Journal for Geometry and Graphics
Volume 24 (2020), No. 2, 175–191

The Pentagrammum Mysticum, Twelve Special Conics and the Twisted Icosahedron

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Abstract. We study an abstract pentagon in planar projective geometry via its unique circumscribing conic, associated to 15 diagonal lines and exagonal points, and describe 12 beautifully inter-related conics. These are closely connected to a distance-regular graph X which is a sister of the icosahedral graph, arising from dihedral orderings of five objects.

Key Words: strongly regular graph, icosahedron, Pascal theorem, conic, pentagon

MSC 2020: 14Nxx, 51A20, 05Exx

1 Introduction

There is an extensive literature on the remarkable configurations resulting from six points on a conic, starting with Pascal's theorem, and generally referred to as the *Hexagrammum Mysticum*. Investigation of this rich phenomenon involved some of the greatest geometers of the nineteenth century [10, page 172].

A Pascal line is determined by an ordering of the six points, and by considering all possible orderings, it turns out that there are in total 60 Pascal lines. There are then an associated 60 Kirkman points each lying on three Pascal lines, with each Pascal line incident with three Kirkman points. The Pascal lines also pass, three at a time, through 20 additional Steiner points. There are 20 Cayley lines which pass through both a Steiner point and three Kirkman points. The Steiner points also lie, four at a time, on 15 Plücker lines. Furthermore, the 20 Cayley lines concur four at a time at 15 Salmon points.

See Levi's text [6] for a systematic description of these classical lines and points, and Figure 6.14 of [3, p. 233] for a visualization of the 60 Pascal lines. This fascinating configuration has also attracted recent interest [1], [2], as it turns out that the combinatorics of the number six is intimately connected with the situation.

In this paper we want to show that with just five points in general position in the projective plane, there is also a panorama of fascinating geometric and combinatorial structure, which we might call the *Pentagrammum Mysticum*, which goes further in the direction set out

ISSN 1433-8157/$ 2.50 © 2020 Heldermann Verlag

by Kasner [5] and Hoffman [4]. We look in some detail at the projective geometry of five general Points, by exploiting properties of the unique conic C that passes through those five points, and the five tangent Lines to C at those points. These five Points and five Lines form a configuration that yields ten *Diagonal lines* and ten *Exagonal points*, which in turn determine fifteen *Diagonal points* and fifteen *Exagonal lines*. This is a more elementary configuration than that obtained from six points on a conic, but it turns out to be quite interesting. There are two classical theorems here which we call the *Diagonal points on Exagonal lines* theorem, and the *Exagonal lines through Dihedral points* theorem, and they are naturally dual. These deserve to be more widely known.

Further we show that the twelve dihedral orderings of five points naturally index twelve *dihedral conics* associated to the original five Points, and that the intersection data for these conics is reflected in a distance transitive graph X which we call the *twisted icosahedron*. We give some reasons why this is indeed a good name for this graph, describe some of its geometry, and conclude with its adjacency matrix.

In [6, p. 213] Levi discusses also how the Pascal configuration reduces when two of the points are identified, where a line between the original points becomes a tangent to the conic, which breaks the symmetry of the original points. This symmetry can be reinstated by considering all six such Pascal pentagons, and the possible connections between this and the current study seems to warrant further investigation. We would like to thank the reviewer/editor for bringing this to our attention.

2 Some projective constructions from a pentagon

The projective plane $\mathbb{P}^2$ can be viewed over a general field, typically for us the rational numbers, but finite or other fields are also allowed. The projective pentagon in this situation is important not only for its intrinsic interest, but also because it is connected to the study of the rectangular spherical triangle and to Euclidean geometry, as pointed out by Motzkin [7]. We begin with some notational conventions and book-keeping with regard to five objects.

Points will generally be associated to square brackets $[\quad]$ and lines to round brackets $(\quad)$. The polarity, or duality, between points and lines determined by a fixed conic C, goes back to Apollonius. To define the dual of a point A with respect to a conic C, we may choose any two lines which pass through A, and meet C in two points each. The quadrangle so determined has two other diagonal points and the dual line $a = A^{\perp}$ is the join of these, and is independent of the choices of initial lines through A. In the case of the point A lying on the conic, the dual line $a = A^{\perp}$ is the tangent line to C through A. Additionally it turns out that if B lies on the dual a of A, then A lies on the dual b of B. This allows us to define the dual of a line as the meet of the duals of any two points on it.

Because we are interested in the combinatorics that arise, we will label objects just by their indices. We also adopt a partial German convention of capitalizing objects that are directly associated to the five original **Points** which will be denoted

$$[0], [1], [2], [3], [4]$$

and which we assume are in general position, so no three are collinear.

The **Point Conic** through these five Points is denoted C. The tangent lines to C at these Points are the respective **Lines**

$$(0), (1), (2), (3), (4)$$

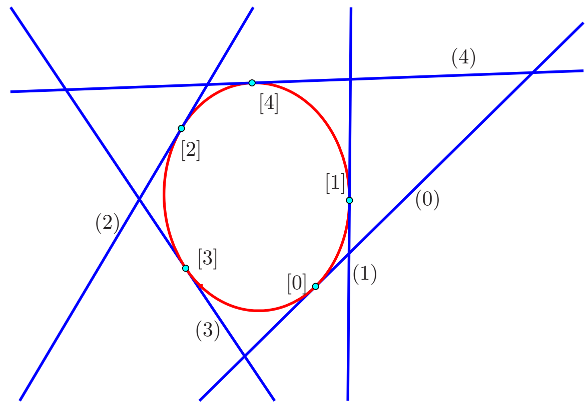

Figure 1: Five Points, the conic through them, and the associated five Lines

so that (0) denotes the tangent line to C at the point [0], and so on. These five Lines will also be in general position, so no three are concurrent.

This fundamental association between five generic Points and five generic Lines which are incident to them distinguishes the number five in projective geometry. The symmetry between the five Points and Lines can also be viewed as given by the duality induced by C.

An example of five Points, the associated Point Conic, and the five Lines are shown in Figure 1. The five Points join to form ten **Diagonal lines**

$$(01)\,,(02)\,,(03)\,,(04)\,,(12)\,,(13)\,,(14)\,,(23)\,,(24)\,,(34)$$

where for example $(01) \equiv [0]\,[1]$ is the join of the Points [0] and [1] . Two such Diagonal lines pass through the same Point precisely when they share an index, so that for example (03) and (13) both pass through the Point [3] . Note that while these labels are intrinsically unordered, we present them in lexicographical order for convenience of identification, and begin ordering with 0.

The five Lines meet at ten **Exagonal points**

$$[01]\,,[02]\,,[03]\,,[04]\,,[12]\,,[13]\,,[14]\,,[23]\,,[24]\,,[34]$$

where for example $[01] \equiv (0)\,(1)$ is the meet of the Lines (0) and (1) . Two such Exagonal points lie on the same Line precisely when they share an index, so that for example [02] and [24] both lie on the Line (2).

