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Permutation Ellipses

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Abstract. We use homogeneous coordinates in the plane of a triangle to define a family of ellipses having the centroid of the triangle as center. The family, which includes the Steiner circumscribed and inscribed ellipses, is closed under many operations, including permutation of coordinates, complements and anticomplements, duality, and inversion.

Key Words: barycentric coordinates, Steiner circumellipse, Steiner inellipse, complement and anticomplement, dual conic, inversion in ellipse, Frégier ellipse

MSC 2020: 51N20, 51N15

1 Introduction

In 1827, something revolutionary happened in triangle geometry. A descendant of Martin Luther named Möbius introduced barycentric coordinates. Suddenly, points and lines in the plane of a triangle ABC with sidelengths a, b, c had a new kind of representation, and traditional geometric relationships found new algebraic kinds of expression. The lengths a, b, c came to be regarded as variables or indeterminates, and a criterion for collinearity of three points became the zeroness of a determinant, as did the concurrence of three lines.

More recently, algebraic concepts have been *imposed* on the objects of triangle geometry. You can make up a suitable function $f(a, b, c)$ and say "Let P be the point with barycentric coordinates $f(a, b, c) : f(b, c, a) : f(c, a, b)$" and then use algebra to find geometrically notable properties of P. For example, the Conway point, $a^4 + b^2c^2 : b^4 + c^2a^2 : c^4 + a^2b^2$, is proved by a zero determinant to lie on the Euler line of ABC. (John Horton Conway made several nifty contributions to triangle geometry.)

The purpose of this article is to introduce another example of the imposing of algebraic concepts and methods in triangle geometry. For the reader already familiar with barycentric coordinates, the example can be stated as follows. Formally, a point $P = p : q : r$ belongs to a set of six *permutation points*:

$$p : q : r, \quad p : r : q, \quad q : r : p, \quad q : p : r, \quad r : p : q, \quad r : q : p.$$

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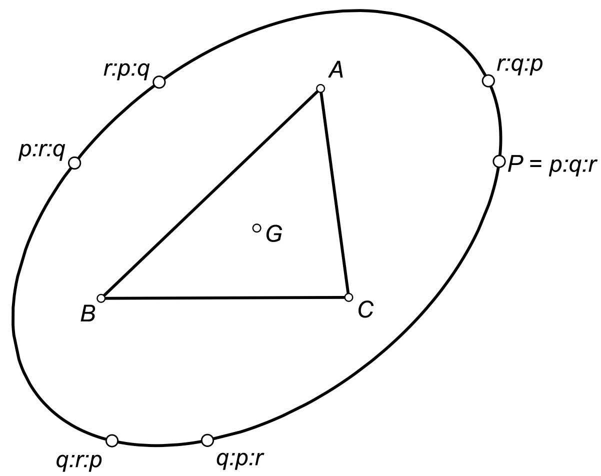

Figure 1: Permutation Ellipse, E(P)

The fact that a certain determinant equals zero proves that the six points lie on a conic—which turns out to be an ellipse having as center the centroid, G, of ABC; see Figure 1. If we include G as a degenerate ellipse, then the family of these *permutation ellipse*s partitions the plane of ABC. *Every* point (e.g., incenter, circumcenter, Fermat point) lies on exactly one permutation ellipse.

Now let's back up to say what it means for a point P to have barycentric coordinates $p : q : r$. Let $\sigma = \text{area}(ABC)$. Then using

$$\sigma_1' = \text{area}(PBC), \quad \sigma_2' = \text{area}(PCA), \quad \sigma_3' = \text{area}(PAB),$$

we define *signed* areas, depending on the fact that the line BC separates the plane into a positive region, containing A, and a negative region—the half-plane not containing A. If P lies in the negative region, then its signed area is $-\sigma_1'$, denoted by σ_1. Otherwise, $\sigma_1 = \sigma_1'$, and similarly for σ_2 and σ_3. The triple $(\sigma_1/\sigma, \sigma_2/\sigma, \sigma_3/\sigma)$, called the *normalized barycentric coordinates* of P, satisfy the simple equation

$$x + y + z = 1.$$

Any triple (p, q, r) proportional to $(\sigma_1, \sigma_2, \sigma_3)$ are called *homogeneous barycentric coordinates*, or simply *barycentrics*, for P. Such a triple is written as $p : q : r$. For example, the normalized barycentrics for G are $(1/3, 1/3, 1/3)$, but also, we can represent G as

$$G = 1 : 1 : 1 = 2 : 2 : 2 = abc : abc : abc.$$

As a second example, the circumcenter (where the perpendicular bisectors of the sides of ABC meet) is

$$O = \sin 2A : \sin 2B : \sin 2C, \text{ alias } a^2(b^2 + c^2 - a^2) :: .$$

You can see here a switch to algebra (a homogeneous polynomial of degree 4), as well as the sufficiency of only the first of the three barycentrics, followed by two colons, if certain algebraic properties are understood. Specifically, if the first barycentric is $f(a, b, c)$, then the second and third are $f(b, c, a)$ and $f(c, a, b)$, respectively. Also, we follow the common practice of using the symbols A, B, C in two different ways: for the vertices of the reference triangle ABC, and also for the angles: $A = \angle CAB$, $B = \angle ABC$, $C = \angle BCA$.

