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The Circumcevian-Inversion Perspector of Two Triangles

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Abstract. Beginning with a point P in the plane of a triangle ABC, reflections and circumcircle-inversions are used to define a triangle $X'Y'Z'$ that is perspective to ABC. The perspector, denoted by $Cip(P)$, defines a transform, Cip, that is applied to selected curves; e.g., Cip maps the Euler line to itself in a manner well represented by Shinagawa coefficients, and in general Cip maps lines to conics. Barycentric coordinates are used to determine properties of Cip and related points and mappings. Four new equilateral triangles are presented.

Key Words: barycentric coordinates, circumcircle, circumcevian, inversion, perspective, Shinagawa coefficients

MSC 2020: 51N20, 51N15

1 Introduction

In the plane of a triangle ABC, let P be a point that is not on one of the sidelines, BC, CA, AB. Let Γ be the circumcircle of ABC, and let DEF be the circumcevian triangle of P (as in ([3], p. 221)); that is, D is the point, other than A, in which the line AP meets Γ, and likewise for the vertices E and F. Let X, Y, Z be the reflections of P in D, E, F, respectively, and let X', Y', Z' be the Γ-inverses of X, Y, Z, respectively. Let

$$P' = BY' \cap CZ'.$$

In Section 2, we prove that P' also lies on AX', so that ABC and $X'Y'Z'$ are perspective triangles, and P' is their perspector, which we call the circumcevian-inversion perspector of $X'Y'Z'$ and ABC.

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The rest of this section gives basics to be used in the remaining sections. Details can be found in Paul Yiu's excellent online book [8].

Homogeneous barycentric coordinates (henceforth simply *barycentrics*) of a point P are written $p : q : r$, whereas normalised, or absolute, barycentrics are written in standard ordered-triple notation: (p, q, r). Thus $p : q : r$ represents (kp, kq, kr) for some (usually unspecified) symmetric function $k = k(a, b, c)$. Barycentrics for many triangle centers are given in the Encyclopedia of Triangle Centers (henceforth ETC) [6]. Among those to appear in this paper are

$$\begin{aligned}
X(1) &= \text{incenter} = a : b : c \\
X(2) &= \text{centroid} = 1 : 1 : 1 \\
X(3) &= \text{circumcenter} = \sin 2A : \sin 2B : \sin 2C \\
&= a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2),
\end{aligned}$$

this being the center of Γ. The classical symbols for these points are also useful: I, G, O, respectively. Also to appear in the sequel is $H = X(4)$, the orthocenter. Since triangle centers necessarily have the form

$$f(a, b, c) : f(b, c, a) : f(c, a, b) \quad \text{or} \quad g(A, B, C) : g(B, C, A) : g(C, A, B),$$

as typified by the barycentrics for $X(3)$ shown above, it is convenient to reduce the notation to

$$f(a, b, c) \;::\quad \text{or} \quad g(A, B, C) \;::\; .$$

We shall also use Conway triangle notation [7]:

$$\begin{aligned}
S &= 2(\text{area of } ABC); \\
S_A &= bc \cos A = (b^2 + c^2 - a^2)/2, \;\; S_B = ca \cos B, \;\; S_C = ab \cos C; \\
S_\omega &= S \cot \omega = \frac{a^2 + b^2 + c^2}{2S}, \text{where } \omega = \text{Brocard angle of } ABC.
\end{aligned}$$

For $P = p : q : r$ not on a sideline BC, CA, AB, the cevians of P are the lines AP, BP, CP.

Several theorems and examples in this paper, discovered and investigated with the help of Mathematica, and have lengthy results —too lengthy to be fully reproduced here.

2 Main Theorem

Continuing the discussion in the first paragraph of Section 1, we have the following theorem, which is fundamental to the rest of the article.

Theorem 1. *Let P be a point that is not on a sideline of ABC. The triangle $X'Y'Z'$ formed by the circumcircle-inverses of the reflections X, Y, Z, of P in the vertices of the circumcevian triangle of P is perspective to ABC. The perspector is the circumcevian-inversion perspector of the triangles $X'Y'Z'$ and ABC (Figure 1).*

