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The Ellipse, Monge's Circle and Other Circles

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Abstract. New developments of the author's research project on the geometry of conics are presented. For any general point P taken on a given ellipse H, some triplets of collinear, peculiar points are described (Theorem 2.1); several angles sharing the same vertex are shown to share the same line as bisector, too (Theorems 2.3 through 2.7). Three new circles (the bridge-circles Theorems 3.1, 3.2 and 3.3) linking points belonging to the ellipse H, Monge's circle and other conics introduced by the author (the symbiotic ellipse H_Σ and the circle Φ_1) are described. Unsuspected relationships linking the newly defined objects with a circle previously introduced by the author (denoted *circle* Ω) are described.

Key Words: ellipse, collinear points, angle bisector, concyclic points, symbiotic conics, Monge's circle, bridge-circle

MSC 2020: 51M04

1 Introduction

New developments of the author's research project [4–8] on the geometry of conics are presented. In an orthogonal reference frame (Figure 1), let H be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \quad (a > b), \tag{1}$$

whose foci are

$$F_1(-c, 0); \quad F_2(c, 0); \quad \left(c = \sqrt{a^2 - b^2}\right) \tag{2}$$

Throughout this paper, the ellipse general point – that is, any point different from the vertexes – is denoted by $P\,(a\cos\varepsilon;\ b\sin\varepsilon)$ or simply P; to avoid the exceeding verbal complexity, I will formulate any statement on the basis of the assumption that P lies in the 1st quadrant $(x > 0,\ y > 0)$. For the reader's convenience, some geometrical objects frequently referred to throughout this paper are listed and previous results are summarized:

1. The ellipse diameters with slope $m_e = \tan\varepsilon$ and $m_{e'} = -\tan\varepsilon$, introduced by Ternullo [4] and named *eccentric line* (3) and *symm-eccentric line* (4), respectively:

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$$y = x \tan\varepsilon \qquad (3) \qquad\qquad y = -x \tan\varepsilon \qquad (4)$$

2. The tangent t (5) to the ellipse H at P:

$$y = -x\frac{b}{a}\cot\varepsilon + \frac{b}{\sin\varepsilon} \tag{5}$$

3. The x- and y-intercepts (T_x (6) and T_y (7), respectively) of the tangent (5) to the ellipse H at P:

$$T_x\Big(\frac{a}{\cos\varepsilon};\ 0\Big) \qquad (6) \qquad\qquad T_y\Big(0;\ \frac{b}{\sin\varepsilon}\Big) \qquad (7)$$

4. The normal n (8) to the ellipse H at P:

$$y = x\frac{a}{b}\tan\varepsilon - \frac{c^2}{b}\sin\varepsilon \tag{8}$$

5. The x- and y-intercepts (N_x (9) and N_y (10), respectively) of the normal n (8) to the ellipse H at P:

$$N_x\Big(\frac{c^2}{a}\cos\varepsilon;\ 0\Big) \qquad (9) \qquad\qquad N_y\Big(0;\ -\frac{c^2}{b}\sin\varepsilon\Big) \qquad (10)$$

6. The following points E (11) and I (12), where the normal (8) meets the eccentric (3) and the symm-eccentric (4) line of P, respectively:[1]

$$E((a+b)\cos\varepsilon;\ (a+b)\sin\varepsilon) \qquad (11) \qquad\qquad I((a-b)\cos\varepsilon;\ -(a-b)\sin\varepsilon) \qquad (12)$$

7. The following circle Φ_1, whose center is the y-intercept T_y (7) of the tangent (5):

$$x^2 + \Big(y - \frac{b}{\sin\varepsilon}\Big)^2 = c^2 + \frac{b^2}{\sin^2\varepsilon} \tag{13}$$

The circle Φ_1 passes through the foci (by definition), as well as through the points E (11) and I (12) ([7], Theorem 2.2) and the following T_{131} (14) and T_{112} (15) ([8], Theorem 1), where the tangent t (5) drawn to the ellipse H (1) at P meets the tangents drawn at the vertexes $V_3(-a,0)$ and $V_1(a,0)$ of the same ellipse, respectively:

$$T_{131}\Big(-a;\ \frac{b(1+\cos\varepsilon)}{\sin\varepsilon}\Big) \qquad (14) \qquad\qquad T_{112}\Big(a;\ \frac{b(1-\cos\varepsilon)}{\sin\varepsilon}\Big), \qquad (15)$$

