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A Rational Trigonometric Relationship Between the Dihedral Angles of a Tetrahedron and its Circumradius

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Abstract. This paper will extend a known relationship between the circumradius and dihedral angles of a tetrahedron in three-dimensional Euclidean space to three-dimensional affine space over a general field not of characteristic two or three, using only the framework of rational trigonometry devised by Wildberger. In this framework, a linear algebraic view of trigonometry is presented, which allows the associated three-dimensional vector space of such a three-dimensional affine space to be equipped with a non-degenerate symmetric bilinear form. This will also generalise the results presented to arbitrary geometries parameterised by such a non-degenerate symmetric bilinear form.

Key Words: rational trigonometry, tetrahedron, circumradius, dihedral angle, symmetric bilinear form

MSC 2020: 51M04, 51N10, 15A63, 11E04

1 Introduction

The following result, from [3], establishes a relationship between the dihedral angles of a tetrahedron in three-dimensional Euclidean space over the real number field and its circumradius.

Theorem 1. *Let A_0, A_1, A_2 and A_3 be the points of a tetrahedron in three-dimensional Euclidean space over the real number field. Let R be its circumradius, and for distinct i and j in the set $\{0,1,2,3\}$, let θ_{ij} be the interior dihedral angle between A_i and A_j. Then, the volume V of the tetrahedron is*

$$V = \frac{32}{3}\frac{N_0N_1N_2N_3}{M^{\frac{3}{2}}}R^3$$

where, for l in the set $\{0,1,2,3\}$,

$$N_l = \det\begin{pmatrix} 1 & \cos\theta_{ij} & \cos\theta_{ik} \\ \cos\theta_{ji} & 1 & \cos\theta_{jk} \\ \cos\theta_{ki} & \cos\theta_{kj} & 1 \end{pmatrix}$$

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and

$$M = -\det \begin{pmatrix} 0 & \sin^2\theta_{01} & \sin^2\theta_{02} & \sin^2\theta_{03} \\ \sin^2\theta_{10} & 0 & \sin^2\theta_{12} & \sin^2\theta_{13} \\ \sin^2\theta_{20} & \sin^2\theta_{21} & 0 & \sin^2\theta_{23} \\ \sin^2\theta_{30} & \sin^2\theta_{31} & \sin^2\theta_{32} & 0 \end{pmatrix}.$$

In this paper, we aim to derive a similar result using only the framework of rational trigonometry from [12]. In rational trigonometry, the classical metrical notions of distance and angle are replaced respectively by quadrance and spread, which have purely linear algebraic definitions. This allows us to define the metrical notions of rational trigonometry in three-dimensional affine space; we can thus equip to its associated three-dimensional vector space a non-degenerate symmetric bilinear form that can be represented by an invertible symmetric 3×3 matrix, for which we then have a generalised definition of perpendicularity. This gives us a general metrical structure by which the results seen in this paper can be generalised to arbitrary geometries. We should also note that these results can also be generalised to arbitrary fields not of characteristic 2 or 3, so that the Zero denominator convention in [12, p. 28] is adopted without constant reference.

We will extend a known result from [4] with regards to the circumradius of a tetrahedron and its volume and side lengths to a general metrical framework, which allows us to obtain a rational analog of the aforementioned result. Proving this result also allows us to obtain a result pertaining to the ratio of the product of opposing dihedral angles to the product of opposing side lengths, a result found in [9] but verified here in a different way, and to express the circumradius explicitly in terms of the dihedral angles, volume and areas of the tetrahedron. The notions of area, volume and dihedral angle from classical trigonometry will be replaced by the rational trigonometric notions of quadrea, quadrume and dihedral spread to suit the demands of the paper.

We will also adopt a novel approach to proving the results presented in this paper, based on the framework of [7]. Here, we take an affine map from a general tetrahedron to a special tetrahedron whose points are

$$X_0 = [0,0,0], \quad X_1 = [1,0,0], \quad X_2 = [0,1,0] \quad \text{and} \quad X_3 = [0,0,1].$$

This special tetrahedron will be named the *Standard tetrahedron*, and it allows us to analyse a specific tetrahedron over a general symmetric bilinear form rather than a general tetrahedron over a specific symmetric bilinear form. Using this tool, a key property of the affine map implies that any result we prove for the Standard tetrahedron can be generalised to a general tetrahedron, by way of the inverse affine map; thus, it is sufficient that any result presented in this paper is proven for the Standard tetrahedron.