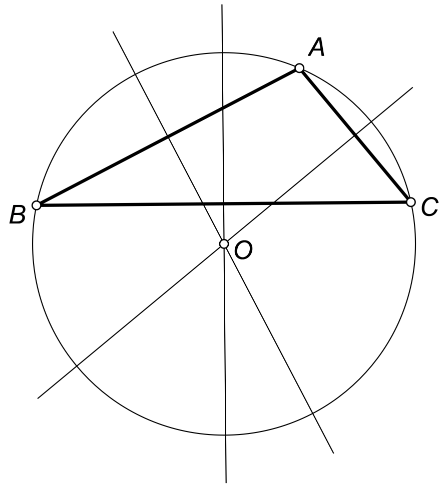

Figure 2: Circumcenter, O

Near the end of the second millennium, Philip J. Davis [1] made some observations and predictions concerning triangle geometry. Along with the influence of algebraic and function-theoretic methods is the role of computers, or, more specifically, powerful programs such as *Mathematica*. Many deep results in triangle geometry depend on computer-algebra systems to simplify elaborate expressions. One source of the complexity of these expressions is the distance between two points. If they are written as $P = p : q : r$ and $U = u : v : w$, then the distance between them is given ([8], p. 90) by

$$|PU| = \frac{1}{(p+q+r)(u+v+w)}\sqrt{\Psi}, \text{ where} \tag{1}$$
$$\Psi = S_A[(v+w)p - u(q+r)]^2 + S_B[(w+u)q - v(r+p)]^2 + S_C[(u+v)r - w(p+q)]^2,$$
$$S_A = (b^2+c^2-a^2)/2, S_B = (c^2+a^2-b^2)/2, S_C = (a^2+b^2-c^2)/2.$$

Computer-dependent reduction of elaborate expressions leads to a certain style for presenting proofs of theorems, in which the steps in a proof are essentially instructions to be followed by a computer. This should be kept in mind regarding some of the proofs in this article.

2 First facts

We call a point P a *regular point* if it is not one of these four:

$$A = 1:0:0, \quad B = 0:1:0, \quad C = 0:0:1, \quad G = 1:1:1.$$

Theorem 1. *Suppose that $P = p : q : r$ is a regular point. The six points*

$$p:q:r, \ \ p:r:q, \ \ q:r:p, \ \ q:p:r, \ \ r:p:q, \ \ r:q:p \tag{2}$$

lie on the curve

$$(qr+rp+pq)(x^2+y^2+z^2) - (p^2+q^2+r^2)(yz+zx+xy) = 0, \tag{3}$$

which is an ellipse, $E(P)$, with center G. If $U = u : v : w$ is a point on $E(P)$, then $E(U) = E(P)$.

Proof. The equation (3) has the form

$$fx^2 + gy^2 + hz^2 + 2f'yz + 2g'zx + 2h'xy = 0,$$

so that it represents a conic (e.g., [8], p. 117). Clearly each of the six permutation points satisfies (3). Using a type detector and equation for the center of a conic ([8], p. 127), we conclude that the points (2) lie on an ellipse with center G.

Now suppose that a point $U = u : v : w$ lies on $E(P)$. Then by (3),

$$\frac{qr + rp + pq}{p^2 + q^2 + r^2} = \frac{vw + wu + uv}{u^2 + v^2 + w^2}. \tag{4}$$

To see that $E(U) = E(P)$, suppose that $X = x : y : z$ is a point on $E(U)$, so that

$$\frac{yz + zx + xy}{x^2 + y^2 + z^2} = \frac{vw + wu + uv}{u^2 + v^2 + w^2}.$$

This equation and (4) imply (3), so that $E(U) \subseteq E(P)$. Likewise, $E(P) \subseteq E(U)$. □

If $qr + rp + pq \neq 0$, then (3) can be written as

$$x^2 + y^2 + z^2 - t(yz + zx + xy) = 0, \tag{5}$$

where

$$t = \frac{p^2 + q^2 + r^2}{qr + rp + pq} \tag{6}$$

is a function $t = t(a, b, c)$. It is natural to ask, if we start with an equation of the form (5), what conditions on t result in a permutation ellipse? We consider this only in the case that t is a constant, as in the next theorem.