The obvious correspondence between Exagonal points and Diagonal lines is again given by the duality induced by the conic C so that for example the Exagonal point [14] is the pole of the Diagonal line (14), while (14) is the polar of [14] . We will use the word ”dual” to express both notions. Some Exagonal points and Diagonal lines are shown in Figure 2, while others will be off the viewing page.

The meet of two Diagonal lines which do not share a common Point is a **Diagonal point**, and there are fifteen, namely

$$\begin{array}{ccccc} (01)\,(23) & (04)\,(12) & (02)\,(14) & (13)\,(24) & (03)\,(24) \\ (02)\,(13) & (12)\,(34) & (03)\,(14) & (01)\,(34) & (04)\,(23) \\ (03)\,(12) & (01)\,(24) & (04)\,(13) & (02)\,(34) & (14)\,(23) \end{array}$$

Here we express the meet of the lines (01) and (23) as the point $(01)\,(23)$ etc. instead of introducing new terminology.

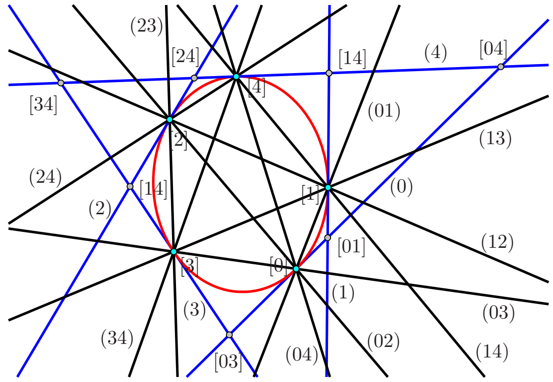

Figure 2: Some Exagonal points and Diagonal lines

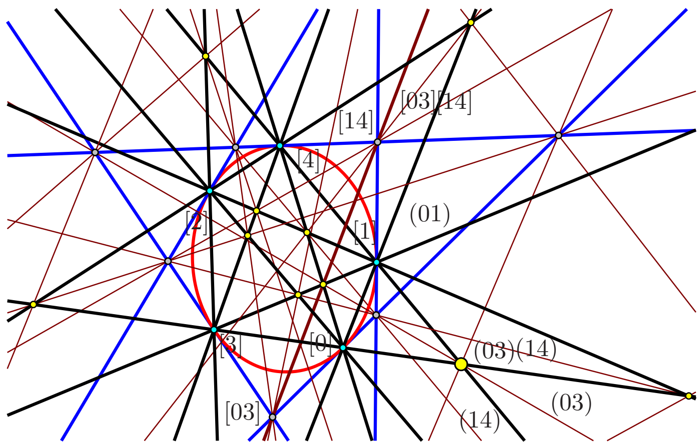

Figure 3: Fifteen Exagonal lines and Diagonal points

The join of two Exagonal points which do not lie on a common Line is an **Exagonal line**, and there are fifteen such, namely

[01] [23]	[04] [12]	[02] [14]	[13] [24]	[03] [24]
[02] [13]	[12] [34]	[03] [14]	[01] [34]	[04] [23]
[03] [12]	[01] [24]	[04] [13]	[02] [34]	[14] [23]

Exagonal lines and Diagonal points are also related in the obvious way via the duality of C, so for example in Figure 3 we see Exagonal lines (in brown) and Diagonal points (in yellow), with both the Diagonal point (03) (14) and its dual Exagonal line [03] [14] labelled and highlighted.

Any four of the five original Points, such as [0] , [1] , [3] and [4] , determine a complete quadrangle, which has six lines, each of which is a Diagonal line, such as (03) or (14). Such a quadrangle is determined by the unique Point which is not included, in this example [2]. The six Diagonal lines determined by such a quadrangle meet also at three Diagonal points, which in this case are exactly those not involving the excluded index, namely (03) (14) , (01) (34) and (04) (13) .

Correspondingly, any four of the five original Lines, such as (0) , (1) , (3) and (4) , determine a complete quadrilateral, which has six points, each of which is an Exagonal point, such as [03] or [14] . Such a quadrilateral is also determined by the unique Line which is not included, in this example (2). The six Exagonal points determined by such a quadrilateral join also to form three Exagonal lines, which in this case are exactly those not involving the excluded index, such as $[03]\,[14]$, $[01]\,[34]$ and $[04]\,[13]$.

Figure 3 shows some unexpected concurrences and collinearities, which are well-known and described in the following pair of dual theorems.

Theorem (Diagonal points on Exagonal lines) *The Exagonal line* $[ij]\,[mn]$ *passes through the Diagonal points* $(im)\,(jn)$ *and* $(in)\,(jm)$.

Theorem (Exagonal lines through Diagonal points) *The Diagonal point* $(ij)\,(mn)$ *lies on the Exagonal lines* $[im]\,[jn]$ *and* $[in]\,[jm]$.

3 Dihedral orderings of five objects

We now make some general comments on the combinatorics associated to the number five, and connect with the symmetries between a pentagon and its associated pentagram, see also [8] and [9]. A **dihedral ordering** of five objects indexed by $0, 1, 2, 3$ and 4 is a (linear) ordering, with the convention that two such orderings are equal if they can be obtained by cyclic shifts as well as reflections. So for example all the following dihedral orderings are equal:

$$01234 = 12340 = 23401 = 34012 = 40123 =$$
$$43210 = 04321 = 10432 = 21043 = 32104.$$

Since there are $5! = 120$ permutations of five objects, and 10 are equal in one dihedral order, there are altogether 12 distinct dihedral orderings. When we write a dihedral order, our convention is to choose the permutation which begins with 0, which is possible due to the assumed cyclic symmetry, and whose next element is the smaller of 0's two neighbouring entries, which is possible by the reflection symmetry. So the primary name of the dihedral order above would be $F = 01234$.

Two dihedral orderings are **opposite** precisely when one of them is obtained by taking every second, or alternating, element from the other. We denote this by a bar. So for example the orderings $F = 01234$ and $\overline{F} = 02413$ are opposite, and the correspondence can be visualized as the dual relation between a pentagon and its associated pentagram, as in Figure 4. By choosing every second entry (again either forwards of backwards) in the pentagram, we recover the original pentagon. So this is an involution, which we denote by σ, on dihedral orderings, so we write $\sigma(F) = \overline{F}$.