With the use of the Standard tetrahedron, we may be able to prove our desired results using this powerful mechanism, which makes computationally-intensive problems more manageable.

2 Preliminaries

We start with the three-dimensional affine space over a general field $\mathbb{F}$ not of characteristic 2 or 3, which we will denote by $\mathbb{A}^3$. The associated three-dimensional vector space $\mathbb{V}^3$, which contains three-dimensional row vectors, is then equipped with a non-degenerate symmetric bilinear form represented by an invertible symmetric 3×3 matrix B and defined by

$$u \cdot_B v = uBv^T$$

for vectors u and v in $\mathbb{V}^3$, which we will call the *B-scalar product* of u and v (see [8]). From this, we also define the *B-quadratic form* by

$$Q_B(v) = v \cdot_B v$$

for a vector v in $\mathbb{V}^3$. If $u \cdot_B v = 0$ then u and v are said to be *B-perpendicular.* We also say that a vector v is *B-null* if $Q_B(v) = 0$; note that if B is positive definite, then $v = \mathbf{0}$ is the only B-null vector.

The primary objects in $\mathbb{A}^3$ are called *points* and are denoted in this paper as a triple enclosed in rectangular brackets, and the primary objects in $\mathbb{V}^3$ are called *vectors* and are typically denoted as a three-dimensional row matrix. The association mentioned above is described by the operation

$$X + v = Y$$

for points X and Y in $\mathbb{A}^3$, and v a vector in $\mathbb{V}^3$, which then allows us to define a vector between X and Y by

$$\overrightarrow{XY} = v \sim Y - X.$$

Here, we denote $Y - X$ to be the affine difference of two points in $\mathbb{A}^3$, which is equivalent to the vector v.

2.1 Tetrahedron in three-dimensional affine space

A *tetrahedron* in $\mathbb{A}^3$ is a set of four points in $\mathbb{A}^3$ and typically denoted as $\overline{A_0A_1A_2A_3}$. An *edge* of a tetrahedron $\overline{A_0A_1A_2A_3}$ is then a subset containing any two distinct points of $\overline{A_0A_1A_2A_3}$ and is denoted by $\overline{A_iA_j}$ for integers i and j satisfying $0 \le i < j \le 3$. Furthermore, a *triangle* of a tetrahedron $\overline{A_0A_1A_2A_3}$ is a subset of any three distinct points of $\overline{A_0A_1A_2A_3}$ and is denoted by $\overline{A_iA_jA_k}$ for integers i, j and k satisfying $0 \le i < j < k \le 3$. Note that there are six edges and four triangles associated to any tetrahedron in $\mathbb{A}^3$, and that there are three edges associated to each triangle of such a tetrahedron. We will also define the *midpoint* of the edge $\overline{A_iA_j}$ to be the point M_{ij} satisfying

$$\overrightarrow{A_iM_{ij}} = \overrightarrow{M_{ij}A_j} = \frac{1}{2}\overrightarrow{A_iA_j}.$$

Associated to each edge of a tetrahedron $\overline{A_0A_1A_2A_3}$ is a *B-quadrance*, which is the number

$$Q_B\left(\overline{A_iA_j}\right) = Q_B\left(\overrightarrow{A_iA_j}\right) = \overrightarrow{A_iA_j} \cdot_B \overrightarrow{A_iA_j}$$

for integers i and j satisfying $0 \le i < j \le 3$. This will be denoted for the rest of this paper by Q_{ij}.