Theorem 2. *If t is a real number, then the equation* (5) *represents a permutation ellipse if and only if $t < -2$ or $t > 1$.*

Proof. Putting $y = z$ in (5), we find

$$x = y(t \pm \sqrt{(t+2)(t-1)}) \tag{7}$$

which is real if and only if $t \leq -2$ or $t \geq 1$. However, if $t = -2$, then (5) reduces to $x + y + z = 0$, not an ellipse, and if $t = 1$, the result is the single point $G = 1 : 1 : 1$.

For the converse, if $t < -2$ or $t > 1$, then there must exist k close enough to 1 that the line $y = kz$ meets the ellipse in a point $x : y : z$ such that x, y, z are distinct, so that the six points

$$x : y : z, \quad x : y : z, \quad y : z : x, \quad y : x : z, \quad z : x : y, \quad z : y : x$$

are distinct. It is clear from the symmetry of x, y, z in (5) that all six points lie on the ellipse, so that it is a permutation ellipse. □

The next theorem may be a surprise—that for any pair of permutation ellipses, each is a dilation from G of the other! Equivalently, we shall prove that every permutation ellipse is a dilation of the Steiner circumellipse, to be discussed in Section 3, given by

$$yz + zx + xy = 0. \tag{8}$$

Let A' denote the point where the ray GA meets $E(P)$. Then $|GA'|/|GA|$ is the *dilation factor* for $E(P)$. Each point U on $E(A)$, which is the Steiner circumellipse, is dilated by the factor $|GA'|/|GA|$ to a point on $E(A')$, which is also $E(P)$.

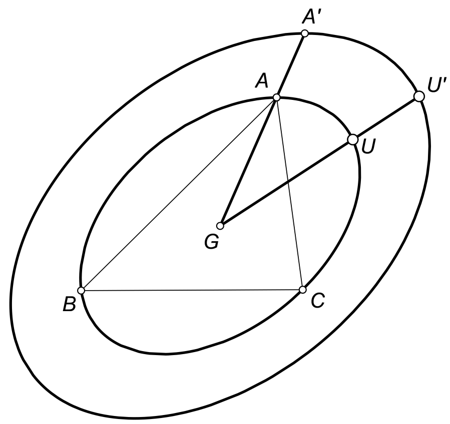

Figure 3: Configuration for Theorem 2

Theorem 3. *Suppose $E(P)$ is a permutation ellipse given by*

$$x^2 + y^2 + z^2 - t(yz + zx + xy) = 0.$$

Then the dilation factor for $E(P)$ is

$$\sqrt{\frac{t-1}{t+2}}. \tag{9}$$

Proof. Let A' be the point where the ray PA meets $E(P)$, so that A' is given by normalized barycentrics

$$(\frac{\Delta}{2+\Delta}, \frac{1}{2+\Delta}, \frac{1}{2+\Delta}),$$

where $\Delta = t + \sqrt{(t-1)(t+2)}$. The distance formula (1) gives

$$|GA|^2 = (1/9)(4S_A + S_B + S_C),$$

and

$$\begin{aligned} |GA'|^2 &= S_A\left(\frac{\Delta}{2+\Delta} - \frac{1}{3}\right)^2 + S_B(\frac{1}{2+\Delta} - \frac{1}{3})^2 + S_C(\frac{1}{2+\Delta} - \frac{1}{3})^2 \\ &= (\frac{\Delta - 1}{3(\Delta+2)})^2(4S_A + S_B + S_C) \end{aligned}$$

Simplifying, we obtain $|GA'|/|GA|$ as in (9). □

Regarding Theorem 3, note first that the domain is the union of the intervals $t < 2$ and $t \geq 1$ and that the ellipse $E(P)$ lies inside the Steiner circumellipse, for which $t = \infty$, if and only if $t > 1$. If $t = 2$, then the dilation factor is $1/2$ and the dilated ellipse is the Steiner inellipse; that is, the ellipse inscribed in ABC.

A further note is that regarding (7), one could choose $\Delta = t - \sqrt{(t-1)(t+2)}$ instead of $\Delta = t + \sqrt{(t-1)(t+2)}$. The resulting dilation maps A to the reflection of A' in G; see Figure 3.