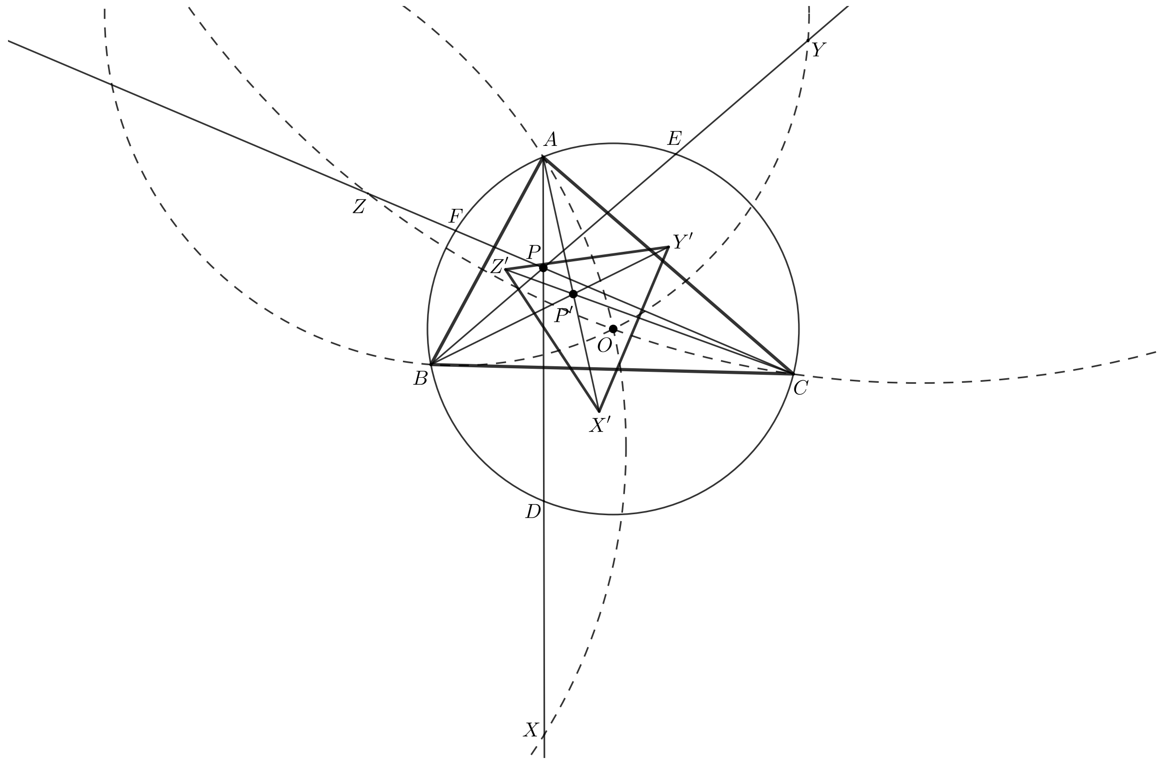

Figure 1: $P' = Cip(P)$

Proof. Referring to the definition of circumcevian-inversion and the associated notation in Section 1, let O_X, O_Y, O_Z be the circumcircles of triangles AOX, BOY, COZ, respectively, and let $\Lambda_X(P), \Lambda_Y(P), \Lambda_Z(P), \Lambda(P)$ be the powers of P with respect to O_X, O_Y, O_Z, Γ, respectively. Then

$$\Lambda_X(P) = AP \cdot PX = AP \cdot 2PD = 2\Lambda(P).$$

Likewise,

$$\Lambda_Y(P) = \Lambda_Z(P) = 2\Lambda(P),$$

so that the circles O_X, O_Y, O_Z are coaxal with the line OP as radical axis. The three circles meet in two points, one of which is O, and the other we denote by Q. Let $\mathcal{I}$ denote inversion in Γ, and note that the inverse of the three circles are lines:

$$\mathcal{I}(O_X) = A\mathcal{I}(X), \quad \mathcal{I}(O_Y) = B\mathcal{I}(Y), \quad \mathcal{I}(O_Z) = C\mathcal{I}(Z).$$

The lines AX', BY', CZ' concur in the point $\mathcal{I}(Q)$; i.e., this point is the perspector of $X'Y'Z'$ and ABC. $\square$

Henceforth, we refer to the point $\mathcal{I}(Q)$ as P' or $Cip(P)$, depending on context.

Example 2. Here we choose $P = I$, the incenter of ABC, with barycentric coordinates $a : b : c$. The circumcevian triangle of I is known as the 2nd circumperp triangle ([3], p. 164), and the triangle XYZ, as the excentral triangle ([3], p. 157); indeed, its vertices are the excenters of ABC, given by

$$X = -a : b : c, \quad Y = a : -b : c, \quad Z = a : b : -c.$$

The A-vertex of the circumcevian-inversion triangle $X'Y'Z'$ is given by

$$X' = a^2(a^2 - b^2 - c^2 + bc) : b^2(b^2 - c^2 - a^2 - ca) : c^2(c^2 - a^2 - b^2 - ab),$$

and $Cip(I)$ by

$$\begin{aligned} X(35) &= a^2(a^2 - b^2 - c^2 - bc) : b^2(b^2 - c^2 - a^2 - ca) : c^2(c^2 - a^2 - b^2 - ab) \\ &= \sin A + \sin 2A : \sin B + \sin 2B : \sin C + \sin 2C. \end{aligned}$$

It can be shown that the locus of a point Q such that the cevian triangle of Q is perspective to $X'Y'Z'$ is the pivotal isocubic having pole $X(35192)$ and pivot $X(21)$; this cubic passes through $X(i)$ for

$$i = 1, 3, 21, 35, 3467, 11107, 35193, 35194, 35195, 35196.$$

(The pivotal isocubic $pK(U, P)$ with pole U and pivot P is given by

$$ux(ry^2 - qz^2) + vy(pz^2 - rx^2) + wz(qx^2 - py^2) = 0.$$

See [5], especially pages 7 and 31.)

The locus of a point Q such that the anticevian triangle of Q is perspective to $X'Y'Z'$ is the pivotal isocubic having pole $X(50)$ and pivot $X(1)$; this cubic passes through $X(i)$ for

$$i = 1, 35, 36, 1094, 1095, 2169, 5353, 5357, 35198, 35199, 35200, 35201.$$

Theorem 3. *Let d be the directed length of segment PO and R the radius of Γ. Then P' is the point on line PO that satisfies*

$$\frac{P'P}{P'O} = 1 - \frac{d^2}{R^2}.$$

Proof.

$$\begin{aligned} \Lambda(P) &= R^2 - d^2 = AP \cdot PD; \\ \Lambda_X(P) &= AP \cdot PX = 2AP \cdot PD = 2(R^2 - d^2) = OP \cdot PQ; \\ PQ &= \frac{2(R^2 - d^2)}{d}, \text{ so that } OQ = \frac{2R^2 - d^2}{d}. \end{aligned}$$

Then

$$\begin{aligned} OQ \cdot OP' = R^2, &\text{ so that } OP' = \frac{R^2 d}{2R^2 - d^2}; \\ \frac{OP}{OP'} = \frac{2R^2 - d^2}{R^2}, &\text{ so that } \frac{P'P}{P'O} = 1 - \frac{d^2}{R^2}. \end{aligned}$$

□

Theorem 4. *P' is the $\{O, P\}$-harmonic conjugate of $\mathcal{I}(P)$.*

Proof. Let $Q' = \mathcal{I}(P)$. Then

$$\begin{aligned} OP \cdot OQ' &= R^2, \text{ so that } OQ' = \frac{R^2}{d}; \\ PQ' &= \frac{R^2 - d^2}{d}, \text{ whence } \frac{PQ'}{OQ'} = \frac{R^2 - d^2}{R^2} \\ &= 1 - \frac{d^2}{R^2}, \text{ so that } \frac{PQ'}{OQ'} = \frac{P'P}{P'O}. \end{aligned}$$

□

The following corollaries refer to the Poncelet porism ([1], [4]).