8. The following circle Φ_2, whose center is the y-intercept N_y (10) of the normal (8):

$$x^2 + \Big(y + \frac{c^2}{b}\sin\varepsilon\Big)^2 = c^2 + \Big(\frac{c^2}{b}\sin\varepsilon\Big)^2. \tag{16}$$

[1]The first mention of the point E (11) known to the author can be found in an exercise of Salmon ([3], Chapter XIII, Article 231, p. 221); Salmon is aware that the locus of such point is a circle concentric with the ellipse. Afterwards, such circle has been studied by A. Barlotti [1] and Ternullo ([4], Theorem 1).

The circle Φ_2 passes through through the foci; the normal (8) meets Φ_2 at the following points N_{21} and N_{22}:

$$N_{22}\left(-c\cos\varepsilon;\ \frac{c(a+c)\sin\varepsilon}{-b}\right) \quad (17) \qquad N_{21}\left(c\cos\varepsilon;\ \frac{c(a-c)\sin\varepsilon}{b}\right) \quad (18)$$

9. The following circle Φ_3 whose center is the x-intercept T_x (6) of the tangent (5):

$$\left(x-\frac{a}{\cos\varepsilon}\right)^2+y^2=\frac{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}{\cos^2\varepsilon}. \tag{19}$$

The circle Φ_3 passes through the points E (11) and I (12) (by definition), as well as through the following T_{321} and T_{342}, where the tangent t (5) drawn to the ellipse H (1) at P meets the tangents to the same ellipse at its vertexes $V_2(0,b)$ and $V_4(0,-b)$, respectively ([8], Theorem 1):

$$T_{321}\left(\frac{a(1-\sin\varepsilon)}{\cos\varepsilon};\ b\right) \quad (20) \qquad T_{342}\left(\frac{a(1+\sin\varepsilon)}{\cos\varepsilon};\ -b\right) \quad (21)$$

The circles Φ_1, Φ_2 and Φ_3 taken pairwise are mutually orthogonal ([7], Theorem 2.1).

The vertexes of the ellipse H (1) are denoted by $V_1(a,\ 0)$, $V_2(0,\ b)$, $V_3(-a,\ 0)$ and $V_4(0,\ -b)$ or simply $V_1,\dots V_4$. The points where the circle Φ_i $(i=1,3)$ meets the tangent drawn at the ellipse vertex V_j $(j=1,4)$ are denoted by $T_{ij\lambda}$ $(\lambda=1,2)$.

2 Some Peculiar Points on the Circles Φ_1, Φ_2 and Φ_3

Let the points lying on the circles Φ_i $(i=1,3)$ at maximal and minimal distance to the H center O (the Φ_i *distal* and *proximal points*), be denoted by Φ_{id} and Φ_{ip}, respectively. Using the following, compact notation

$$(asbc)=a^2\sin^2\varepsilon+b^2\cos^2\varepsilon \tag{22}$$

such points are represented as follows:

$$\Phi_{1d}\left(0;\ \frac{b+\sqrt{(asbc)}}{\sin\varepsilon}\right) \quad (23) \qquad \Phi_{1p}\left(0;\ \frac{b-\sqrt{(asbc)}}{\sin\varepsilon}\right) \quad (24)$$

$$\Phi_{2d}\left(0;\ -c\frac{c\sin\varepsilon+\sqrt{(asbc)}}{b}\right) \quad (25) \qquad \Phi_{2p}\left(0;\ c\frac{-c\sin\varepsilon+\sqrt{(asbc)}}{b}\right) \quad (26)$$

$$\Phi_{3d}\left(\frac{a+\sqrt{(asbc)}}{\cos\varepsilon};\ 0\right) \quad (27) \qquad \Phi_{3p}\left(\frac{a-\sqrt{(asbc)}}{\cos\varepsilon};\ 0\right) \quad (28)$$