Given a triangle $\overline{A_iA_jA_k}$ of a tetrahedron $\overline{A_0A_1A_2A_3}$, for integers i, j and k satisfying $0 \le i < j < k \le 3$, we have the three edges $\overline{A_iA_j}$, $\overline{A_iA_k}$ and $\overline{A_jA_k}$ associated to it, with respective B-quadrances Q_{ij}, Q_{ik} and Q_{jk}. This allows to associate to $\overline{A_iA_jA_k}$ the number

$$\mathcal{A}_B\left(\overline{A_iA_jA_k}\right) = A\left(Q_{ij}, Q_{ik}, Q_{jk}\right)$$

where

$$A(a,b,c) = (a+b+c)^2 - 2\left(a^2+b^2+c^2\right)$$

is *Archimedes's function* (see [12, p. 64]). This quantity is called the *B-quadrea* and will be denoted for the rest of this paper by $\mathcal{A}_{ijk}$.

Associated to a tetrahedron $\overline{A_0A_1A_2A_3}$ itself is the B-*quadrume*, which is the number

$$\mathcal{V}_B\left(\overline{A_0A_1A_2A_3}\right) = \frac{1}{2}E\left(Q_{23}, Q_{13}, Q_{12}, Q_{01}, Q_{02,}Q_{03}\right)$$

where

$$E\left(q_1, q_2, q_3, p_1, p_2, p_3\right) = \det\begin{pmatrix} 2p_1 & p_1+p_2-q_3 & p_1+p_3-q_2 \\ p_1+p_2-q_3 & 2p_2 & p_2+p_3-q_1 \\ p_1+p_3-q_2 & p_2+p_3-q_1 & 2p_3 \end{pmatrix}$$

is *Euler's four-point function* (see [12, p. 191]). This function is essentially the *Cayley-Menger determinant* (see [2], [5] and [11]) and it satisfies the properties

$$E\left(q_1, q_2, q_3, p_1, p_2, p_3\right) = E\left(p_1, p_2, p_3, q_1, q_2, q_3\right)$$

and

$$E\left(q_1, q_2, q_3, p_1, p_2, p_3\right) = E\left(q_i, q_j, q_k, p_i, p_j, p_k\right)$$

for any permutation i, j and k of the integers 1, 2 and 3. For the rest of this paper, this quantity is denoted by $\mathcal{V}$.

For $0 \leq i < j \leq 3$ and indices k and l distinct from i and j, we can associate to a pair of triangles $\overline{A_iA_jA_k}$ and $\overline{A_iA_jA_l}$ of a tetrahedron $\overline{A_0A_1A_2A_3}$ the number

$$E_{ij} = \frac{4Q_{ij}\mathcal{V}}{\mathcal{A}_{ijk}\mathcal{A}_{ijl}}.$$

This quantity will be called the B-*dihedral spread* (see [9]) between the triangles $\overline{A_iA_jA_k}$ and $\overline{A_iA_jA_l}$, with the edge $\overline{A_iA_j}$ being common in these two triangles. Here, the result of the Tetrahedron dihedral spread formula in [9] is used as a definition to simplify our discussion.

2.2 Standard tetrahedron

Consider an affine map which sends a general tetrahedron $\overline{A_0A_1A_2A_3}$ to the tetrahedron $\overline{X_0X_1X_2X_3}$, where

$$X_0 = [0,0,0], \quad X_1 = [1,0,0], \quad X_2 = [0,1,0] \quad \text{and} \quad X_3 = [0,0,1].$$

Such a tetrahedron will be called the *Standard tetrahedron* (see [7]). If we have a C-scalar product on $\mathbb{V}^3$, the affine map induces a new scalar product given by

$$\begin{aligned} u \cdot_C v &= uCv^T = u\left(LL^{-1}\right)C\left(LL^{-1}\right)^T v^T \\ &= (uL)\left[\left(L^{-1}\right)C\left(L^{-1}\right)^T\right](vL)^T \end{aligned}$$

where L is the matrix representing the linear component of the affine map. For $M = L^{-1}$, we set the matrix MCM^T to be the matrix B, so that

$$u \cdot_C v = (uL) \cdot_B (vL).$$

With this tool we may prove a result for a general tetrahedron by verifying it for the Standard tetrahedron without any loss of generality, due to the preservation of various geometric objects under an affine map.