Corollary 5. *The circumcevian-inversion perspector of a poristically fixed point is poristically fixed.*

Proof. Suppose that a point P is poristically fixed, and let $P' = Cip(P)$. Let $\mathcal{I}(P)$ be the Γ-inverse of P. By Theorem 4,

$$P' = \{O, P\}\text{-harmonic conjugate of } \mathcal{I}(P).$$

Since O, P, and $\mathcal{I}(P)$ are poristically fixed, the same holds for P'. □

Corollary 6. *There are infinitely many poristically fixed triangle centers.*

Proof. Let I denote the incenter. Then $Cip(I), Cip(Cip(I)), Cip(Cip(Cip(I))), \ldots$ are all poristically fixed, since I lies inside Γ and $1 - d^2/R^2 > 0$. □

3 Barycentrics for P' and X'

In this section, we use Theorems 3 and 4 to derive barycentric coordinates for the circumcevian-inversion perspector P' of a point P having normalised barycentric coordinates (p, q, r).

The normalised barycentrics for the circumcenter are given by

$$O = (\frac{a^2 S_A}{2S^2}, \frac{b^2 S_B}{2S^2}, \frac{c^2 S_C}{2S^2}).$$

The barycentric distance formula ([8], p. 89) gives

$$\begin{aligned} d^2 = (PO)^2 &= -\sum_{cyc} a^2 (\frac{b^2 S_B}{2S^2} - q)(\frac{c^2 S_C}{2S^2} - r); \\ \frac{PP'}{P'O} = 1 - \frac{d^2}{R^2} &= 1 + \frac{\sum_{cyc} a^2 (\frac{b^2 S_B}{2S^2} - q)(\frac{c^2 S_C}{2S^2} - r)}{R^2}, \end{aligned}$$

so that

$$\begin{aligned} &P' = (p', q', r'), \text{ where } q' \text{ and } r' \text{ are defined cyclically from} \\ &p' = 8S^6 R^2 p + a^2 S_A \Big(4S^4 R^2 + \sum_{cyc} a^2 (b^2 S_B - 2qS^2)(c^2 S_C - 2rS^2)\Big). \end{aligned}$$

Switching from Conway notation to the variables a, b, c, and simplifying, we obtain the following homogeneous barycentrics:

$$P' = p(p+q+r)a^2b^2c^2 + a^2(b^2+c^2-a^2)(a^2qr+b^2rp+c^2pq) :: .$$

These barycentrics enable rapid identifications, by computer, of pairs (P, P'), such as these:

$$(X(1), X(35)), (X(2), X(7496)), (X(4), X(3520)),$$
$$(X(5), X(34864)), (X(6), X(574)).$$

Many other pairs (P, P') are known ([6]; see the preamble just before X(34864)).

Barycentrics for the $A-$vertex of the circumcevian-inversion triangle are given by

$$X' = a^2h(a,b,c) : b^2k(b,c,a) : c^2k(c,a,b), \text{where}$$

$$\begin{aligned} h(a,b,c) = {} & b^4(2a^2-2b^2-c^2)p^2r^2 + (c^4(2a^2-b^2-2c^2)p^2q^2 \\ & + b^4(a^2-b^2)pr^3 + c^4(a^2-c^2)pq^3 \\ & + b^2(3a^4-2a^2b^2-b^4+2a^2c^2-b^2c^2-c^4)pqr^2 \\ & + c^2(3a^4+2a^2b^2-b^4-2a^2c^2-b^2c^2-c^4)pq^2r \\ & + a^2c^2(a^2+b^2-c^2)q^3r + a^2b^2(a^2-b^2+c^2)qr^3; \\ & + b^2c^2(5a^2-3b^2-3c^2)p^2qr + a^2(a^2+b^2-c^2)(a^2-b^2+c^2)q^2r^2 \\ k(b,c,a) = {} & \Big(-c^2(2a^2-b^2+c^2)pq - a^2c^2q^2 - b^2(a^2-b^2+c^2)pr \\ & - a^2(a^2-b^2+2c^2)qr\Big)\Big(2c^2pq + c^2q^2 + 2b^2pr + (a^2+b^2+c^2)qr + b^2r^2\Big). \end{aligned}$$

In addition to X' in Example 2, we have, for $P = X(3)$,

$$X' = a^2(2a^2-b^2-c^2) - (b^2-c^2)^2 : 3b^2(b^2-c^2-a^2) : 3c^2(c^2-a^2-b^2).$$

4 Shinagawa coefficients

The Shinagawa coefficients of a triangle center

$$X = f(a,b,c) : f(b,c,a) : f(c,a,b)$$

on the Euler line are defined ([6], Introduction) as functions $g(a,b,c)$ and $h(a,b,c)$ such that

$$f(a,b,c) = g(a,b,c)S^2 + h(a,b,c)S_BS_C, \tag{1}$$

In (1), with regard to the homogeneity of barycentrics, we can (and do) assume that g and h are symmetric in (a,b,c), so that we can represent X unambiguously by

$$X = gS^2 + hS_BS_C : gS^2 + hS_CS_A : gS^2 + hS_AS_B$$

and represent X simply as (g, h). Since the circumcevian-inversion perspector X' also lies on the Euler line, we have $X' = (g', h')$ for some g' and h'. Our objective in this section is to determine g' and h'.

We begin by writing

$$\frac{OX}{XH} = \frac{r_1}{r_2} \text{ so that } X = r_2 S^2 + (2r_2 - r_1) S_B S_C : : ,$$

leading to

$$\frac{OX}{XH} = \frac{g+h}{2g}, \text{ and } OX = \frac{(g+h)(OH)}{3g+h}.$$

Next, we introduce two functions symmetric in a, b, c :

$$E = \frac{(S_B + S_C)(S_C + S_A)(S_A + S_B)}{S^2} = (\frac{abc}{S})^2 = 4R^2$$
$$F = \frac{S_A S_B S_C}{S^2} = \frac{(a^2 + b^2 + c^2)}{2} - 4R^2 = S_\omega - 4R^2.$$

By Theorem 4 ,

$$\begin{aligned}\frac{X'X}{X'O} &= 1 - \frac{d^2}{R^2} \\ &= 1 - \frac{(g+h)^2 (OH)^2}{(3g+h)^2 R^2} \\ &= 1 - \frac{(g+h)^2 (E - 8F)}{(3g+h)^2 E} \\ &= \frac{(3g+h)^2 E - (g+h)^2 (E - 8F)}{(3g+h)^2 E}.\end{aligned}$$

Now converting to barycentrics, we find

$$\begin{aligned}X' &= (g', h'), \text{ where} \\ g' &= 2g(3g+h)E + (3g+h)^2 E - (g+h)^2 (E - 8F), \\ h' &= 2h(3g+h)E - (3g+h)^2 E + (g+h)^2 (E - 8F).\end{aligned}$$

Example 7. The Shinagawa coefficients of the nine-point center, $X(5)$, are $(1, 1)$, so that the Shinagawa coefficients of $X(34864)$, the circumcevian-inversion perspector of $X(5)$, are $(5E + 8F, -E - 8F)$.

