quad FX = -Y, \\ &= \frac{F^2(X)+\alpha F(X)}{\beta}, \quad \text{as} \quad F^2X = -X, \\ &= \frac{-\beta X}{\beta}, \quad \text{as} \quad F^2+\alpha F+\beta I = 0, \\ \frac{F+\alpha I}{\beta}F(X) &= -X, \\ (F+\alpha I)F &= -\beta I. \end{aligned}$$

Hence, $F^2 + \alpha F + \beta I = 0$. □

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