Here are the twelve dihedral orderings written using these conventions:

$$\begin{array}{llllll} A \equiv 01432 & B \equiv 01342 & C \equiv 01243 & D \equiv 01423 & E \equiv 01324 & F \equiv 01234 \\ \overline{A} \equiv 03124 & \overline{B} \equiv 03214 & \overline{C} \equiv 02314 & \overline{D} \equiv 02134 & \overline{E} \equiv 02143 & \overline{F} \equiv 02413. \end{array}$$

Two dihedral orderings are **adjacent** precisely when they differ by swapping a (cyclically) neighbouring pair of points, so that for example the dihedral orderings adjacent to 01234 are

$$01243 \qquad 01324 \qquad 02134 \qquad 01432 \qquad 03214.$$

With this notion of adjacency, the 12 dihedral orderings form a regular graph X of degree 5 which we call the **twisted icosahedron**—the reason for the name will be justified as we

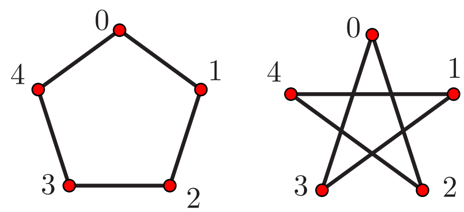

Figure 4: A pentagon and its associated pentagram

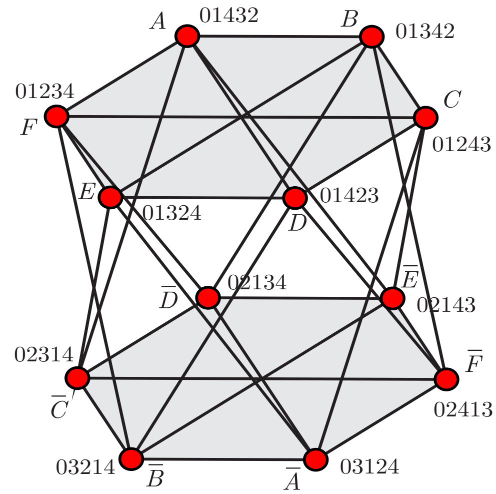

Figure 5: The pentagon dihedral ordering graph

proceed. We may view X as in Figure 5, consisting of two hexagons with opposing vertices, such as A and D, connected, and which lie in parallel planes and are connected by edges in an alternating up and down fashion: each vertex on one hexagon is joined also to the vertices in the other hexagon which are directly above or below its two nearest non-opposing neighbours. Here the shading of the hexagons is just for illustrative purposes.The number of edges of this graph is $9+9+12=30$. This is however not the graph of an icosahedron, since for example two adjacent vertices have no common neighbours, while in an icosahedron graph they have two.

Recall that a hyperboloid of one sheet in three-dimensional space may be obtained by taking two parallel circles, one directly above the other, and joining each point on the top circle with the two points on the bottom circle which are a fixed turn, or angle, from the vertical projection of the given point. This makes a ruled surface with two rulings. The twisted icosahedron can thus be viewed as a discretization of such a hyperboloid, and as such one could readily investigate generalizations where the hexagon is replaced by another polygon.

In Figure 5 the top shaded hexagon consists exactly of the dihedral orderings

$$01432 \qquad 01342 \qquad 01243 \qquad 01423 \qquad 01324 \qquad 01234$$

characterized by containing the adjacent pair 01, and so we could denote it by F_{01}. The opposite bottom shaded hexagon consists of the dihedral orderings characterized by *not* having $0,1$ as neighbours, and we could denote it by G_{01}. There are

$$2 \times \binom{5}{2} = 20$$

such hexagonal ”faces” of the twisted icosahedron.

Each edge occurs in three such faces of both kinds; for example the edge between $A = 01432$ and $B = 01342$, which is determined by the adjacent pair 34, is contained in the faces F_{01}, F_{02} and F_{34} as well as G_{04}, G_{03} and G_{12}, and these can be easily read off from the overlap of the two dihedral orderings: $01, 02, 34$ are common adjacencies, while $04, 03, 12$ are common non-adjacencies.

So if we loosen our definition of ”face” of a graph to include the kind of hexagons we are here discussing (in the context of simplicial complexes), then both the icosahedron graph I obtained from the vertices and edges of a regular icosahedron and the twisted icosahedron X have 12 vertices, 30 edges and 20 faces. It is well known that the icosahedron graph is **distance transitive**, meaning that the symmetry group of graph automorphisms acts transitively on the set of pairs of vertices a distance k apart, for any k from 2 to the diameter of the graph.

Both X and the icosahedron graph I have automorphism groups that active transitively on the vertices; X has symmetry group S_5 acting in the obvious way on dihedral orders, while the automorphism group of the icosahedral graph is also of order 120, but it is rather the Coxeter group $H_3 = A_5 \times Z_2$. Less obvious is that the graph X shares the stronger property of being distance transitive.

Theorem 1. *The twisted icosahedron X is a distance transitive graph.*

Proof. Because of the transitivity of the automorphism group S_5 on vertices, the distance transitivity follows from showing that S_5 permutes the vertices of some fixed distance from a given fixed vertex, say $D = 01432$. The adjacent orderings to D are 01432, 01243, 03214, 01324 and 02413. If τ denotes the permutation (01423), which sends 0 to 1, 1 to 4, 4 to 2, 2 to 3 and 3 to 0, then $\tau(D) = D$, but

$$\begin{aligned} \tau(01432) &= 14203 = 02413 & \qquad \tau(03214) &= 10342 = 01243 \\ \tau(02413) &= 13240 = 01324 & \qquad \tau(01243) &= 14320 = 01432 \\ \tau(01324) &= 14032 = 03214 & & \end{aligned}$$

so powers of τ permute these five adjacent dihedral orders cyclically, and in particular transitively.

Similarly the orderings of distance two to D, which are the orderings we will say are **distant** to D, are 01234, 01342, 03124, 02143 and 02314. The action of $\tau = (01423)$ on these is

$$\begin{aligned} \tau(01234) &= 14302 = 02143 & \qquad \tau(02314) &= 13042 = 03124 \\ \tau(02143) &= 13420 = 01342 & \qquad \tau(03124) &= 10432 = 01234 \\ \tau(01342) &= 14023 = 02314 & & \end{aligned}$$

and so this also generates a cyclic, and therefore transitive, action. Since there is only one vertex of distance 3 from D, namely the opposite vertex, the automorphism group permutes all the sets of vertices a fixed distance D, and hence X is distance transitive. □

The diameter of both graphs is three; but the adjacencies are different. In the twisted icosahedron X, adjacent vertices have no common neighbours, while in the icosahedron graph adjacent vertices have two common neighbours. In X if x and y are vertices of distance 2, then they have exactly 3 common neighbours, while for the icosahedron such vertices have only 2 common neighbours. For example in X the vertices $F = 01234$ and $B = 01342$ are of distance two apart, and have 3 common neighbours, namely

$$A = 01432 \qquad E = 01324 \qquad \overline{D} = 02134.$$

In both graphs each vertex has a single opposite vertex of distance three from it, and two vertices of distance 3 must have no common neighbours. On X the opposite map σ sends a vertex v to the vertex directly above or below the opposing vertex of v in its hexagon.

One important difference between the two graphs is that only X is bipartite; given by the division of its vertices into disjoint subsets $\left\{A\ C\ E\ \overline{B}\ \overline{D}\ \overline{F}\right\}$ and the opposite subset $\left\{\overline{A}\ \overline{C}\ \overline{E}\ B\ D\ F\right\}$. Each vertex in one subset is joined to all the vertices in the opposite set except for its opposite vertex. So the twisted icosahedron can be regarded as a slight modification of the complete bipartite graph $K(6,6)$, with edges missing from vertices to their opposites.