Let the lines joining the focus F_1 with the distal and proximal points of the circle Φ_1 [Φ_{1d} (23) and Φ_{1p} (24), respectively] be drawn; such lines $F_1\Phi_{1d}$ and $F_1\Phi_{1p}$ meet the circle Φ_2 in the following points Φ_{2l} and Φ_{2r}, respectively:

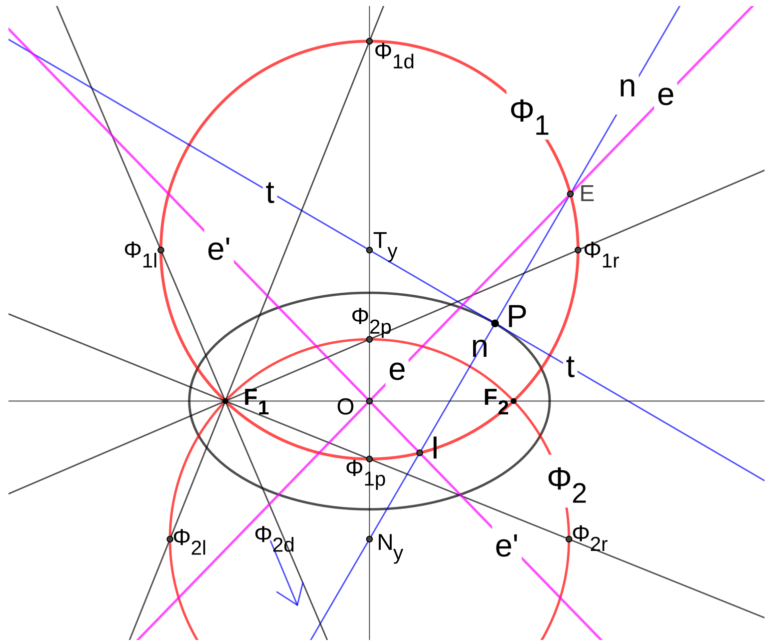

Figure 1: For a general P taken on the ellipse, the tangent t and normal n [blue], the eccentric line e and symm-eccentric line e' [magenta], the circles Φ_1 and Φ_2 [red], the points E, I, Φ_{id}, Φ_{ip}, Φ_{il}, Φ_{ir} $(i = 1, 2)$ are shown.

$$\Phi_{2l}\left(-\frac{c}{b}\sqrt{(asbc)};\ -\frac{c^2}{b}\sin\varepsilon\right) \qquad (29) \qquad\qquad \Phi_{2r}\left(\frac{c}{b}\sqrt{(asbc)};\ -\frac{c^2}{b}\sin\varepsilon\right) \qquad (30)$$

One can easily check that Φ_{2l} (29) and Φ_{2r} (30) are also the points where the lines $x = \pm\frac{c}{b}\sqrt{(asbc)}$ touch the circle Φ_2; therefore, they are the Φ_2 points lying at the maximal distance to the y-axis. Quite similarly, the lines $F_1\Phi_{2d}$ and $F_1\Phi_{2p}$ meet the circle Φ_1 in the following points Φ_{1l} and Φ_{1r}, lying at the maximal distance to the y-axis, respectively:

$$\Phi_{1l}\left(-\frac{\sqrt{(asbc)}}{\sin\varepsilon};\ \frac{b}{\sin\varepsilon}\right) \qquad (31) \qquad\qquad \Phi_{1r}\left(\frac{\sqrt{(asbc)}}{\sin\varepsilon};\ \frac{b}{\sin\varepsilon}\right) \qquad (32)$$

Accordingly, the following holds:

Theorem 2.1. *The focus F_1 is collinear with the distal point Φ_{1d}* (23) *of the circle Φ_1 and the point Φ_{2l}* (29) *of the circle Φ_2, as well as with the proximal point Φ_{1p}* (24) *of Φ_1 and the point Φ_{2r}* (30) *of Φ_2 on the following lines* (33) *and* (34)*, respectively:*

$$y = \frac{b + \sqrt{(asbc)}}{c\sin\varepsilon}(x + c) \qquad (33) \qquad\qquad y = \frac{b - \sqrt{(asbc)}}{c\sin\varepsilon}(x + c) \qquad (34)$$