Theorem 4. *Suppose that $P = p : q : r$ is a regular point, so that the six points*

$$\begin{aligned} P_1 &= p : q : r, & P_2 &= p : r : q, \\ Q_1 &= q : r : p, & Q_2 &= q : p : r, \\ R_1 &= r : p : q, & R_2 &= r : q : p \end{aligned}$$

are distinct. In each of the three sets of four lines listed below, each pair are parallel:

$$\begin{gathered} \{BC, P_1P_2, Q_1Q_2, R_1R_2\} \\ \{CA, P_1R_2, Q_1P_2, R_1Q_2\} \\ \{AB, P_1Q_2, Q_1R_2, R_1P_2\}. \end{gathered}$$

Proof. The lines P_1P_2 and Q_1Q_2 are given by the equations $(q + r)x - py - pz = 0$ and $(p + r)x - qy - qz = 0$, and the line BC, by $x = 0$. Clearly, the point $A^* = 0 : 1 : -1$ lies on all three lines, which lines must be parallel because A^* lies on the line at infinity, given by the equation

$$x + y + z = 0. \tag{10}$$

The same argument applies to all the asserted parallelisms. □

Theorem 5. *Let M and m be the lengths of the semi-major and semi-minor axes of the permutation ellipse $E(P)$ given by*

$$(qr + rp + pq)(x^2 + y^2 + z^2) - (p^2 + q^2 + r^2)(yz + zx + xy) = 0,$$

respectively. Then

$$\begin{aligned} M^2 &= \frac{p_2}{9p_1^2}\left(a_1 + \sqrt{a_1^2 - 48\sigma^2}\,\right) \quad \textit{and} \quad m^2 = \frac{p_2}{9p_1^2}\left(a_1 - \sqrt{a_1^2 - 48\sigma^2}\,\right), \ \textit{where} \\ p_1 &= p + q + r \\ p_2 &= p^2 + q^2 + r^2 - qr - rp - pq \\ a_1 &= a^2 + b^2 + c^2 \\ \sigma &= \operatorname{area}(ABC). \end{aligned}$$

Proof. Starting with any five of the six points (2), construct the major and minor axes of $E(P)$, as in ([8], p. 150). Then find the points of intersection of $E(P)$ and the two axes ([8], p. 151). Finally, apply (1) to find the distance from G to an endpoint of each axis. □

The lengths M and m for a permutation ellipse

$$(x^2 + y^2 + z^2) - t(yz + zx + xy) = 0$$

can be more compactly written than in Theorem 5 in terms of the Brocard angle, ω, of ABC and the symbol e so effectively used by Gallatly ([2], p. 96):

$$e = \sqrt{1 - 4\sin^2\omega},$$

$$M = \frac{2}{3}\sqrt{\frac{t-1}{t+2}\sigma(\cos\omega + e)\csc\omega}, \quad m = \frac{2}{3}\sqrt{\frac{t-1}{t+2}\sigma(\cos\omega - e)\csc\omega},$$

where $\sigma = \operatorname{area}(ABC)$. The eccentricity of all permutation ellipses is

$$\frac{M}{m} = \sqrt{\frac{2e}{e + \cos\omega}}.$$

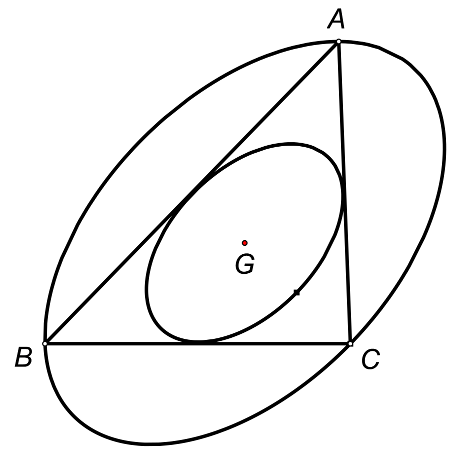

Figure 4: Steiner circumellipse (SCE) and Steiner inellipse (SIE)

3 Special ellipses: SCE and SIE

The Steiner circumellipse, given by (8), henceforth simply SCE, passes through the vertices A, B, C, as does the circumcircle, and it also passes through a fourth point of intersection with the circumcircle, the Steiner point, indexed as $X(99)$ in the Encyclopedia of Triangle Centers (ETC) [7]. Barycentrics for the Steiner point are

$$(c^2 - a^2)(a^2 - b^2) : (a^2 - b^2)(b^2 - c^2) : (b^2 - c^2)(c^2 - a^2).$$

ETC includes more than 100 triangle centers on SCE, so that there are more than 600 "known points" on SCE. This ellipse is given by an especially simple equation:

$$yz + zx + xy = 0, \text{ or, equivalently, } 1/x + 1/y + 1/z = 0.$$

A second well-known ellipse that is a permutation ellipse is the Steiner inellipse (SIE), which is $E(X(115))$, given by

$$x^2 + y^2 + z^2 - 2(yz + zx + xy) = 0.$$

According to Marden's theorem [5] if the vertices A, B, C have complex coordinates z_a, z_b, z_c, and

$$f(z) = (z - z_a)(z - z_b)(z - z_c),$$

then the roots of the derivative $f'(z)$ are the foci of SIE. One could switch to barycentrics and find the foci of all permutation ellipses. However, the results are too long to appear here; see ([7]: $X(39158)$–$X(39165)$ for the foci of SCE and SCI, and $X(39202)$–$X(39209)$ for the vertices.)