2.2.1 Trigonometric quantities of the Standard tetrahedron

In what follows, let

$$B = \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix}$$

we define

$$r_1 = a_2 + a_3 - 2b_1, \quad r_2 = a_1 + a_3 - 2b_2, \quad r_3 = a_1 + a_2 - 2b_3$$

and

$$\Delta = \det B = a_1 a_2 a_3 + 2b_1 b_2 b_3 - a_1 b_1^2 - a_2 b_2^2 - a_3 b_3^2.$$

We will also define

$$\operatorname{adj} B = \begin{pmatrix} a_2 a_3 - b_1^2 & b_1 b_2 - a_3 b_3 & b_1 b_3 - a_2 b_2 \\ b_1 b_2 - a_3 b_3 & a_1 a_3 - b_2^2 & b_2 b_3 - a_1 b_1 \\ b_1 b_3 - a_2 b_2 & b_2 b_3 - a_1 b_1 & a_1 a_2 - b_3^2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix}$$

to be the *adjoint matrix* (see [1]) of B, so that we may define

$$D = \alpha_1 + \alpha_2 + \alpha_3 + 2\beta_1 + 2\beta_2 + 2\beta_3.$$

The B-quadrances of $\overline{X_0 X_1 X_2 X_3}$ are by definition

$$Q_{01} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T = a_1$$

and similarly

$$Q_{02} = a_2, \quad Q_{03} = a_3, \quad Q_{23} = r_1, \quad Q_{13} = r_2 \quad \text{and} \quad Q_{12} = r_3.$$

The B-quadreas of $\overline{X_0 X_1 X_2 X_3}$ are by definition

$$\begin{aligned} \mathcal{A}_{012} &= A\left(Q_{01}, Q_{02}, Q_{12}\right) = \left(a_1 + a_2 + r_3\right)^2 - 2\left(a_1^2 + a_2^2 + r_3^2\right) \\ &= 4\left(a_1 a_2 - b_3^2\right) = 4\alpha_3 \end{aligned}$$

and similarly

$$\mathcal{A}_{013} = 4\alpha_2, \quad \mathcal{A}_{023} = 4\alpha_1 \quad \text{and} \quad \mathcal{A}_{123} = 4D$$

and, by definition, the B-quadrume of $\overline{X_0 X_1 X_2 X_3}$

$$\begin{aligned} \mathcal{V} &= \frac{1}{2} \det \begin{pmatrix} 2Q_{01} & Q_{01} + Q_{02} - Q_{12} & Q_{01} + Q_{03} - Q_{13} \\ Q_{01} + Q_{02} - Q_{12} & 2Q_{02} & Q_{02} + Q_{03} - Q_{23} \\ Q_{01} + Q_{03} - Q_{13} & Q_{02} + Q_{03} - Q_{23} & 2Q_{03} \end{pmatrix} \\ &= \frac{1}{2} \det \begin{pmatrix} 2a_1 & 2b_3 & 2b_2 \\ 2b_3 & 2a_2 & 2b_1 \\ 2b_2 & 2b_1 & 2a_3 \end{pmatrix} = 4\Delta. \end{aligned}$$

Finally, the B-dihedral spreads of $\overline{X_0 X_1 X_2 X_3}$ are by definition

$$E_{01} = \frac{4Q_{01}\mathcal{V}}{\mathcal{A}_{012}\mathcal{A}_{013}} = \frac{4a_1\left(4\Delta\right)}{\left(4\alpha_3\right)\left(4\alpha_2\right)} = \frac{a_1\Delta}{\alpha_2\alpha_3}$$

and similarly

$$E_{02} = \frac{a_2\Delta}{\alpha_1\alpha_3}, \quad E_{03} = \frac{a_3\Delta}{\alpha_1\alpha_2}, \quad E_{23} = \frac{r_1\Delta}{\alpha_1 D}, \quad E_{13} = \frac{r_2\Delta}{\alpha_2 D} \quad \text{and} \quad E_{12} = \frac{r_3\Delta}{\alpha_3 D}.$$

2.3 Circumquadrance of tetrahedron

One of the most important centres of a tetrahedron is its *circumcentre* (see [6, pp. 82-83]). In a general metrical framework, the circumcentre will end up being dependent on the choice of non-degenerate symmetric bilinear form. With this in mind, we proceed to find a suitable generalisation.