The focus F_1 is collinear with the proximal point Φ_{2p} (26) *of the circle Φ_2 and the point Φ_{1r}* (32) *of the circle Φ_1 as well as with the distal point Φ_{2d}* (25) *of Φ_2 and the point Φ_{1l}* (31) *of Φ_1 on the following lines* (35) *and* (36)*, respectively:*

$$y = -\frac{c\sin\varepsilon + \sqrt{(asbc)}}{b}(x+c) \qquad (35) \qquad\qquad y = \frac{-c\sin\varepsilon + \sqrt{(asbc)}}{b}(x+c) \qquad (36)$$

Of course, $\measuredangle \Phi_{1d}F_1\Phi_{1p} = \measuredangle \Phi_{2p}F_1\Phi_{2d} = \pi/2$, because both angles are inscribed in semicircles. On the other hand, the lines $F_1\Phi_{2p}\Phi_{1r}$ (35) and $F_1\Phi_{1p}\Phi_{2r}$ (34) bisect the angles $\angle\Phi_{1d}F_1\Phi_{1p}$ and $\angle\Phi_{2p}F_1\Phi_{2d}$, because pass through the midpoints (Φ_{1r} and Φ_{2r}) of the arcs $\Phi_{1d}\Phi_{1p}$ and $\Phi_{2p}\Phi_{2d}$, respectively. Accordingly, we can state the following:

Theorem 2.2. *For any P taken on the ellipse H, the following identities hold:*

$$\measuredangle \Phi_{1d}F_i\Phi_{2p} = \measuredangle \Phi_{2p}F_i\Phi_{1p} = \measuredangle \Phi_{1p}F_i\Phi_{2d} = \pi/4$$
$$\measuredangle \Phi_{1d}F_i\Phi_{2d} = 3\pi/4; \quad (i = 1, 2)$$

In 2016, Ternullo [7] introduced the *symbiotic conics* (Figure 2): taken a point P on the ellipse H (1), the symbiotic conics of the ellipse H about P are the ellipse H_Σ and the hyperbola Y_Σ whose center is P and whose axes are the tangent and normal to H at P; moreover, both H_Σ and Y_Σ pass through the center O of H, where admit the axes of symmetry of H as tangent and normal. The equation of the ellipse H_Σ follows:

$$x^2\frac{a^2 - b^2\cos^2\varepsilon}{a^2\cos^2\varepsilon} - 2xy\frac{b}{a}\tan\varepsilon - 2x\frac{c^2}{a\cos\varepsilon} + y^2 = 0 \qquad (37)$$

The symbiotic conics H_Σ and Y_Σ are confocal; their foci are the points E (11) and I (12) ([7], Theorem 3.1). The symbiotic ellipse of H_Σ about O is the ellipse H (1). For any object Ω (point, line, angle etc.) playing a certain role w.r.t. the ellipse H, there exists a homologous object Ω' playing the same role w.r.t. the ellipse H_Σ. Accordingly, from any statement involving a special set of objects, we may generate a twin statement involving the set of homologous objects. In many cases [7, 8], the new statements, far from being trivial duplicates of the original ones, reveal new, worth mentioning facts. In Table 1 some couples of objects homologous to each other are listed.

Let the symbiotic ellipse H_Σ (37) of the ellipse H (1) about the point P be constructed. Both tangents drawn to H_Σ from the focus F_1 are represented in the following, compact form (the notation $R = a^2 - b^2\cos^2\varepsilon + 2ac\cos\varepsilon$ is used):

$$y = \frac{b\sin\varepsilon \pm \sqrt{R}}{c + 2a\cos\varepsilon}(x+c) \qquad (38)$$

The lines (38) touch the ellipse H_Σ in the points Σ_1, Σ_2 represented as follows:

$$\Sigma_i\left(\frac{\left(R \pm b\sin\varepsilon\sqrt{R}\right)ac\cos\varepsilon}{(c + a\cos\varepsilon)R \mp ab\sin\varepsilon\cos\varepsilon\sqrt{R}};\ \frac{R\left(b\sin\varepsilon \pm \sqrt{R}\right)c}{(c + a\cos\varepsilon)R \mp ab\sin\varepsilon\cos\varepsilon\sqrt{R}}\right)(i = 1, 2) \qquad (39)$$