Among points in ETC that lie on SIE are these:

$$\begin{aligned} X(115) &= (b^2 - c^2)^2 :: \\ X(1015) &= a^2(b - c)^2 :: \\ X(1084) &= a^4(b^2 - c^2)^2 :: \\ X(1086) &= (b - c)^2 :: \end{aligned}$$

It is easy to prove that if a point $P = p : q : r$ is on the line (10) at infinity, then its barycentric square, $p^2 : q^2 : r^2$ is on SIE; indeed, algebraically, SIE is the square of the line at infinity.

4 Complements and anticomplements

Suppose that $P = p : q : r$ is a point other than G. The *complement* of P, denoted by $m_1(P)$, is defined algebraically by

$$m_1(P) = q + r : r + p : p + q,$$

and the *anticomplement* of P, denoted by $m_{-1}(G)$, is defined by

$$m_{-1}(P) = -p + q + r : p - q + r : p + q - r.$$

The classical geometric definitions are as follows: the complement of P is the point P' satisfying the vector equation $PG = 2GP'$; the anticomplement of P is the point P'' satisfying $P''G = 2GP$. Letting $m_0(P) = P$, inductively we define, for $i \geq 1$,

$$m_i(P) = m_1(m_{i-1}(P)) \text{ and } m_{-i}(P) = m_{-1}(m_{-i+1}(P)),$$

so that

$$\begin{aligned} m_2(P) &= 2p + q + r : p + 2q + r : p + q + 2r \\ m_3(P) &= 2p + 3q + 3r : 3p + 2q + 3r : 3p + 3q + 2r \\ m_4(P) &= 6p + 5q + 5r : 5p + 6q + 5r : 5p + 5q + 6r \end{aligned}$$

and

$$\begin{aligned} m_{-2}(P) &= 3p - q - r : -p + 3q - r : -p - q + 3r \\ m_{-3}(P) &= -5p + 3q + 3r : 3p - 5q + 3r : 3p + 3q - 5r \\ m_{-4}(P) &= -11p - 5q - 5r : -5p + 11q - 5r : -5p - 5q + 11r. \end{aligned}$$

As a corollary to Theorem 1, if P is a regular point, then $E(m_i(P)) = m_i(E(P))$ for every integer i. The permutation ellipses $E(m_i(P))$ are nested, as shown here:

$$\cdots \subset E(m_2(P)) \subset E(m_1(P)) \subset E(P) \subset E(m_{-1}(P)) \subset E(m_{-2}(P)) \subset \cdots$$

5 Dual ellipses

The *dual* of a conic Γ is the conic consisting of points $p : q : r$ such that the line $px+qy+rz = 0$ is tangent to Γ (e.g., [8], p. 125).

Theorem 6. *The dual of a permutation ellipse is a permutation ellipse. Specifically, if $E(P)$ is given by*

$$x^2 + y^2 + z^2 - t(yz + zx + xy) = 0, \tag{11}$$

then the dual of $E(P)$ is given by

$$(t - 2)(x^2 + y^2 + z^2) - 2t(yz + zx + xy) = 0. \tag{12}$$

Proof. This is obtained directly from a formula ([8], p. 125) for a dual conic. □

corollary 1. *There is exactly one self-dual permutation ellipse:*

$$x^2 + y^2 + z^2 - 4(yz + zx + xy) = 0.$$

Proof. Self-duality is equivalent to $-t = 2t/(t+2)$ with $t \neq 0$. □

If $P = p : q : r$ is a point on SCE, then the point

$$\Delta(P) = 3p + \delta : 3q + \delta : 3r + \delta,$$

where $\delta = (\sqrt{2}-1)(p+q+r)$, is on the self-dual ellipse. Examples in ETC include $X(39103) = \Delta(X(99))$ and $X(39105) = \Delta(X(671))$.

corollary 2. *The ellipse* (11) *lies in the interior of its dual* (12) *if and only if* $1 < t < 4$.