Example 8. The Shinagawa coefficients of the de Longchamps points, $X(20)$, are $(1, -2)$, so that the Shinagawa coefficients of $X(7488)$, the circumcevian-inversion perspector of $X(20)$, are $(E + 4F, -2E - 4F)$.

5 Circumcircle of $X'Y'Z'$

Suppose that P, P', and $X'Y'Z'$ have the same meanings as in Section 2. The circumcenter of $X'Y'Z'$ for arbitrary $P = p : q : r$ is the point O' given by

$$a^2 f(a, b, c) : b^2 f(b, c, a) : c^2 f(c, a, b), \text{ where}$$

$$f(a, b, c) = \beta_1 p^2 + \beta_2 q^2 + \beta_3 r^2 + \beta_4 qr + \beta_5 rp + \beta_6 pq,$$

$$\begin{aligned}
\beta_1 &= b^2c^2(3a^4 - 4a^2b^2 + b^4 - 4a^2c^2 - 2b^2c^2 + c^4)\\
\beta_2 &= 2a^2b^2c^2(a^2 - b^2 - c^2)\\
\beta_4 &= -a^2(a^2 - b^2 - c^2)(a^4 - 2a^2b^2 + b^4 - 2a^2c^2 - 6b^2c^2 + c^4)\\
\beta_5 &= -b^2(a^6 - b^6 - 2c^2 - 3a^4b^2 + 3a^2b^4 + 3b^2c^4 + 9a^2c^4 - 8a^4c^2 + 8a^2b^2c^2),
\end{aligned}$$

and β_3 is obtained from β_2, and β_6 from β_5, by interchanging b and c.

Theorem 9. *The circumcenter of triangle $X'Y'Z'$ satisfies the following distance-ratio identity:*

$$\frac{OP}{OP'} = 2\frac{OO'}{O'P'}.$$

Proof. This follows easily (by computer) from the barycentrics for O' and the barycentric distance formula ([8], p. 89). □

Theorem 10. *Let $|\rho|$ denote the radius of the circumcircle of $X'Y'Z'$. Then*

$$\begin{aligned}
|\rho| &= (abc)^{3/2}\frac{p+q+r}{d(a,b,c)S}, \quad \textit{where}\\
d(a,b,c) &= a^2(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 8b^2c^2)qr\\
&\quad + b^2(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 8c^2a^2)rp\\
&\quad + c^2(a^4 + b^4 + c^4 - 2a^2c^2 - 2b^2c^2 - 8a^2b^2)pq\\
&\quad - 3a^2b^2c^2(p^2 + q^2 + r^2).
\end{aligned}$$

Theorem 10 can be used to show that P' is a center of similitude of Γ and the circumcircle of $X'Y'Z'$, here denoted by Γ'. Care is required for an interpretation, since the radius ρ can be negative. The various cases can be summarised as follows: P is an exsimilicenter (i.e., external center of similitude) when Γ' lies entirely within Γ; otherwise, P' is the insimilicenter. Or, if the triangles ABC and $X'Y'Z'$ have the same orientation, then P' is the insimilicenter, and otherwise, the P' is the exsimilicenter.

6 The *Cip* transform applied to curves

In previous sections, we discussed the images under the circumcevian-inversion perspector transform of individual points. Using the notation *Cip* for that transform, we note that *Cip* is a rational, but not birational, quadratic transformation. We turn now to images of lines and circular cubics.

First, suppose that L is a line, represented as $px + qy + rz = 0$. Then $Cip(L)$ is the conic given by

$$a^2(\beta_1 b^2c^2px^2 + \beta_2 yz) + (\text{cyclic}) = 0, \text{where}$$

$$\begin{aligned}
\beta_1 &= (a^4 + a^2b^2 - 2b^4 + a^2c^2 + 4b^2c^2 - 2c^4)p - 3b^2(a^2 - b^2 + c^2)q - 3c^2(a^2 + b^2 - c^2)r\\
\beta_2 &= a^4(a^2 - b^2 - c^2)^2p^2 + b^4(a^2 - b^2 - 2c^2)(a^2 - b^2 + c^2)q^2 + c^4(a^2 - 2b^2 - c^2)(a^2 + b^2 - c^2)r^2\\
&\quad - b^2c^2(2a^4 - 5a^2b^2 + 3b^4 - 5a^2c^2 - 6b^2c^2 + 3c^4)qr\\
&\quad - a^2c^2(2a^2 - b^2 - 2c^2)(a^2 - b^2 - c^2)rp - a^2b^2(2a^2 - 2b^2 - c^2)(a^2 - b^2 - c^2)pq.
\end{aligned}$$

Example 11. If L = Euler line, then the conic $Cip(L)$ is degenerate; indeed, it is the Euler line, as in Section 4.

Example 12. If L = Nagel line (the line X(1)X(2)), then $Cip(L)$ is the conic that passes through $X(i)$ for $i = 3, 35, 7496, 34758, 34868, 34875, 34876$.

The center of the conic $Cip(L)$, if not degenerate, is the point

$$a^2 f(a, b, c) : b^2 f(b, c, a) : c^2 f(c, a, b)$$

given by

$$f(a, b, c) = \beta_1 p^2 + \beta_2 q^2 + \beta_3 r^2 + \beta_4 qr + \beta_5 rp + \beta_6 pq = 0, \text{where}$$

$$\begin{aligned}
\beta_1 &= a^4(a^2 - b^2 - c^2)(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 4b^2c^2)\\
\beta_2 &= b^4(a^6 - b^6 - 2c^6 - 3a^4b^2 + 3a^2b^4 + 3b^2c^4 - 6a^4c^2 + 7a^2c^4 + 6a^2b^2c^2)\\
\beta_4 &= -b^2c^2(6a^6 - 3b^6 - 3c^6 - 15a^4b^2 - 15a^4c^2 + 12a^2b^4 + 12a^2c^4 + 3b^4c^2 + 3b^2c^4 + 8a^2b^2c^2)\\
\beta_5 &= -a^2c^2(2a^6 - 5b^6 - 2c^6 - 9a^4b^2 + 12a^2b^4 - 6a^4c^2 + 6a^2c^4 + 4b^4c^2 + 3b^2c^4 + 6a^2b^2c^2),
\end{aligned}$$

and β_3 is obtained from β_2, and β_6 from β_5, by interchanging b and c.