The lines F_1E and F_1I, linking the focus $F_1(-c, 0)$ with the points E (11) and I (12) are represented by the following (40) and (41) equations, respectively:

$$y = \frac{(a+b)\sin\varepsilon}{(a+b)\cos\varepsilon + c}(x+c); \qquad (40) \qquad\qquad y = -\frac{(a-b)\sin\varepsilon}{(a-b)\cos\varepsilon + c}(x+c) \qquad (41)$$

Table 1: Couples of homologous objects

Objects defined w.r.t the ellipse H	**Homologous objects**
P: point of H	O: point of H_Σ
$\vec{x}$ (x-axis): major axis of H	n: normal to H at P, major axis of H_Σ
$\vec{y}$ (y-axis): minor axis of H	t: tangent to H at P, minor axis of H_Σ
e: ecc. line OE of P (3)	ecc. line PF_1 of O
e': symm-ecc. line OI of P (4)	symm-ecc. line PF_2 of O
n: normal to H at P (8)	x-axis: normal to H_Σ at O
t: tangent to H at P (5)	y-axis; tangent to H_Σ at O
E: $e \cap n$ (11)	F_1: $PF_1 \cap \vec{x}$
I: $e' \cap n$ (12)	F_2: $PF_2 \cap \vec{x}$
F_1, F_2: foci of H	E, I: foci of H_Σ
T_y: $t \cap \vec{y}$ (7)	T_y: $\vec{y} \cap t$
N_y: $n \cap \vec{y}$ (10)	T_x: $\vec{x} \cap t$
T_x: $t \cap \vec{x}$ (6)	N_y: $\vec{y} \cap n$
N_x: $n \cap \vec{x}$ (9)	N_x: $\vec{x} \cap n$
Circle Φ_1 (about T_y, through $F_1,\ F_2$) (13)	Circle Φ_1 (about T_y, through $E,\ I$)
Circle Φ_2 (about N_y, through $F_1,\ F_2$) (16)	Circle Φ_3 (about T_x, through $E,\ I$)
Circle Φ_3 (about T_x, through $E,\ I$) (19)	Circle Φ_2 (about N_y, through $F_1,\ F_2$)
T_{131}, T_{112}: $t \cap \Phi_1$	$\Phi_{1d},\ \Phi_{1p}$: $\vec{y} \cap \Phi_1$
T_{321}, T_{342}: $t \cap \Phi_3$	$\Phi_{2p},\ \Phi_{2d}$: $\vec{y} \cap \Phi_2$
$N_{21},\ N_{22}$: $n \cap \Phi_2$	$\Phi_{3p},\ \Phi_{3d}$: $\vec{x} \cap \Phi_3$
line $F_1N_{21}T_{112}$ (43)	line $E\Phi_{3p}\Phi_{1p}$
line $T_{131}F_1N_{22}$ (44)	line $\Phi_{1d}E\Phi_{3d}$
line $F_2N_{21}T_{131}$	line $I\Phi_{3p}\Phi_{1d}$
line $T_{112}F_2N_{22}$	line $\Phi_{1p}I\Phi_{3d}$

The collinearity of the triplets $F_1N_{21}T_{112}$, $T_{131}F_1N_{22}$, $F_2N_{21}T_{131}$, $T_{112}F_2N_{22}$, $E\Phi_{3p}\Phi_{1p}$, $\Phi_{1d}E\Phi_{3d}$, $I\Phi_{3p}\Phi_{1d}$ and $\Phi_{1p}I\Phi_{3d}$ has been proved by Ternullo ([8], Theorems 2.10 and 2.13).

The focal radius F_1P is:

$$y = \frac{b \sin\varepsilon}{a\cos\varepsilon + c}(x + c) \tag{42}$$

Finally, we should remember that the focus $F_1(-c, 0)$ is collinear with the points N_{21} (18) and T_{112} (15), as well as with the points T_{131} (14) and N_{22} (17), on the following orthogonal lines $F_1N_{21}T_{112}$ (43) and $T_{131}F_1N_{22}$ (44), respectively ([8], Theorem 2.10):

$$y = \frac{(a-c)\sin\varepsilon}{b(1+\cos\varepsilon)}(x + c) \tag{43}$$

$$y = -\frac{b(1+\cos\varepsilon)}{(a-c)\sin\varepsilon}(x + c) \tag{44}$$

Bearing these facts in mind, we can state the following:

Theorem 2.3. *[Fig. 2] The line $F_1N_{21}T_{112}$ (43) [black, dashed] bisects the following angles sharing the focus F_1 as vertex: (i) $\angle PF_1F_2$ [blue], (ii) $\angle EF_1I$ [red], (iii) $\angle\Sigma_1F_1\Sigma_2$ [green] and (iv) $\angle T_{131}F_1N_{22}$ [red].*

Proof. The Theorem 2.3 is equivalent to the following four statements:

$$(i)\ \measuredangle PF_1F_2 = 2\measuredangle T_{112}F_1F_2; \quad (ii)\ \measuredangle EF_1T_{112} = \measuredangle T_{112}F_1I;$$
$$(iii)\ \measuredangle \Sigma_1F_1T_{112} = \measuredangle T_{112}F_1\Sigma_2; \quad (iv)\ \measuredangle T_{131}F_1T_{112} = \measuredangle T_{112}F_1N_{22}.$$

Taking the slopes of the lines $F_1N_{21}T_{112}$ (43) and F_1P (42) into account, we may express the 1st statement as follows:

$$\frac{b\sin\varepsilon}{(a\cos\varepsilon + c)} = \tan\left(2\arctan\frac{(a-c)\sin\varepsilon}{b(1+\cos\varepsilon)}\right) \tag{45}$$

Indeed, if we rewrite the r.h.s. (right hand side) of (45) as follows:

$$\frac{2\frac{(a-c)\sin\varepsilon}{b(1+\cos\varepsilon)}}{1-\left(\frac{(a-c)\sin\varepsilon}{b(1+\cos\varepsilon)}\right)^2} = \frac{2(a-c)b(1+\cos\varepsilon)\sin\varepsilon}{(a+c)(a-c)(1+\cos\varepsilon)^2-(a-c)^2(1+\cos\varepsilon)(1-\cos\varepsilon)}$$

one can easily check that (45) is an identity.

As regards the item (ii), observe that (a) the tangent to the ellipse H at P meets the circle Φ_1 in T_{112} ([8], Theorem 1) and (b) the points E and I symmetrically lie about such tangent ([4], Theorem 2) on the circle Φ_1 ([7], Theorem 2.3); it follows that the arcs ET_{112} and $T_{112}I$ are congruent; this conclusion implies, in turn, that $\angle EF_1T_{112}$ and $\angle T_{112}F_1I$ are congruent because are inscribed in the circle Φ_1 and are subtended by congruent arcs.

As regards the item (iii), let us rewrite the thesis $\measuredangle\Sigma_1F_1T_{112} = \measuredangle T_{112}F_1\Sigma_2$ as follows: $\measuredangle\Sigma_1F_1P + \measuredangle PF_1T_{112} = \measuredangle T_{112}F_1F_2 + \measuredangle F_2F_1\Sigma_2$; accordingly, by virtue of the item (i) of the present Theorem 2.3 (namely, $\measuredangle PF_1T_{112} = \measuredangle T_{112}F_1F_2$), to prove the thesis it is enough to prove: $\measuredangle\Sigma_1F_1P = \measuredangle F_2F_1\Sigma_2$; remembering the equations (38) representing the tangents drawn to the ellipse H_Σ from the focus $F_1(-c,0)$, we can write:

$$\measuredangle\Sigma_1F_1P = \arctan\frac{\frac{b\sin\varepsilon+\sqrt{R}}{c+2a\cos\varepsilon}-\frac{b\sin\varepsilon}{a\cos\varepsilon+c}}{1+\frac{b\sin\varepsilon+\sqrt{R}}{c+2a\cos\varepsilon}\frac{b\sin\varepsilon}{a\cos\varepsilon+c}}; \tag{46}$$
$$\measuredangle F_2F_1\Sigma_2 = \arctan\left|\frac{b\sin\varepsilon-\sqrt{R}}{c+2a\cos\varepsilon}\right| \tag{47}$$