4 Dihedral conics

A given dihedral order such as $D = 01423$ determines five Diagonal points, obtained by choosing distinct pairs of cyclically adjacent indices in that order, namely

$$[14]\,[23] \qquad [03]\,[24] \qquad [01]\,[23] \qquad [03]\,[14] \qquad [01]\,[24]\,.$$

These have been listed so that the excluded points follow the same order as in D. We refer to them as the **Diagonal points of the dihedral order** D. The same dihedral order D also determines five Exagonal lines, namely those obtained by choosing distinct pairs of non-adjacent pairs. In the case of $D = 01423$, that would be

$$(12)\,(34) \qquad (02)\,(34) \qquad (02)\,(13) \qquad (04)\,(13) \qquad (04)\,(12)\,.$$

It is worth noting that equivalently these correspond to distinct pairs of cyclically adjacent indices in the opposite dihedral order $\overline{D} = 02134$. We refer to them as the **Exagonal lines of the dihedral order** D.

We can record both the Diagonal points and Exagonal lines of a dihedral order, by putting them together as a column vector, so that every vector only involves four of the five indices $0, 1, 2, 3$ and 4. Thus for the order $D = 01423$ above we get the five pairs:

$$\begin{pmatrix}(14)\,(23)\\ [12]\,[34]\end{pmatrix} \qquad \begin{pmatrix}(03)\,(24)\\ [02]\,[34]\end{pmatrix} \qquad \begin{pmatrix}(01)\,(23)\\ [02]\,[13]\end{pmatrix} \qquad \begin{pmatrix}(03)\,(14)\\ [04]\,[13]\end{pmatrix} \qquad \begin{pmatrix}(01)\,(24)\\ [04]\,[12]\end{pmatrix}. \tag{1}$$

More generally the dihedral order $ijklm$ with opposite order $ikmjl$ determines the five Diagonal points

$$(jk)\,(lm) \qquad (im)\,(kl) \qquad (ij)\,(lm) \qquad (im)\,(jk) \qquad (ij)\,(kl)$$

along with the five respectively corresponding Exagonal lines

$$[jl]\,[km] \qquad [il]\,[km] \qquad [il]\,[jm] \qquad [ik]\,[jm] \qquad [ik]\,[jl]\,.$$

It is well known that five points in general position determine a unique conic passing through them, and that five lines in general position determine a unique conic tangent to them. Here is our main theorem.

Theorem 2 (Dihedral conics)**.** *The unique conic c which passes through the Diagonal points of a given dihedral order is tangent at those Diagonal points to the corresponding Exagonal lines of that dihedral order.*

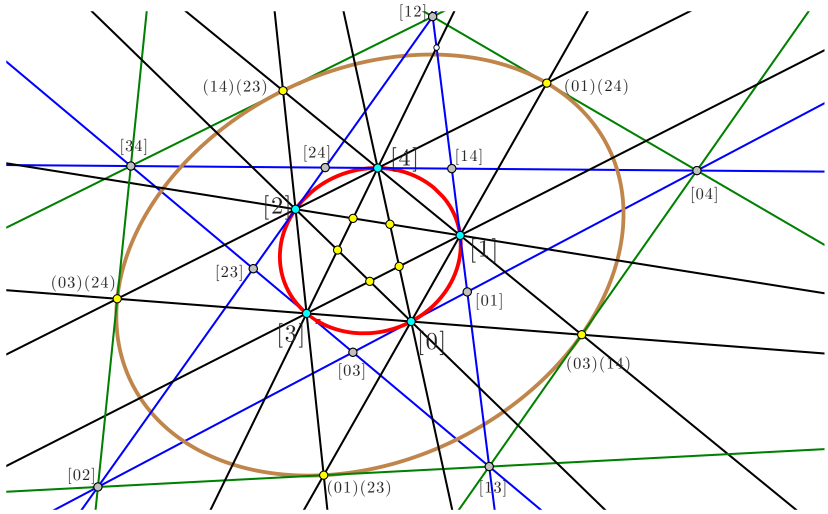

Figure 6: The dihedral conic $c(D) = c(01423)$

In Figure 6 we see the conic $c = c(D) = c(01423)$, which for example passes through the Diagonal point $(14)(23)$ and has tangent there the Exagonal line $[12][34]$. The reader should verify that this conic does pass through the Diagonal points of (1) and is tangent at those points to the associated Exagonal lines.

More generally if the dihedral order is $ijklm$, then the unique conic $c(ijklm)$ through the Diagonal points $(jk)(lm)$, $(im)(kl)$, $(ij)(lm)$, $(im)(jk)$, $(ij)(kl)$ of that dihedral order has respective tangents $[jl][km]$, $[il][km]$, $[il][jm]$, $[ik][jm]$, $[ik][jl]$ at those Diagonal points.

We call the conic $c(ijklm)$ a **dihedral conic.** There are then twelve dihedral conics associated to our original pentagon, and we list them along with their respective Diagonal points and Exagonal line pairs. In the diagrams that follow, they are referred to by the colors

$c(01432)$: [light green],	$c(01342)$: [dark green],	$c(01243)$: [light blue],
$c(01423)$: [light brown],	$c(01324)$: [dark blue],	$c(01234)$: [dark gray],
$c(03124)$: [light purple],	$c(03214)$: [orange],	$c(02314)$: [pink],
$c(02134)$: [dark brown],	$c(02143)$: [dark purple],	$c(02413)$: [black].

First we have the six conics associated to A, B, C, D, E and F.

$$c(A) = c(01432) : \begin{pmatrix} (14)(23) \\ [13][24] \end{pmatrix}, \begin{pmatrix} (02)(34) \\ [03][24] \end{pmatrix}, \begin{pmatrix} (01)(23) \\ [03][12] \end{pmatrix}, \begin{pmatrix} (02)(14) \\ [04][12] \end{pmatrix}, \begin{pmatrix} (01)(34) \\ [04][13] \end{pmatrix}$$

$$c(B) = c(01342) : \begin{pmatrix} (13)(24) \\ [14][23] \end{pmatrix}, \begin{pmatrix} (02)(34) \\ [04][23] \end{pmatrix}, \begin{pmatrix} (01)(24) \\ [04][12] \end{pmatrix}, \begin{pmatrix} (02)(13) \\ [03][12] \end{pmatrix}, \begin{pmatrix} (01)(34) \\ [03][14] \end{pmatrix}$$

$$c(C) = c(01243) : \begin{pmatrix} (12)(34) \\ [14][23] \end{pmatrix}, \begin{pmatrix} (03)(24) \\ [04][23] \end{pmatrix}, \begin{pmatrix} (01)(34) \\ [04][13] \end{pmatrix}, \begin{pmatrix} (03)(12) \\ [02][13] \end{pmatrix}, \begin{pmatrix} (01)(24) \\ [02][14] \end{pmatrix}$$

$$c(D) = c(01423) : \begin{pmatrix} (14)(23) \\ [12][34] \end{pmatrix}, \begin{pmatrix} (03)(24) \\ [02][34] \end{pmatrix}, \begin{pmatrix} (01)(23) \\ [02][13] \end{pmatrix}, \begin{pmatrix} (03)(14) \\ [04][13] \end{pmatrix}, \begin{pmatrix} (01)(24) \\ [04][12] \end{pmatrix}$$

$$c(E) = c(01324) : \begin{pmatrix} (13)(24) \\ [12][34] \end{pmatrix}, \begin{pmatrix} (04)(23) \\ [02][34] \end{pmatrix}, \begin{pmatrix} (01)(24) \\ [02][14] \end{pmatrix}, \begin{pmatrix} (04)(13) \\ [03][14] \end{pmatrix}, \begin{pmatrix} (01)(23) \\ [03][12] \end{pmatrix}$$

$$c(F) = c(01234) : \begin{pmatrix} (12)(34) \\ [13][24] \end{pmatrix}, \begin{pmatrix} (04)(23) \\ [03][24] \end{pmatrix}, \begin{pmatrix} (01)(34) \\ [03][14] \end{pmatrix}, \begin{pmatrix} (04)(12) \\ [02][14] \end{pmatrix}, \begin{pmatrix} (01)(23) \\ [02][13] \end{pmatrix}$$