We start by defining a *plane* through a point A and B-perpendicular to a vector v to be the space of points X satisfying the equation

$$v \cdot_B \overrightarrow{AX} = 0.$$

Then we may define a *B-midplane* associated to an edge $\overline{A_iA_j}$ of a tetrahedron $\overline{A_0A_1A_2A_3}$ to be a plane through the midpoint M_{ij} and B-perpendicular to the vector $\overrightarrow{A_iA_j}$, for $0 \leq i < j \leq 3$. There are six B-midplanes in total for a general tetrahedron. The following result establishes the concurrency of each of the six B-midplanes of a tetrahedron.

Theorem 2 (Tetrahedron B-circumcentre theorem)**.** *The six B-midplanes associated to each edge of a tetrahedron $\overline{A_0A_1A_2A_3}$ meet at a single point.*

Proof. Without loss of generality, transform the tetrahedron $\overline{A_0A_1A_2A_3}$ to the Standard tetrahedron $\overline{X_0X_1X_2X_3}$. Note that this will induce a new non-degenerate symmetric bilinear form, but since we started with a general symmetric bilinear form we may use the same one without any loss of generality. For $0 \leq i < j \leq 3$, let M_{ij} be the midpoint of $\overline{X_iX_j}$, so that any point $C = [x, y, z]$ on each B-midplane of $\overline{X_0X_1X_2X_3}$ satisfies the equation

$$\overrightarrow{X_iX_j} \cdot_B \overrightarrow{M_{ij}C} = 0.$$

This yields six equations, namely

$$a_1x + b_3y + b_2z = \frac{1}{2}a_1,$$

$$b_3x + a_2y + b_1z = \frac{1}{2}a_2,$$

$$b_2x + b_1y + a_3z = \frac{1}{2}a_3,$$

$$(b_3 - a_1)\,x + (a_2 - b_3)\,y + (b_1 - b_2)\,z = \frac{1}{2}\,(a_2 - a_1)\,,$$

$$(b_2 - a_1)\,x + (b_1 - b_3)\,y + (a_3 - b_2)\,z = \frac{1}{2}\,(a_3 - a_1)$$

and

$$(b_2 - b_3)\,x + (b_1 - a_2)\,y + (a_3 - b_1)\,z = \frac{1}{2}\,(a_3 - a_2)\,.$$

From the first three equations,

$$C = \left[\frac{\alpha_1a_1 + \beta_3a_2 + \beta_2a_3}{2\Delta}, \frac{\beta_3a_1 + \alpha_2a_2 + \beta_1a_3}{2\Delta}, \frac{\beta_2a_1 + \beta_1a_2 + \alpha_3a_3}{2\Delta}\right].$$

Substitute the co-ordinates of this point into the last three equations to get the desired result. □

The intersection point obtained from the proof of the Tetrahedron circumcentre theorem will be called the *B-circumcentre* of the Standard tetrahedron $\overline{X_0X_1X_2X_3}$; to obtain the B-circumcentre of a general tetrahedron, we merely perform the inverse affine map on such a point.

The B-quadrance between the B-circumcentre of a tetrahedron and any point of a tetrahedron is called the *B-circumquadrance*, and will be denoted by R. The B-circumquadrance of the Standard tetrahedron $\overline{X_0X_1X_2X_3}$ is then

$$R = Q_B\left(\overrightarrow{X_0C}\right) = \frac{A\left(a_1b_1, a_2b_2, a_3b_3\right) + a_1a_2a_3\left(D - 4\left(b_1+b_2+b_3\right)\right)}{4\Delta}$$

The following result, which is an extension of Crelle's result in [4] from the Euclidean setting to a more general setting, links the B-circumquadrance of a tetrahedron with its B-quadrume and B-quadrances.

Theorem 3 (Crelle's circumquadrance formula)**.** *For a tetrahedron $\overline{A_0A_1A_2A_3}$ with B-quadrances Q_{ij}, for $0 \le i < j \le 3$, B-quadrume $\mathcal{V}$ and B-circumquadrance R, the relation*

$$4\mathcal{V}R = A\left(Q_{01}Q_{23}, Q_{02}Q_{13}, Q_{03}Q_{12}\right)$$

is satisfied.