Proof. This follows from a comparison of the dilation factors for the two ellipses:

$$\sqrt{\frac{t-1}{t+2}} \quad \text{and} \quad \sqrt{\frac{t+2}{4t-4}}.$$

□

6 Inverses of ellipses in ellipses

Suppose that U and P are distinct points. Let $\mathbb{E}$ be an ellipse with center U. Let Q be the point of intersection of the ray UP and $\mathbb{E}$. The $\mathbb{E}$-*inverse* of P is defined [4] as the point P' satisfying

$$|UP||UP'| = |UQ|^2. \tag{13}$$

Theorem 7. *In the plane of a triangle ABC, the inverse of a permutation ellipse in a permutation ellipse is a permutation ellipse. Specifically, if $\mathbb{E}$ is given by*

$$x^2 + y^2 + z^2 - h(yz + zx + xy) = 0 \tag{14}$$

and $E(P)$ by

$$x^2 + y^2 + z^2 - k(yz + zx + xy) = 0, \tag{15}$$

then the inverse of $E(P)$ in $\mathbb{E}$ is given by

$$x^2 + y^2 + z^2 - t(yz + zx + xy) = 0, \tag{16}$$

where

$$t = -\frac{h^2k + h^2 + 2k - 4h}{h^2 - 2hk - k + 2}. \tag{17}$$

Proof. In (13), put $U = G$, and choose Q to be the point of intersection of the ray GA and the ellipse (14); i.e.,

$$Q = h + \sqrt{(h-1)(h+2)} : 1 : 1.$$

Next, for any P on the line GA except G, we have $P = x : 1 : 1$ for some $x \neq 1$. The distance formula (1) gives

$$|GP|^2 = (\frac{x-1}{3(x+2)})^2(4S_A + S_B + S_C).$$

Similarly, writing $P' = w : 1 : 1$, we have

$$|GP'|^2 = (\frac{w-1}{3(w+2)})^2(4S_A + S_B + S_C).$$

and

$$|GQ|^2 = (\frac{H-1}{3(H+2)})^2(4S_A + S_B + S_C).$$

where

$$H = h + \sqrt{(h-1)(h+2)}.$$

Consequently,

$$\frac{(x-1)(w-1)}{(x+2)(w+2)} = (\frac{H-1}{H+2})^2,$$

which leads to

$$w = \frac{2-h-hx}{h-x}. \tag{18}$$

As $x:1:1$ is on $E(P)$, we find

$$x = k + \sqrt{(k-1)(k+2)}. \tag{19}$$

Next, we need some t such that $w:1:1$ satisfies (16), where

$$t = \frac{p'^2 + q'^2 + r'^2}{q'r' + r'p' + p'q'}$$

for any $p':q':r'$ on $\mathbb{E}$. Taking $p':q':r' = w:1:1$ gives

$$t = \frac{w^2+2}{2w+1}. \tag{20}$$

Substitute the right side of (19) into (18) and then substitute w into (20). The result simplifies to (17). □

corollary 3. *The inverse of an ellipse* (15) *in SCE is given by* (16) *with* $t = -k-1$.

Proof.

$$\lim_{h\to\infty} -\frac{h^2k + h^2 + 2k - 4h}{h^2 - 2hk - k + 2} = -k-1.$$ □

corollary 4. *The inverse of SCE in an ellipse* (14) *is given by* (16) *with* $t = \frac{h^2+2}{2h+1}$.

Proof.

$$\lim_{k\to\infty} -\frac{h^2k + h^2 + 2k - 4h}{h^2 - 2hk - k + 2} = \frac{h^2+2}{2h+1}.$$ □

The inverse of SIE in SCE is given by $t(a,b,c) = -3$ in (3). This ellipse, which is the anticomplement of SCE, is $E(X(148))$. The inverse of SCE in SIE is given by $t(a,b,c) = 6/5$ in (3). This ellipse, which is the complement of the complement of SCE, is $E(X(620))$.

7 Frégier ellipses

Frégier's theorem starts with any point P on a conic $\mathbb{C}$. Let U be any point on $\mathbb{C}$ except P, and let U' be the point on $\mathbb{C}$ such that the segment UU' is the hypotenuse of the right triangle UPU'. As U traverses $\mathbb{C}$ the segments UU' all pass through a fixed point, $F(P)$, called the *Frégier point* of P. It is known (e.g., [6], p. 279) that if $\mathbb{C}$ is an ellipse, then the locus of $F(P)$ as P traverses $\mathbb{C}$ is an ellipse. We call it the *Frégier ellipse* of $\mathbb{C}$.

Theorem 8. *Suppose that E is an ellipse with center G. Let M and m denote, respectively, the lengths of the semi-major and semi-minor axes of E. Then the Frégier ellipse $F(E)$ is the dilation from G of E, and the dilation factor is*

$$\frac{M^2 - m^2}{M^2 + m^2}.$$

(Thus, the Frégier ellipse of a permutation ellipse is a permutation ellipse.)