It may seem surprising that $Cip(\Lambda)$ can be a conic when Λ is a curve other than a line. In order to account for such a curve, we note that if the target conic, $Cip(\Lambda)$, passes through the circumcenter, O, then a degree-reducing cancellation of $x+y+z$ occurs, and the remaining equation is quadratic, as in the next three examples: Brocard circle (Example 13), Jerabek circumhyperbola (Example 14), and Thomson-Gibert-Moses hyperbola (Example 15, introduced at X(5642) in [6]). Four more examples, in the form of Geogebra files, can be accessed from the preamble just before X(39371) in [6].

Example 13. Let K_1 be the circular cubic (defined in [2]) given by

$$\begin{aligned}
&a^2\Big(b^4c^4x^3 - a^2c^2(a^4 + 2b^4 + c^4 - 2a^2c^2 - 2b^2c^2)y^2z\\
&+ a^2b^2(a^4 + b^2 + 2c^4 - 2a^2b^2 - 2b^2c^2)yz^2\Big) + (\text{cyclic})\\
&- a^2b^2c^2(5a^4 + 5b^4 + 5c^4 - 4a^2b^2 - 4a^2c^2 - 4b^2c^2)xyz\\
&= 0.
\end{aligned}$$

Then $Cip(K_1)$ is the Brocard circle.

Example 14. Let K_2 be the circular cubic given by

$$\begin{aligned}
&a^4(a^2 - b^2 - c^2)\Big(c^2(a^2 + b^2 - c^2 - ab)(a^2 + b^2 - c^2 + ab)y^2z\\
&- b^2(a^2 - b^2 + c^2 - ac)(a^2 - b^2 + c^2 + ac)yz^2\Big) + (\text{cyclic})\\
&- 6a^2b^2c^2(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)xyz\\
&= 0.
\end{aligned}$$

Then $Cip(K_2)$ is the Jerabek circumhyperbola. For example, the point $X(35372)$ on K_2 maps to the point $X(35373)$ on the Jerabek circumhyperbola.

Example 15. Let K_3 be the circular cubic given by

$$\begin{aligned}
&2a^2b^4(b - c)c^4(b + c)x^3\\
&+ a^4b^2(a^6 - 3a^4b^2 + 3a^2b^4 - b^6 + 4a^4c^2 + a^2b^2c^2 - 5b^4c^2 - 4a^2c^4 + 5b^2c^4 - c^6)yz^2\\
&- a^4c^2(a^6 + 4a^4b^2 - 4a^2b^4 - b^6 - 3a^4c^2 + a^2b^2c^2 + 5b^4c^2 + 3a^2c^4 - 5b^2c^4 - c^6)y^2z\\
&+ (\text{cyclic}) - 14a^2(a - b)b^2(a + b)(a - c)(b - c)c^2(a + c)(b + c)xyz\\
&= 0.
\end{aligned}$$

Then $Cip(K_3)$ is the Thomson-Gibert-Moses hyperbola. The cubic K_3 (as well as K_1 and K_2) appears to be new to the literature. Notably, K_3 passes through the vertices of the Thomson triangle, the points X(3), X(110), X(10620), X(14915), and the circular points at infinity. The singular focus of K_3 lies on the lines X(3)X(541), X(476)X(7464), X(1302)X(7496), and X(9003)X(32305).

The preceding three examples can be extended to other conics that pass through O, such as the circumconic that passes through O and I, the Kiepert circumhyperbola of the medial triangle, the Jerabek circumhyperbola of the medial triangle, the Feuerbach circumhyperbola of the tangential triangle, the Lester circle, and the Hung circle.

7 Secondary pre-circumcevian-inversion points

Suppose that $X = Cip(P)$, where $P = p : q : r$. Barycentric coordinates for a second point, $\hat{X}$, such that $\hat{X} = Cip(P)$, can be found by computer:

$$a^2 f(a, b, c) : b^2 f(b, c, a) : c^2 f(c, a, b), \text{ where}$$

$$\begin{aligned} f(a, b, c) = {} & (b^2c^2(a^4 - 2b^4 - 2c^4 + a^2b^2 + a^2c^2 + 4b^2c^2)p^2 \\ & + 3a^2b^2c^2(a^2 - b^2 - c^2)q^2 + 3a^2b^2c^2(a^2 - b^2 - c^2)r^2 \\ & + a^2(a^2 - b^2 - c^2)(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 + 4b^2c^2)qr \\ & + b^2(a^6 - b^6 - 3c^6 - 3a^4b^2 + 3a^2b^4 + a^4c^2 + a^2c^4 - b^4c^2 + 5b^2c^4)rp \\ & + c^2(a^6 - c^6 - 3b^6 - 3a^4c^2 + 3a^2c^4 + a^4b^2 + a^2b^4 - b^2c^4 + 5b^4c^2)pq. \end{aligned}$$

The point $\hat{X}$ is introduced in ([6]; see the preamble just before $X(35000)$) as the secondary pre-circumcevian-inversion point of P.

Example 16. Not only is $Cip(X(1)) = X(35)$, but also, $Cip(X(35000)) = X(35)$, so that $X(35000)$ is the secondary pre-circumcevian-inversion point of the incenter.

Example 17. Not only is $Cip(X(2)) = X(7496)$, but also, $Cip(X(35001)) = X(7496)$, so that $X(35001)$ is the secondary pre-circumcevian-inversion point of the centroid.

Next we introduce two circles, named, by the third author, the Suren circle and the Moses-Suren circle. The two circles play a central role in the connections between X and $\hat{X}$.