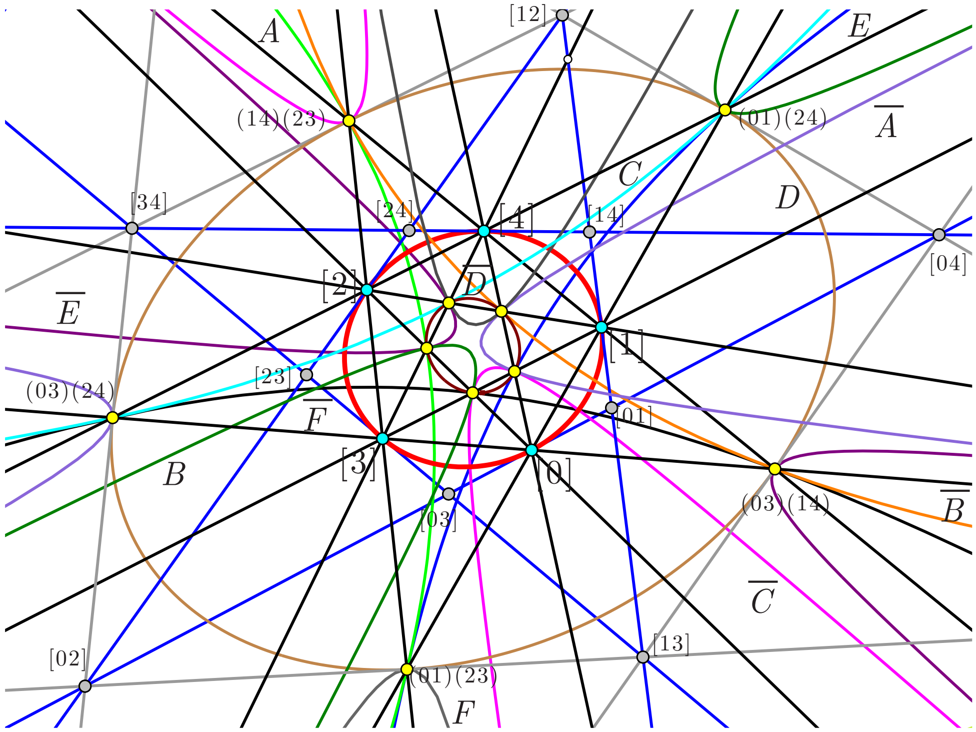

Figure 7: The twelve dihedral conics of the Points $[0]\,,[1]\,,[2]\,,[3]$ and $[4]$

Here are the other six conics associated to the opposites orderings $\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{E}$ and $\overline{F}$.

$$c\left(\overline{A}\right) = c\,(03124) : \begin{pmatrix} (13)\,(24) \\ [14]\,[23] \end{pmatrix}, \begin{pmatrix} (04)\,(12) \\ [02]\,[14] \end{pmatrix}, \begin{pmatrix} (03)\,(24) \\ [02]\,[34] \end{pmatrix}, \begin{pmatrix} (04)\,(13) \\ [01]\,[34] \end{pmatrix}, \begin{pmatrix} (03)\,(12) \\ [01]\,[23] \end{pmatrix}$$

$$c\left(\overline{B}\right) = c\,(03214) : \begin{pmatrix} (14)\,(23) \\ [13]\,[24] \end{pmatrix}, \begin{pmatrix} (04)\,(12) \\ [01]\,[24] \end{pmatrix}, \begin{pmatrix} (03)\,(14) \\ [01]\,[34] \end{pmatrix}, \begin{pmatrix} (04)\,(23) \\ [02]\,[34] \end{pmatrix}, \begin{pmatrix} (03)\,(12) \\ [02]\,[13] \end{pmatrix}$$

$$c\left(\overline{C}\right) = c\,(02314) : \begin{pmatrix} (14)\,(23) \\ [12]\,[34] \end{pmatrix}, \begin{pmatrix} (04)\,(13) \\ [01]\,[34] \end{pmatrix}, \begin{pmatrix} (02)\,(14) \\ [01]\,[24] \end{pmatrix}, \begin{pmatrix} (04)\,(23) \\ [03]\,[24] \end{pmatrix}, \begin{pmatrix} (02)\,(13) \\ [03]\,[12] \end{pmatrix}$$

$$c\left(\overline{D}\right) = c\,(02134) : \begin{pmatrix} (12)\,(34) \\ [14]\,[23] \end{pmatrix}, \begin{pmatrix} (04)\,(13) \\ [03]\,[24] \end{pmatrix}, \begin{pmatrix} (02)\,(34) \\ [03]\,[24] \end{pmatrix}, \begin{pmatrix} (04)\,(12) \\ [01]\,[24] \end{pmatrix}, \begin{pmatrix} (02)\,(13) \\ [01]\,[23] \end{pmatrix}$$

$$c\left(\overline{E}\right) = c\,(02143) : \begin{pmatrix} (12)\,(34) \\ [13]\,[24] \end{pmatrix}, \begin{pmatrix} (03)\,(14) \\ [04]\,[13] \end{pmatrix}, \begin{pmatrix} (02)\,(34) \\ [04]\,[23] \end{pmatrix}, \begin{pmatrix} (03)\,(12) \\ [01]\,[23] \end{pmatrix}, \begin{pmatrix} (02)\,(14) \\ [01]\,[24] \end{pmatrix}$$

$$c\left(\overline{F}\right) = c\,(02413) : \begin{pmatrix} (13)\,(24) \\ [12]\,[34] \end{pmatrix}, \begin{pmatrix} (03)\,(14) \\ [01]\,[34] \end{pmatrix}, \begin{pmatrix} (02)\,(13) \\ [01]\,[23] \end{pmatrix}, \begin{pmatrix} (03)\,(24) \\ [04]\,[23] \end{pmatrix}, \begin{pmatrix} (02)\,(14) \\ [04]\,[12] \end{pmatrix}$$

5 Adjacency of dihedral conics and the twisted icosahedron

If we study the list of Diagonal point and Exagonal line pairs associated to each dihedral order, we observe three possible relations between any two given dihedral conics. Without loss of generality, by the obvious S_5 symmetry we will illustrate the situation with respect to the fixed dihedral conic $c\,(D) = c\,(01423)$. Then there are exactly 5 other dihedral conics that share two Diagonal points and two (non-associated) Exagonal lines, these are $c\,(C)$, $c\,(E)$, $c\,(A)$, $c\left(\overline{C}\right)$ and $c\left(\overline{E}\right)$. We will say that two dihedral conics that have this relation