Proof. Without loss of generality, transform the tetrahedron $\overline{A_0A_1A_2A_3}$ to the Standard tetrahedron $\overline{X_0X_1X_2X_3}$; it is sufficient to prove the required result for this tetrahedron. So,

$$\begin{aligned} R &= \frac{A\left(a_1b_1, a_2b_2, a_3b_3\right) + a_1a_2a_3\left(a_1+a_2+a_3-2b_1-2b_2-2b_3\right)}{4\Delta} \\ &= \frac{A\left(a_1r_1, a_2r_2, a_3r_3\right)}{16\Delta} = \frac{A\left(Q_{01}Q_{23}, Q_{02}Q_{13}, Q_{03}Q_{12}\right)}{4\mathcal{V}}. \end{aligned}$$

The required result follows. □

Crelle's circumquadrance formula allows us to conveniently write the B-circumquadrance of $\overline{X_0X_1X_2X_3}$ as

$$R = \frac{A\left(a_1r_1, a_2r_2, a_3r_3\right)}{16\Delta}.$$

3 Main result

We now present the main result of this paper, which generalises Theorem 1 to the rational trigonometric setting over an arbitrary symmetric bilinear form.

Theorem 4 (Circumquadrance dihedral spread theorem)**.** *For a tetrahedron $\overline{A_0A_1A_2A_3}$ in $\mathbb{A}^3$, let $\mathcal{V}$ be its B-quadrume, $\mathcal{A}_{ijk}$ be the B-quadrea of the triangle $\overline{A_iA_jA_k}$, E_{ij} be the B-dihedral spread between $\overline{A_iA_jA_k}$ and $\overline{A_iA_jA_l}$, and R be its B-circumquadrance. Then,*

$$\left(\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}\right)^2 M = 1024\mathcal{V}^5R,$$

where

$$M = -\det\begin{pmatrix} 0 & E_{01} & E_{02} & E_{03} \\ E_{01} & 0 & E_{12} & E_{13} \\ E_{02} & E_{12} & 0 & E_{23} \\ E_{03} & E_{13} & E_{23} & 0 \end{pmatrix} = A\left(E_{01}E_{23}, E_{02}E_{13}, E_{03}E_{12}\right).$$

Proof. Perform an affine map on $\overline{A_0A_1A_2A_3}$ to the Standard tetrahedron $\overline{X_0X_1X_2X_3}$, so that we only require to prove this result on $\overline{X_0X_1X_2X_3}$. We then have

$$E_{01}E_{23} = \frac{a_1r_1\Delta^2}{\alpha_1\alpha_2\alpha_3 D}, \quad E_{02}E_{13} = \frac{a_2r_2\Delta^2}{\alpha_1\alpha_2\alpha_3 D}, \quad E_{03}E_{12} = \frac{a_3r_3\Delta^2}{\alpha_1\alpha_2\alpha_3 D}$$

and

$$\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123} = 256\alpha_1\alpha_2\alpha_3 D$$

so that

$$\begin{aligned} M &= A\left(E_{01}E_{23}, E_{02}E_{13}, E_{03}E_{12}\right) \\ &= \left(\frac{\Delta^2\left(a_1r_1 + a_2r_2 + a_3r_3\right)}{\alpha_1\alpha_2\alpha_3 D}\right)^2 - 2\left(\frac{\Delta^4\left(a_1^2r_1^2 + a_2^2r_2^2 + a_3^2r_3^2\right)}{\left(\alpha_1\alpha_2\alpha_3 D\right)^2}\right) \\ &= \frac{\Delta^4\left(a_1r_1 + a_2r_2 + a_3r_3\right)^2}{\left(\alpha_1\alpha_2\alpha_3 D\right)^2} - \frac{2\Delta^4\left(a_1^2r_1^2 + a_2^2r_2^2 + a_3^2r_3^2\right)}{\left(\alpha_1\alpha_2\alpha_3 D\right)^2} \\ &= \frac{\Delta^4 A\left(a_1r_1, a_2r_2, a_3r_3\right)}{\left(\alpha_1\alpha_2\alpha_3 D\right)^2} \end{aligned}$$

and thus

$$\begin{aligned} \left(\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}\right)^2 M &= \frac{\Delta^4 A\left(a_1r_1, a_2r_2, a_3r_3\right)}{\left(\alpha_1\alpha_2\alpha_3 D\right)^2}\left(256\alpha_1\alpha_2\alpha_3 D\right)^2 \\ &= 65536\Delta^4 A\left(a_1r_1, a_2r_2, a_3r_3\right). \end{aligned}$$

By Crelle's circumquadrance formula,

$$\left(\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}\right)^2 M = 1024R\left(1024\Delta^5\right) = 1024\left(4\Delta\right)^5 R = 1024\mathcal{V}^5 R$$

as required. □

We can now provide an alternative expression for the Circumquadrance dihedral spread theorem.