The Suren circle has center O and radius ρ satisfying $\rho^2 = -2R^2$, where R is the circumradius of ABC. The Suren circle is imaginary. Its perspector ([8], p. 127) is $X(3526)$, and an equation for the Suren circle is

$$a^2yz + b^2zx + c^2xy - 3R^2(x + y + z)^2 = 0.$$

The Moses-Suren circle has center O and radius ρ satisfying $\rho^2 = 2R^2$. The Moses-Suren circle is real. Its perspector is $X(3527)$, and an equation for the Moses-Suren circle is

$$a^2yz + b^2zx + c^2xy + 3R^2(x + y + z)^2 = 0.$$

Theorem 18. *The point $\hat{X}$ is the Suren-circle-inverse of X; thus $\hat{X} = X$ if and only if X lies on the Suren circle.*

Proof. As in the proof of Theorem 3,

$$OP' = \frac{R^2 d}{2R^2 - d^2}.$$

Let $x = OP'$. Then

$$xd^2 + R^2 d - 2R^2 x = 0. \tag{2}$$

As a quadratic equation, there are formally two solutions for d. As the discriminant, $R\sqrt{R^2+8x}$ is positive, the two solutions are real; that is, for each length OP', there are two values of d. The product of the roots of (2) is $-2R^2 = \rho^2$. This and the collinearity of P, O, P' establish that $\hat{X}$ is the Suren-circle-inverse of X □

Theorem 19. *The point $\hat{X}$ is the Moses-Suren-circle-inverse of the reflection of X in O, so that if X is on the Moses-Suren circle, then $\hat{X}$ is the antipode of X.*

Theorems 18 and 19 can be routinely verified by computer.

8 Source points

In Theorem 1, we started with a point P and constructed the point $P' = Cip(P)$. Then in Section 7, we discussed a second point, $\hat{X}$ such that $Cip(X) = Cip(\hat{X})$. In this section, we invert the Cip transform; that is, we start with any point $U = u : v : w$ not in $\{A, B, C, X(3)\}$ and find barycentrics for the two points $P_1 = p_1 : q_1 : r_1$ and $P_2 = p_2 : q_2 : r_2$ satisfying

$$U = Cip(P_1) = Cip(P_2).$$

We call the points P_1 and P_2 the source points for U and have these formulas:

$$P_1 = H - K\sqrt{\Delta} \text{ and } P_2 = H + K\sqrt{\Delta},$$

where

$$\begin{aligned}
H &= bc\Big(a^2b^2 + a^2c^2 - (b^2-c^2)^2\Big)p + a^2bc(b^2+c^2-a^2)(q+r);\\
K &= a\Big(b^2c^2(2a^4 - a^2b^2 - b^4 - a^2c^2 + 2b^2c^2 - c^4)p^2\\
&\quad + 3a^2b^2c^2(a^2-b^2-c^2)q^2 + 3a^2b^2c^2(a^2-b^2-c^2)r^2)\\
&\quad + 2a^2(a^2-b^2-c^2)(a^2-b^2-bc-c^2)(a^2-b^2+bc-c^2)qr\\
&\quad + b^2(2a^6 - 2b^6 - 3c^6 - 6a^4b^2 + 6a^2b^4 - a^4c^2 + 2a^2c^4 + b^4c^2 + 4b^2c^4)rp\\
&\quad + c^2(2a^6 - 2c^6 - 3b^6 - 6a^4c^2 + 6a^2c^4 - a^4c^2 + 2a^2b^4 + c^4b^2 + 4c^2b^4)pq\Big);\\
\Delta &= 9a^2b^2c^2(p^2+q^2+r^2) + f(a,b,c)qr + f(b,c,a)rp + f(c,a,b)pq,
\end{aligned}$$

where

$$f(a,b,c) = 2a^2(4a^4 + 4b^4 + 4c^4 + b^2c^2 - 8a^2b^2 - 8a^2c^2)qr.$$

The least i for which the source points of $X(i)$ are in [6] is 15, as indicated in the following table.

i	P_1	P_2
15	36243	36244
16	36241	36242
35	1	35000
574	35002	6
3520	4	18859
5210	33878	187
7280	12702	36
7488	2070	20
7492	3534	23
7496	2	35001
8722	2080	1350
9130	35447	351
11012	22765	40

i	P_1	P_2
11510	1319	35448
14379	6760	64
15020	32609	15054
21855	186	1657
25042	54	35449
33924	34147	35450
34758	35450	8
34864	35452	5
34866	115	35453
34867	9	35454
34868	10	35455
34870	35456	35456
34871	35457	35

An unexpected observation from the above list is that there seems to be no easily recognised pattern for distinguishing between "negative" source points, P_1, and "positive", P_2; for example,

$$X(1), X(2), X(4), X(6), X(20) \text{ are of the form } P_1 = H - K\sqrt{\Delta}, \text{ whereas}$$
$$X(5), X(8), X(9), X(10), X(35) \text{ are of the form } P_1 = H + K\sqrt{\Delta}.$$

9 The center $\Psi(P)$ of a conic

Suppose that P is a point and $L(P)$ is the line through P perpendicular to OP. As in Section 6, $Cip(L(P))$ is a conic. Its center we denote by $\Psi(P)$.

Theorem 20. *If $L(P)$ is tangent to the circumcircle, then $Cip(L(P))$ is a right hyperbola that passes through O and P.*
If $L(P)$ is tangent to the Moses-Suren circle (i.e., $(\sqrt{2}R, O)$), then $Cip((L(P))$ is a parabola.
If $L(P)$ lies outside the Moses-Suren circle, then $Cip((L(P))$ is an ellipse.
If $L(P)$ lies inside the Moses-Suren circle, then $Cip((L(P))$ is a hyperbola.

Theorem 20 is similar to the well-known fact that the isogonal conjugate of a line is an ellipse, parabola, or hyperbola according as the line meets Γ in 0,1, or 2 points.

Theorem 21. *$\Psi(P)$ is the midpoint of O and $Cip(P)$.*

Although Theorem 21 can be routinely checked by computer, an intuitive proof can be described as follows: as a point W moves on $L(P)$ away from O, the point $Cip(W)$ approaches O; thus, regarding Cip as inversion in a circle centered at O, images of Cip are symmetrical for points equally distant on $L(P)$ from OP, so that the center of the conic must be the midpoint of O and $Cip(P)$.

10 A generalisation: $(t)Cip$ transformations

Returning now to Theorem 1, the points X, Y, Z are the images of D, E, F under the homothety with ratio -1 centered at P. If the ratio is changed to an arbitrary nonzero number t, then the resulting lines AX', BY', CZ' concur, as in Theorem 1. We write the resulting point of concurrence as $(t)Cip(P)$.