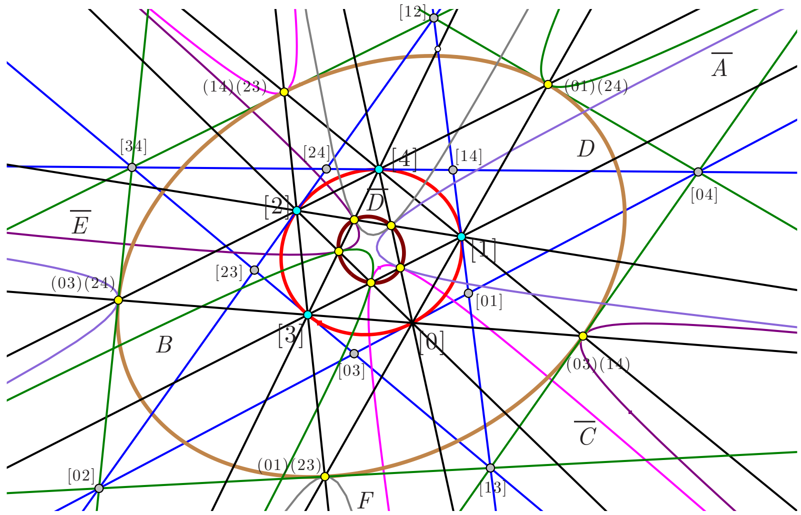

Figure 8: Dihedral conics adjacent to $c(D)$ (light brown)

are **adjacent**, and note that this corresponds exactly to adjacency in the twisted icosahedral graph X whose vertex labels are the corresponding dihedral orderings.

Let us make some observations about this adjacency in more detail for the case of $c\,(D)$ (light brown) and $c\,(C)$ (light blue), and by symmetry these remarks hold more generally for any two adjacent dihedral conics. The respective Diagonal points and Exagonal lines pairs for $c\,(C)$ and $c\,(D)$ are as follows:

$$c\,(C) = c\,(01243) : \begin{pmatrix} (12)\,(34) \\ [14]\,[23] \end{pmatrix}, \begin{pmatrix} (03)\,(24) \\ [04]\,[23] \end{pmatrix}, \begin{pmatrix} (01)\,(34) \\ [04]\,[13] \end{pmatrix}, \begin{pmatrix} (03)\,(12) \\ [02]\,[13] \end{pmatrix}, \begin{pmatrix} (01)\,(24) \\ [02]\,[14] \end{pmatrix}$$

$$c\,(D) = c\,(01423) : \begin{pmatrix} (14)\,(23) \\ [12]\,[34] \end{pmatrix}, \begin{pmatrix} (03)\,(24) \\ [02]\,[34] \end{pmatrix}, \begin{pmatrix} (01)\,(23) \\ [02]\,[13] \end{pmatrix}, \begin{pmatrix} (03)\,(14) \\ [04]\,[13] \end{pmatrix}, \begin{pmatrix} (01)\,(24) \\ [04]\,[12] \end{pmatrix}.$$

Both dihedral conics pass through the Diagonal points $(03)\,(24)$ and $(01)\,(24)$, and are tangent to the Exagonal lines $[04]\,[13]$ and $[02]\,[13]$. Four of the five column vectors are involved here, all except

$$\begin{pmatrix} (12)\,(34) \\ [14]\,[23] \end{pmatrix} \quad \text{for} \quad c\,(C)\,, \quad \text{and} \quad \begin{pmatrix} (14)\,(23) \\ [12]\,[34] \end{pmatrix} \quad \text{for} \quad c\,(D)\,.$$

We may note the symmetry between these two, both excluding the point $[0]$, which is the only point in the dihedral orders $c\,(D) = c\,(01423)$ and $c\,(C) = c\,(01243)$ which have the same neighbours in both orders.

There is another important relation between dihedral conics: we say two such are **distant** when they share a Diagonal point/Exagonal line pair. The conics distant to $c\,(D)$ are shown in the Figure, they are $c\left(\overline{C}\right) = c\,(02314)$ in pink, $c\left(\overline{A}\right) = c\,(03124)$ in purple, $c\,(F) = c\,(01234)$ in gray, $c\,(F) = c\,(01234)$ in green and $c\left(\overline{E}\right) = c\,(02143)$ in magenta. The respective Diagonal points and Exagonal lines pairs for $c\,(D)$ and $c\left(\overline{C}\right)$ are as follows:

$$c\,(D) = c\,(01423) : \begin{pmatrix} (14)\,(23) \\ [12]\,[34] \end{pmatrix}, \begin{pmatrix} (03)\,(24) \\ [02]\,[34] \end{pmatrix}, \begin{pmatrix} (01)\,(23) \\ [02]\,[13] \end{pmatrix}, \begin{pmatrix} (03)\,(14) \\ [04]\,[13] \end{pmatrix}, \begin{pmatrix} (01)\,(24) \\ [04]\,[12] \end{pmatrix}$$

$$c\left(\overline{C}\right) = c\,(02314) : \begin{pmatrix} (14)\,(23) \\ [12]\,[34] \end{pmatrix}, \begin{pmatrix} (04)\,(13) \\ [01]\,[34] \end{pmatrix}, \begin{pmatrix} (02)\,(14) \\ [01]\,[24] \end{pmatrix}, \begin{pmatrix} (04)\,(23) \\ [03]\,[24] \end{pmatrix}, \begin{pmatrix} (02)\,(13) \\ [03]\,[12] \end{pmatrix}.$$

These share exactly the same pair associated to the vertex 0 which was not overlapping in the case of the adjacent dihedral conics $c(D)$ and $c(C)$. Geometrically the distant relation corresponds to two conics being tangent at two points, but in a strong sense with regard to the original Points and Lines, with the common points and common tangents being Diagonal points and Exagonal lines respectively.

There are two other possible relations between Dihedral conics: that they are actually *identical*, and also that they are *opposite*. Opposite dihedral conics are illustrated by the case of $c(D)$ and $c\left(\overline{D}\right)$, here are the respective Diagonal points and Exagonal lines pairs:

$$\begin{aligned} c(D) = c(01423) &: \begin{pmatrix} (14)\,(23) \\ [12]\,[34] \end{pmatrix}, \begin{pmatrix} (03)\,(24) \\ [02]\,[34] \end{pmatrix}, \begin{pmatrix} (01)\,(23) \\ [02]\,[13] \end{pmatrix}, \begin{pmatrix} (03)\,(14) \\ [04]\,[13] \end{pmatrix}, \begin{pmatrix} (01)\,(24) \\ [04]\,[12] \end{pmatrix} \\ c\left(\overline{D}\right) = c(02134) &: \begin{pmatrix} (12)\,(34) \\ [14]\,[23] \end{pmatrix}, \begin{pmatrix} (04)\,(13) \\ [03]\,[24] \end{pmatrix}, \begin{pmatrix} (02)\,(34) \\ [03]\,[24] \end{pmatrix}, \begin{pmatrix} (04)\,(12) \\ [01]\,[24] \end{pmatrix}, \begin{pmatrix} (02)\,(13) \\ [01]\,[23] \end{pmatrix} \end{aligned}$$

We see that these share neither Diagonal points nor Exagonal lines. But nevertheless there is an obvious symmetry between these pairs: if we replace a Diagonal point/Exagonal line pair for one of these dihedral conics with the dual objects coming from the original Conic C, then we get a Diagonal point/Exagonal line pair for the opposite dihedral Conic.