Proof. See [3], pp. 256-257. □

Theorem 9. *Suppose that $P = p : q : r$ is a point on SCE. If $P \notin \{A, B, C\}$, then the Frégier point of P is given by*

$$F(P) = (p^2 + q^2 + pq)(p^2 + r^2 + pr)p' ::, \text{ where}$$
$$p' = \left((3a^2 - b^2 - c^2)qr + (a^2 + b^2 - c^2)q^2 + (a^2 - b^2 + c^2)r^2\right).$$

For the remaining cases,

$$F(A) = a_1 : c_1 : b_1$$
$$F(B) = c_1 : b_1 : a_1$$
$$F(C) = b_1 : a_1 : c_1,$$

where

$$a_1 = b^2 + c^2 - a^2, \quad b_1 = c^2 + a^2 - b^2, \quad c_1 = a^2 + b^2 - c^2.$$

Proof. Let B' be the point, other than B, where the line through B perpendicular to line PB meets SCE. Let C' be the point, other than C, where the line through C perpendicular to PC meets SCE. Then $F(P) = BB' \cap CC'$. □

A second, more laborious, way to prove Theorem 9 is to dilate P from G using the dilation factor in Theorem 8.

The Frégier ellipse of SCE is the ellipse $E(X(69))$, since

$$F(X(99)) = X(69) = a_1 : b_1 : c_1.$$

Likewise, the Frégier ellipse of SIE is $E(X(6))$, where X(6) is the symmedian point of ABC. An equation for $E(X(6))$ is

$$(b^2c^2 + c^2a^2 + a^2b^2)(x^2 + y^2 + z^2) - (a^4 + b^4 + c^4)(yz + zx + xy) = 0.$$

(For Frégier points of selected rectangular hyperbolas, see [7], points X(30181)–X(30257) and the preamble just before X(30182), contributed by César Eliud Lozada. See also the preamble just before X(34341).)

8 Axes of permutation ellipses

Since every permutation ellipse is a dilation from G of SCE, these ellipses all have the same major and minor axes. Points in ETC that lie on the major axis include X(i) for

$$i = 2, 1341, 1348, 2542, 3413, 5638, 13722, 14899, 30509, 31863, 35607, 35913,$$
$$39158, 39159, 39162, 39163, 39202, 39203, 39206, 39207, 39300, 39301, 39304, 39305.$$

Points on the minor axis include $X(i)$ for

$$i = 2, 1340, 1349, 2543, 3414, 5639, 13636, 30508, 31862, 35608, 35609, 35914,$$
$$39160, 39161, 39164, 39165, 39204, 39205, 39208, 39209, 39302, 39303, 39306, 39307.$$

Theorem 10. *The major axis of SCE, SCI, and all permutation ellipses, lies in the line given by*

$$f(a,b,c)x + f(b,c,a)y + f(c,a,b)z = 0, \tag{21}$$

where

$$\begin{aligned} f(a,b,c) &= (b^2 - c^2)(a^2b^2 + a^2c^2 - b^4 - c^4 - (a^2 - b^2 - c^2)\sqrt{D}), \\ D &= a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2. \end{aligned}$$

Equation (21) *also gives the line containing the minor axis if, in the equation for $f(a,b,c)$ just above, $\sqrt{D}$ is replaced by $-\sqrt{D}$.*

Proof. Let $U = u(a,b,c) : v(a,b,c) : w(a,b,c)$ be an endpoint of the major axis, constructed as in the proof of Theorem 5. The major axis lies in the line GU, given by (21), where

$$f(a,b,c) = v(a,b,c) - w(a,b,c).$$

The same steps, starting with an endpoint of the minor axis, give the line of the minor axis. □

9 Antipodes

Suppose that $P = p : q : r$ is a regular point and $U = u : v : w$ is a point other than P. The line PU meets the permutation ellipse $E(P)$ formally in two points. One of them is P, and the other we call the *U-antipode of P*. If $U = G$, the antipode is quite simple (see Corollary 5), but in general, the result is not so simple, as indicated in the next theorem.