Theorem 22. *If $P = p : q : r$ and $t \neq 0$, then*

$$\begin{aligned}(t)Cip(P) &= f(a,b,c) : f(b,c,a) : f(c,a,b), \text{ where}\\ f(a,b,c) &= a^2b^2c^2p(p+q+r) - ta^2(-a^2+b^2+c^2)(a^2qr+b^2rp+c^2pq).\end{aligned}$$

As in Section 7, there is a secondary pre-(t)circumcevian-inversion point, for which, if written as $h(a,b,c) : h(b,c,a) : h(c,a,b)$, then

$$h(a,b,c) = \frac{p(p+q+r)}{a^2qr+b^2rp+c^2pq-uR^2(p+q+r)^2} + \frac{-a^2+b^2+c^2}{b^2c^2(u-1)},$$

where $u = (2t-1)/t$. The two points $f(a,b,c) ::$ and $h(a,b,c) ::$ are an inverse pair in the circle centered at O with radius ρ satisfying $\rho^2 = (1-t)R^2/t$.

Example 23. The appearance of (t, j, k) in the following list means that $(t)Cip(X(j)) = X(k)$:

$$\begin{aligned}&(-1,1,35), (-1/2,1,55), (-1/3,1,3746), (3/2,2,22),\\ &(3/4,2,25), (2/3,6,305), (3/4,186,25).\end{aligned}$$

11 A further generalisation

Suppose that $\Gamma(U)$ is the circumconic with perspector $U = u : v : w$. The $\Gamma(U)$-inverse of a point $F = f : g : h$ is the point

$$ghu^2(u-v-w) + f^2uvw + fu\Big(h(u-v)v + g(u-w)w\Big) \ :: \ .$$

($\Gamma(U)$ is an ellipse if and only if U lies inside the Steiner inellipse.)

As a generalisation of circumcevian triangle (i.e., when $\Gamma(U)$ is the circumcircle, as in the preceding sections), the A-vertex of the $\Gamma(U)$-cevian triangle of a point $P = p : q : r$ is the point

$$D = -qru : q(rv+qw) : r(rv+qw).$$

Let $\sigma = p+q+r$ and $\tau = qru+rpv+pqw$. Then the t-reflection of P in D (with $t = -1$ for ordinary reflection) is the point

$$X = qr\sigma u - t(q+r)\tau : -q\sigma(rv+qw) + tq\tau : -r\sigma(rv+qw) + tr\tau.$$

From those basics, the $\Gamma(U)$-inverse of X is then found to be

$$\begin{aligned}X' = {} & qr\sigma^2u^2vw + t^2u(u-v-w)\tau^2 - t\sigma u\tau(ruv - rv^2 + quw - qw^2)\\ & : -q\sigma^2uvw(rv+qw) + t^2v(-u+v-w)\tau^2 - t\sigma v\tau(-ruv + rv^2 - 2quw + qvw - rvw - qw^2)\\ & : -r\sigma^2uvw(rv+qw) + t^2w(-u-v+w)\tau^2 - t\sigma w\tau(-2ruv - rv^2 - quw - qvw + rvw + qw^2),\end{aligned}$$

the perspector of $X'Y'Z'$ and ABC, by

$$P' = p\sigma uvw + tu(u-v-w)\tau \ ::,$$

and the radius of the circumcircle of $X'Y'Z'$ by

$$\frac{(t-1)a^3b^3c^3\sigma^2}{2S\Big((2t-1)a^2b^2c^2\sigma^2 - 4t^2(a^2qr+b^2rp+c^2pq)S^2\Big)}.$$

For comparison with the results in Section 7, the secondary pre-$\Gamma(U,t)$-cevian-inversion point is given by

$$\begin{aligned}
&(p+q+r)uvw\Big((q+r)u\xi + p(uv - v^2 + uw + 2vw - w^2)\Big)\\
&- tu\Big(qru\xi^3 + 2q^2uvw\xi + 2r^2uvw\xi\\
&+ p^2vw(u+v-w)(u-v+w) + prv(u^3 - 3u^2v + 3uv^2 - v^3 + uw^2 + 3vw^2 - 2w^3)\\
&+ pqw(u^3 + uv^2 - 2v^3 - 3u^2w + 3v^2w + 3uw^2 - w^3)\Big) \ ::,
\end{aligned}$$

where $\xi = u+v+w$.

12 Four equilateral triangles

Next we apply the generalisation in Section 11 to a question raised by Suren: "What is the locus of a point $P = p : q : r$ such that the circumcevian-inversion triangle of P is equilateral?" After elaborate Mathematica computations followed by extensive simplifications, Moses found that for t-reflection (as a generalisation of ordinary reflection, given by $t = -1$), the locus consists of four points. In order to present them in the form

$$a^2 f(a,b,c) : b^2 f(b,c,a) : c^2 f(c,a,b), \tag{3}$$

let

$$\begin{aligned}
\beta_1 &= a^2 + b^2 + c^2\\
\beta_2 &= a^2b^2 + a^2c^2 - b^4 - c^4\\
\beta_3 &= b^2c^2 + c^2a^2 + a^2b^2\\
\beta_4 &= a^4 + b^4 + c^4\\
\delta_1 &= -1,\ \ \delta_2 = 1,\ \ \epsilon_1 = -1,\ \ \epsilon_2 = 1\\
\rho_{ij} &= \epsilon_i\sqrt{2}\sqrt{\sqrt{12}\delta_j\beta_1 S + 4\beta_3 - \beta_4 - 8(\beta_4 - \beta_3 t(1-t))}.
\end{aligned}$$

Then

$$f(a,b,c) = \beta_1\beta_2 + 4(\beta_4 - \beta_3)(-a^2+b^2+c^2)t + \beta_2(\rho_{ij} + \delta_j\sqrt{12}S). \tag{4}$$

The four points P are now given by (3) and (4) using $\delta_i, \epsilon_j = (\pm 1, \pm 1)$. These are the source points for the isodynamic points, $X(15)$ and $X(16)$ (Figures 2 and 3).