6 Main theorems and proofs

To prove the various assertions, we resort to the familiar strategy of choosing suitable coordinates and making computations to verify claims. Since we explicitly give all the relevant formulas for points, lines and conics, the reader can check our computations directly. To help further investigations in this direction, we follow exactly the labelling used by E. Kasner in [5], this helps us in keeping the necessary book-keeping down to a manageable level. This also minimizes, in a reasonably simple way, the complexity of the formulas that occur.

Without loss of generality, we may apply a projective linear transformation to place a set of five generic points into a standard position. Using the usual homogeneous coordinates for the projective plane over a field (which we do not specify), and the Fundamental theorem which allows us to place four points where we like, we may assume our five basic Points to be:

$$[0] = [1:0:0], [1] = [0:1:0], [2] = [0:0:1], [3] = [1:1:1], [4] = [a:b:1].$$

We now proceed to make computations, whose results we summarize.

The conic through these five Points is:

$$(ab-a)\,yz + (b-ab)\,xz + (a-b)\,xy = 0.$$

The tangent lines to the conic at the Points are the Lines:

$$\begin{aligned} (0) &= \langle 0 : a-b : b-ab \rangle & (3) &= \langle ab-a : b-ab : a-b \rangle \\ (1) &= \langle a-b : 0 : ab-a \rangle & (4) &= \langle b(1-b) : a(a-1) : ab(b-a) \rangle \\ (2) &= \langle b-ab : ab-a : 0 \rangle \end{aligned}$$

The five Lines meet at the ten Exagonal points:

$$[01] = [a(1-b) : b(a-1) : a-b] \qquad [13] = [a(b-1) : b-2a+ab : b-a]$$
$$[02] = [a(b-1) : b(a-1) : a-b] \qquad [14] = [a(1-b) : b(a-2b+1) : a-b]$$
$$[03] = [a-2b+ab : b(a-1) : a-b] \qquad [23] = [a(b-1) : b(a-1) : -a-b+2ab]$$
$$[04] = [-a(b-2a+1) : b(a-1) : a-b] \qquad [24] = [a(b-1) : b(a-1) : a+b-2]$$
$$[12] = [a(b-1) : b(a-1) : a(b-a)] \qquad [34] = [a(b+1) : b(a+1) : a+b].$$

The five Points join at the ten Diagonal lines:

$$(01) = \langle 0:0:1\rangle \qquad (12) = \langle 1:0:0\rangle \qquad (23) = \langle 1:-1:0\rangle$$
$$(02) = \langle 0:1:0\rangle \qquad (13) = \langle 1:0:-1\rangle \qquad (24) = \langle -b:a:0\rangle$$
$$(03) = \langle 0:-1:1\rangle \qquad (14) = \langle -1:0:a\rangle \qquad (34) = \langle b-1:1-a:a-b\rangle.$$
$$(04) = \langle 0:-1:b\rangle$$

The join of two Exagonal points which do not lie on a common Tangent is an Exagonal line, and there are fifteen:

$$[01][23] = \langle 1:1:-1\rangle \qquad [03][14] = \langle b-1:1-a:ab-2b+1\rangle$$
$$[01][24] = \langle b:a:-ab\rangle \qquad [03][24] = \left\langle b(b-1):2b-a-b^2:b(a-b)\right\rangle$$
$$[01][34] = \langle b-1:a-1:1-ab\rangle \qquad [04][12] = \langle -b(a+1):a(2a-b+1):2ab\rangle$$
$$[02][13] = \langle 1:-1:1\rangle \qquad [04][13] = \langle 1-b:a-1:(-2a+ab+1)\rangle$$
$$[02][14] = \langle b:-a:ab\rangle \qquad [04][23] = \left\langle b(b-1):b^2-2ab+a:b(a-b)\right\rangle$$
$$[02][34] = \left\langle b(b-1):a-b^2:b(b-a)\right\rangle \qquad [13][24] = \left\langle a^2-2a+b:a(1-a):a(a-b)\right\rangle$$
$$[03][12] = \langle -b(a+1):a(b+1):2b\rangle \qquad [14][23] = \left\langle a^2-2ba+b:a(a-1):a(b-a)\right\rangle,$$

as well as

$$[12][34] = \left\langle b\left(a+b+a^2b-2a^2-a^3\right):a(a-1)\left(a+b+ab-b^2\right):2ab(a-b)\right\rangle.$$

The join of two Diagonal lines which do not share a common initial Point is a Diagonal point, and there are fifteen:

$$(01)(23) = [1:1:0] \qquad (02)(14) = [a:0:1]$$
$$(01)(24) = [a:b:0] \qquad (02)(34) = [a-b:0:1-b]$$
$$(01)(34) = [a-1:b-1:0] \qquad (03)(12) = [0:1:1]$$
$$(02)(13) = [1:0:1] \qquad (03)(14) = [a:1:1]$$

and

$$(03)(24) = [a:b:b] \qquad (12)(34) = [0:b-a:1-a]$$
$$(04)(12) = [0:b:1] \qquad (13)(24) = [a:b:a]$$
$$(04)(13) = [1:b:1] \qquad (14)(23) = [a:a:1].$$
$$(04)(23) = [b:b:1]$$

We may now verify the two well-known theorems in this situation that we began our discussion with.

Theorem 3. *The Exagonal line* $[ij][mn]$ *passes through the Diagonal points* $(im)(jn)$ *and* $(in)(jm)$.

Proof. The proof is straightforward, as we have the equations of the Exagonal lines and the Diagonal points. □

Theorem 4. *The Diagonal point* $(ij)(mn)$ *lies on the Exagonal lines* $[im][jn]$ *and* $[in][jm]$.

Proof. The proof is straightforward, as we have the equations of the Diagonal points and the Exagonal lines. □

And then here is our main result, using our earlier definition of the twelve dihedral conics associated to the twelve dihedral orderings.

Theorem 5 (Dihedral conics)**.** *For any dihedral order* $ijklm$, *the dihedral conic* $c = c(ijklm)$ *passes through the Diagonal points*

$$(ij)(kl) \qquad (jk)(lm) \qquad (im)(kl) \qquad (ij)(lm) \qquad (im)(jk)$$

and is tangent at those points to the Exagonal lines

$$[ik][jl] \qquad [jl][km] \qquad [il][km] \qquad [il][jm] \qquad [ik][jm]$$

respectively.