Corollary 5. *If $N = A\left(Q_{01}Q_{23}, Q_{02}Q_{13}, Q_{03}Q_{12}\right)$, where, for $0 \le i < j \le 3$, Q_{ij} denotes the B-quadrances of a tetrahedron $\overline{A_0A_1A_2A_3}$, then the Circumquadrance dihedral spread theorem can alternatively be expressed as*

$$\left(\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}\right)^2 M = 256\mathcal{V}^4 N.$$

Proof. From Crelle's circumquadrance formula, we have that

$$R = \frac{N}{4\mathcal{V}}.$$

Substitute into the Circumquadrance dihedral spread theorem to obtain

$$\left(\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}\right)^2 M = 1024\mathcal{V}^5\left(\frac{N}{4\mathcal{V}}\right) = 256\mathcal{V}^4 N,$$

as required. □

We can now derive a relationship between M and N, which originates from [9], we use the tools presented in this paper to provide an alternate proof.

Theorem 6 (Dihedral spread ratio theorem)**.** *Given a tetrahedron* $\overline{A_0A_1A_2A_3}$ *in* $\mathbb{A}^3$ *with* B*-dihedral spreads* E_{ij} *and* B*-quadrances* Q_{ij}*, where* $0 \leq i < j \leq 3$*, the relation*

$$\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}}$$

is satisfied.

Proof. We start with the fact that

$$A(\lambda a, \lambda b, \lambda c) = \lambda^2 A(a, b, c)$$

which can be easily verified by the reader. Now, let $\mathcal{V}$ be the B-quadrume of $\overline{A_0A_1A_2A_3}$ and let $\mathcal{A}_{ijk}$ be their B-quadreas, for $0 \leq i < j < k \leq 3$. Then rearrange the equation of Corollary 5 to get

$$M = \frac{256\mathcal{V}^4}{(\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123})^2} N = \left(\frac{16\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}\right)^2 N$$

for

$$M = A(E_{01}E_{23}, E_{02}E_{13}, E_{03}E_{12}) \quad \text{and} \quad N = A(Q_{01}Q_{23}, Q_{02}Q_{13}, Q_{03}Q_{12}).$$

So,

$$M = A\left(\frac{16\mathcal{V}^2 Q_{01}Q_{23}}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}, \frac{16\mathcal{V}^2 Q_{02}Q_{13}}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}, \frac{16\mathcal{V}^2 Q_{03}Q_{12}}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}\right).$$

and thus by comparison

$$\frac{E_{01}E_{23}}{Q_{01}Q_{23}} = \frac{E_{02}E_{13}}{Q_{02}Q_{13}} = \frac{E_{03}E_{12}}{Q_{03}Q_{12}} = \frac{16\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}$$

as required. □

From the proof of the Dihedral spread ratio theorem, we can define

$$K = \frac{16\mathcal{V}^2}{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}$$

which in [9] is called the *Richardson number* of a tetrahedron; this quantity originates from [10] and its geometric meaning is yet to be fully understood (though the author of the paper uses it frequently). From the Circumquadrance dihedral spread theorem, we see that

$$R = \frac{(\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123})^2 M}{1024\mathcal{V}^5} = \frac{M}{4\mathcal{V}}\left(\frac{\mathcal{A}_{012}\mathcal{A}_{013}\mathcal{A}_{023}\mathcal{A}_{123}}{16\mathcal{V}^2}\right)^2 = \frac{M}{4\mathcal{V}K^2}.$$

Crelle's circumquadrance formula is immediate from this, as $M = K^2 N$ from the proof of the Dihedral spread ratio theorem.

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