In order to see that the four points are real for all ABC, it suffices to show that $\rho_{ij}^2 > 0$ for all a, b, c. We write ρ_{ij}^2 in terms of the Brocard angle ω:

$$\rho_{ij}^2 = 4S^2(\csc^2\omega)\Big(2 - 4t(t-1) + (8t^2 - 8t - 1)\cos 2\omega + \delta_j\sqrt{3}\sin 2\omega\Big).$$

Then

$$\min\Big((8t^2 - 8t - 1)\cos 2\omega\Big) = (8t^2 - 8t - 1)/2,$$

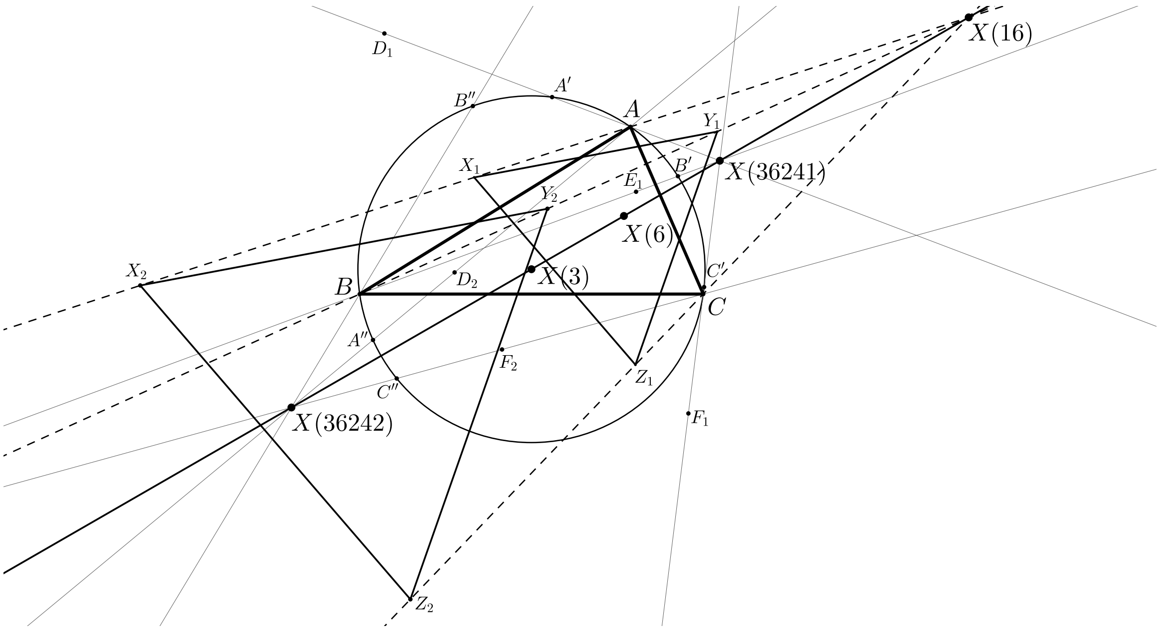

Figure 2: Equilateral triangles with source points for $X(16)$

which occurs when $\omega = \pi/6$. Since

$$2 - 4t(t-1) + (8t^2 - 8t - 1)/2 = 3/2,$$

and since $0 < \sqrt{3}\sin 2\omega < 3/2$ for all a, b, c, we conclude that $\rho_{ij}^2 > 0$.

The four points can also be nicely represented in terms of the Brocard angle ω:

$$(\sin^2 A)\Big(\cos(A+\omega)(\csc A \csc\omega)(\delta_i\sqrt{3} + \cot\omega + \epsilon_j\hat{\rho}\csc\omega) - 2t(\cot A)(\csc^2\omega - 4)\Big) :: ,$$

where

$$\hat{\rho} = \sqrt{2 + 4t(1-t) + (8t^2 - 8t - 1)(\cos 2\omega + \sqrt{3}\sin 2\omega)}.$$

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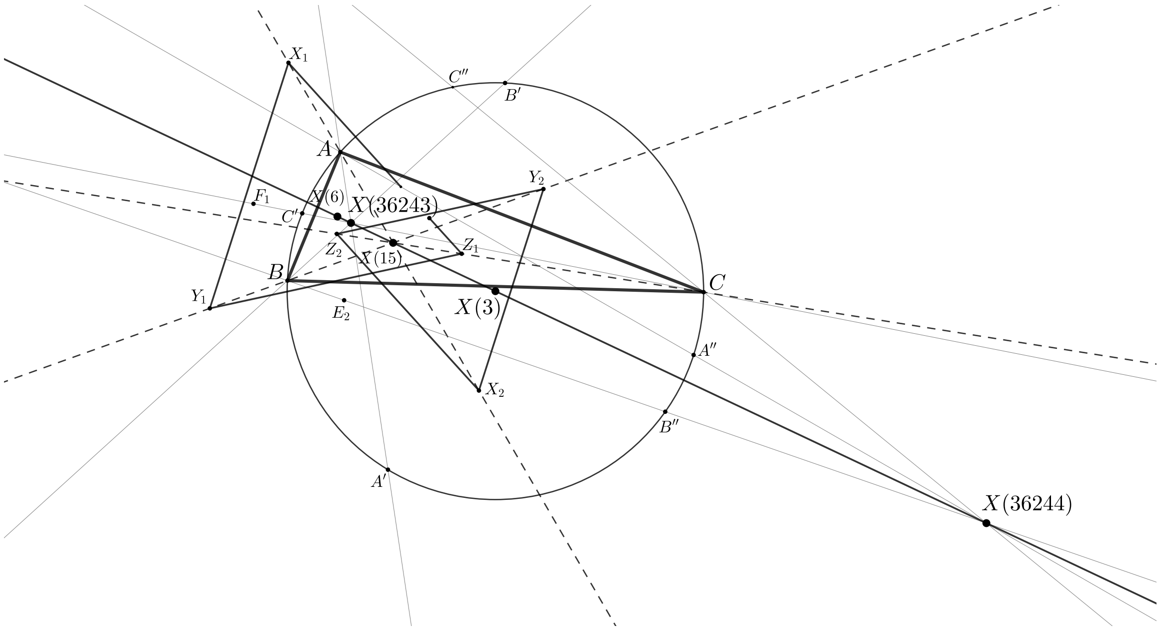

Figure 3: Equilateral triangles with source points for $X(15)$

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