Theorem 11. *If P and U are distinct regular points, then the U-antipode of P is the point*

$$f(p,q,r,u,v,w) : f(q,r,p,v,w,u) : f(r,p,q,w,u,v), \ \textit{where}$$

$$\begin{aligned} f(p,q,r,u,v,w) = {} & [(q+r)(q^2+r^2) - pqr]u^2 \\ & + p(qr + rp + pq)(v^2 + w^2) \\ & - p(p^2 + q^2 + r^2)vw \\ & + [q(p^2 + q^2 + r^2) - 2pr(q+r)]wu \\ & + [r(p^2 + q^2 + r^2) - 2pq(q+r)]uv. \end{aligned}$$

Proof. The line PU is given by

$$(qw - rv)x + (ru - pw)y + (pv - qu)z = 0. \tag{22}$$

Solve this for z and substitute the result into (3). Then solve the resulting equation for y, obtaining a multiple of x other than $y = px/q$, so that, using (22) to express z, we have the required antipode $x : y : z$. Simplification gives the required form. □

corollary 5. *In the permutation ellipse $E(P)$, the G-antipode of a point U is the point*

$$p - 2q - 2r : q - 2r - 2p : r - 2p - 2q.$$

Proof. In Theorem 11, put $(u, v, w) = (1, 1, 1)$. □

corollary 6. *In the permutation ellipse $E(P)$, the A-antipode of P is the point*

$$(q + r)(q^2 + r^2) - pqr : q(qr + rp + pq) : r(qr + rp + pq).$$

Proof. In Theorem 11, put $(u, v, w) = A = (1, 0, 0)$. □

10 A special family

In the previous sections, most of the permutation ellipses $E(P)$ are "central" in the sense that the point P is a triangle center—that is, P has barycentrics of the form

$$f(a, b, c) : f(b, c, a) : f(c, a, b), \tag{23}$$

where the function f is homogeneous in a, b, c and satisfies $f(a, c, b) = f(a, b, c)$. However, there are some interesting permutation ellipses $E(P)$ for which P is not a triangle center. One can start with any three integers, i, j, k, not all zero, and ask: which "integer points" $i' : j' : k'$ lie on the ellipse $E(i : j : k)$? For starters, it is easy to verify that all integer points

$$-n : n + 1 : n^2 + n$$

lie on SCE, and all integer points

$$n^2 : k^2 : (n + k)^2$$

lie on SIE.

The trisector ellipse. Consider the permutation ellipse that passes through these six points:

$$0 : 1 : 2, \quad 0 : 2 : 1, \quad 1 : 0 : 2, \quad 2 : 0 : 1, \quad 1 : 2 : 0, \quad 2 : 1 : 0.$$

We call this the *trisector ellipse* because the six points are the trisectors of the sides of ABC. As shown in Figure 5, the same six points are the points in which lines through G parallel to lines BC, CA, AB meet these same lines. An equation for the trisector ellipse is

$$2(x^2 + y^2 + z^2) - 5(yz + zx + xy) = 0,$$

and the dilation factor is $\sqrt{1/3}$. Among integer points on this ellipse are

$$-2 : 24 : 35, \quad -1 : 10 : 12, \quad 2 : 4 : 15, \quad 3 : 14 : 40, \quad 5 : 6 : 28.$$

Points on its complement and anticomplement are $1 : 2 : 3$ and $1 : -1 : 3$, respectively, as in Section 4.

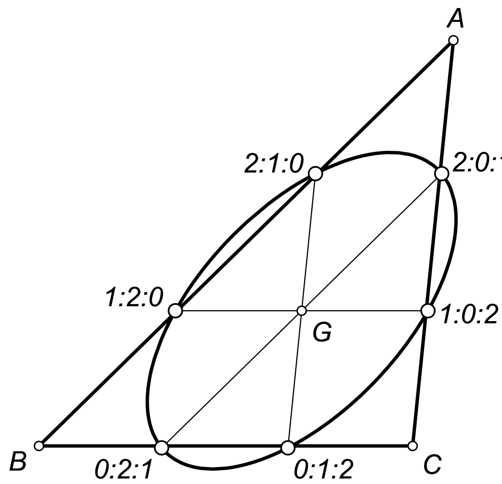

Figure 5: The Trisector Ellipse, E(0:1:1)

The Steiner midway ellipse. For each P on SCI, let P' be the midpoint of G and P. The set of all such midpoints comprise the *Steiner midway ellipse*, SME, given by the equation

$$7(x^2 + y^2 + z^2) - 34(yz + zx + xy) = 0.$$

The dilation factor is 3/4, and among integer points on SME are these:

$$-11 : 34 : 61, \quad -11 : 106 : 421, \quad -2 : 43 : 187, \quad 1 : 154 : 721, \quad 7 : 70 : 367.$$

11 Open problems

Thousands of triangle centers in ETC are *polynomial centers* in the sense that they have a representation (23) in which $f(a, b, c)$ is a polynomial. One wonders if a reasonable characterization can be found for integer points P such that the permutation ellipse $E(P)$ passes through at least one polynomial triangle center. Examples are SCE and SIE, whereas it appears that SME and the trisector ellipse pass through no polynomial triangle centers.

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