Proof. The equations of the six dihedral conics

$$\begin{array}{lll} c(A) = c(01432) & c(B) = c(01342) & c(C) = c(01243) \\ c(D) = c(01423) & c(E) = c(01324) & c(F) = c(01234) \end{array}$$

are

$$\begin{aligned}
c(A):\ & (1-b)x^2 + (1-a)y^2 + a(a-b)z^2 + a(1-b)yz + (b+ab-2a)zx \\
& \qquad + (a+b-2)xy = 0 \\
c(B):\ & b(b-1)x^2 + a(a-1)y^2 + b(b-a)z^2 + a(b-1)yz + \left(b+ab-2b^2\right)zx \\
& \qquad + (a+b-2ab)xy = 0 \\
c(C):\ & b(b-1)x^2 + a(a-1)y^2 + a(a-b)z^2 + \left(a+ab-2a^2\right)yz + b(a-1)zx \\
& \qquad + (a+b-2ab)xy = 0 \\
c(D):\ & -bx^2 - ay^2 + a(b-a)z^2 + a(1-b)yz + b(a-1)zx + (a+b)xy = 0 \\
c(E):\ & -bx^2 - ay^2 + b(a-b)z^2 + a(b-1)yz + b(1-a)zx + (a+b)xy = 0 \\
c(F):\ & (1-b)x^2 + (1-a)y^2 + b(b-a)z^2 + (a-2b+ab)yz + b(1-a)zx \\
& \qquad + (a+b-2)xy = 0
\end{aligned}$$

and the equations of the six dihedral conics

$$\begin{array}{lll} c\left(\overline{A}\right) = c(04213) & c\left(\overline{B}\right) = c(03214) & c\left(\overline{C}\right) = c(02314) \\ c\left(\overline{D}\right) = c(04312) & c\left(\overline{E}\right) = c(03412) & c\left(\overline{F}\right) = c(02413) \end{array}$$

are

$$c\left(\overline{A}\right): b\,(b-1)\,x^2+ay^2+abz^2-a\,(b+1)\,yz+b\,(1-a)\,zx+(a-b)\,xy=0$$
$$c\left(\overline{B}\right): (b-1)\,x^2-ay^2-abz^2+a\,(b+1)\,yz+b\,(1-a)\,zx+(a-b)\,xy=0$$
$$c\left(\overline{C}\right): bx^2+(1-a)\,y^2+abz^2+a\,(b-1)\,yz+b\,(-a-1)\,zx+(a-b)\,xy=0$$
$$c\left(\overline{D}\right): b\,(1-b)\,x^2+(a-1)\,y^2+b\,(a-b)\,z^2+(2b-a-ab)\,yz+b\,(2b-1-a)\,zx + (a-b)\,xy=0$$
$$c\left(\overline{E}\right): (1-b)\,x^2+a\,(a-1)\,y^2+a\,(a-b)\,z^2+a\,(1+b-2a)\,yz+(b-2a+ab)\,zx + (a-b)\,xy=0$$
$$c\left(\overline{F}\right): -bx^2+a\,(1-a)\,y^2-abz^2+a\,(b-1)\,yz+b\,(1+a)\,zx+(a-b)\,xy=0.$$

The general homogeneous equation of the second degree in x, y, z has the form

$$\alpha x^2+\beta y^2+\gamma z^2+2\delta yz+2\lambda zx+2\mu xy=0$$

and if $[x_1 : y_1 : z_1]$ is a point on the conic then the equation of the tangent line at $[x_1 : y_1 : z_1]$ is

$$\langle \alpha x_1+\mu y_1+\lambda z_1 : \mu x_1+\beta y_1+\delta z_1 : \lambda x_1+\delta y_1+\gamma z_1\rangle\,. \tag{2}$$

So we may now verify that each of the twelve dihedral conics does pass through the stated Diagonal points and is tangent at those points to the stated Exagonal lines. This is made simpler by utilizing the inherent symmetry in the situation: because S_5 acts as a symmetry group on the dihedral orderings, it is sufficient to verify the above claim for any one dihedral conic, as the others are obtained from that one by a suitable permutation of the indices. □

7 Spheres in the twisted icosahedron and icosahedron

We can visualize the combinatorics of the adjacent and distant relations between dihedral conics with the twisted icosahedron. If we fix a vertex in this graph, such as A, then we can consider partitioning X into distance sets from A. Let S_i denote those vertices of graph distance i from A, so that

$$S_0=\{A\} \qquad S_1=\left\{B\ D\ F\ \overline{C}\ \overline{E}\right\} \qquad S_2=\left\{\overline{B}\ \overline{D}\ \overline{F}\ C\ E\right\} \qquad S_3=\left\{\overline{A}\right\}$$

So S_0 is A itself, S_1 consists of adjacent points of A, S_2 consists of distant points from A, and S_3 consists of the opposite point $\overline{A}$ of A.

We call S_1 the **adjacent sphere** around A, and S_2 the **distant sphere** around A. Clearly S_1 is also the distant sphere around $\overline{A}$, while S_2 is also the adjacent sphere around $\overline{A}$. In Figure 9 we see the adjacent sphere around A in blue, the distant sphere around A in green, as well as the opposite point $\overline{A}$ (in orange).

It is useful to compare this graph with that of the regular icosahedron I, showing also a point A, its adjacent sphere and distant sphere, and the unique opposite point $\overline{A}$.

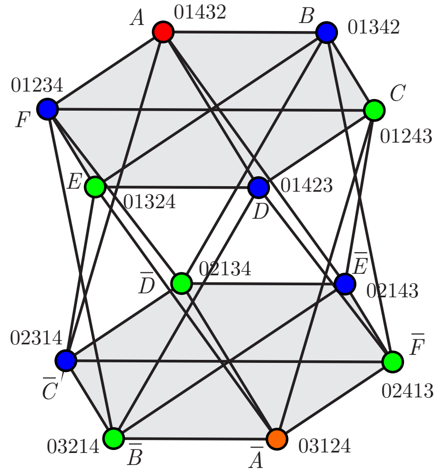

Figure 9: Circles centred at A in the twisted icosahedron X

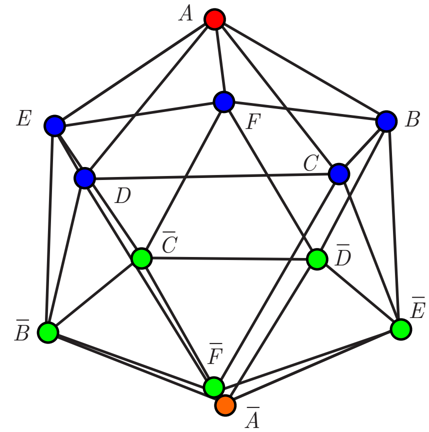

Figure 10: Circles centred at A in the icosahedral graph I

8 Adjacency matrix of the twisted icosahedral graph X

With respect to the ordering $A, B, C, D, E, F, \overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{E}, \overline{F}$ of the vertices, the adjacency matrix of X is

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

with characteristic polynomial

$$(X-5)\,(X+5)\,(X-1)^5\,(X+1)^5 .$$

For comparison, the characteristic polynomial of the icosahedral graph I is

$$(X-5)\,(X+1)^5\left(X^2-5\right)^3 .$$

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Received February 20, 2020; final from August 28, 2020.