Our thesis amounts to state that the r.h.s.'s of (46) and (47) are equal and to write, therefore, the following:

$$\frac{(b\sin\varepsilon+\sqrt{R})(a\cos\varepsilon+c)-b\sin\varepsilon(c+2a\cos\varepsilon)}{(c+2a\cos\varepsilon)(a\cos\varepsilon+c)+(b\sin\varepsilon+\sqrt{R})b\sin\varepsilon} = \frac{\sqrt{R}-b\sin\varepsilon}{c+2a\cos\varepsilon} \tag{48}$$

Indeed, (48) can be written, after clearing and simplifying, as follows:

$$2(c+a\cos\varepsilon)(c+2a\cos\varepsilon) = (c+2a\cos\varepsilon)^2 + a^2 - b^2\cos^2\varepsilon + 2ac\cos\varepsilon - b^2\sin^2\varepsilon$$

which is an identity.

As regards the item (iv) ($\measuredangle T_{131}F_1T_{112} = \measuredangle T_{112}F_1N_{22}$), it is enough to remember the afore mentioned result ([8], Theorem 2.10) ensuring us that the line $F_1N_{21}T_{112}$ (43) orthogonally meets $T_{131}F_1N_{22}$ (44). The same Theorem ensures us that the line $T_{131}F_1N_{22}$ (44) bisects the external angles associated with $\angle PF_1F_2$, $\angle EF_1I$ and $\angle\Sigma_1F_1\Sigma_2$. □

Before stating the next result, let us remember that the focus $F_2(c,0)$ is collinear with the points T_{131} (14) and N_{21} (18), as well as with the points T_{112} (15) and N_{22} (17) on the following orthogonal lines $F_2N_{21}T_{131}$ (49) and $T_{112}F_2N_{22}$ (50), respectively ([8], Theorem 2.10):

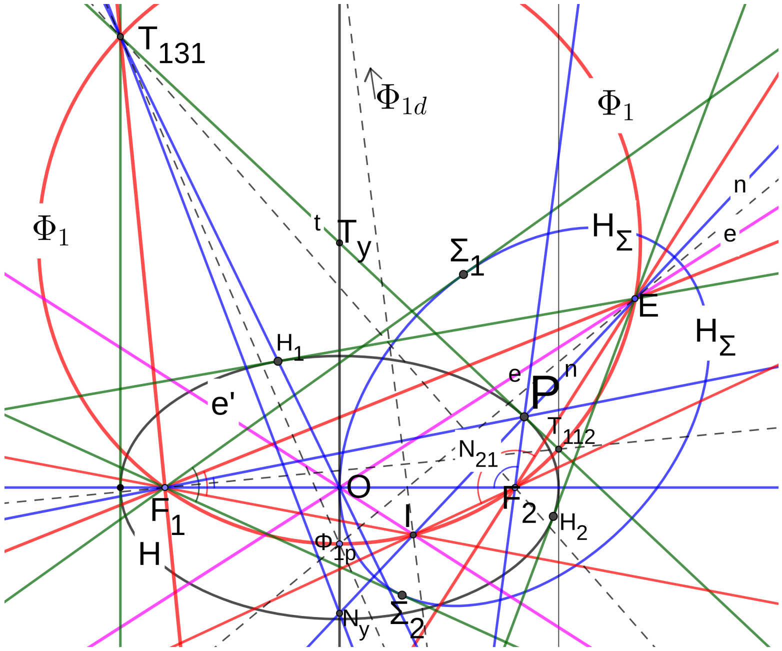

Figure 2: The ellipse H, its point P, the symbiotic ellipse H_Σ [blue], the circle Φ_1 [red], the tangents [green] from F_1 to H_Σ, from E to H and from T_{131} to H, the lines F_1E, F_1I, F_2E, F_2I [red], F_1P, F_1F_2, F_2P [blue], the eccentric line OE [magenta], the normal EPI [blue] the lines $T_{131}F_1$ [red] and $T_{131}F_2$ [black, dashed], $T_{131}N_y$, $T_{131}O$ [blue]; dashed lines are bisectors.

$$y = -\frac{b(1+\cos\varepsilon)}{(a+c)\sin\varepsilon} \qquad (49) \qquad\qquad y = \frac{b(1-\cos\varepsilon)}{(a-c)\sin\varepsilon}(x-c) \qquad (50)$$

Theorem 2.4. *[Fig. 2] The line $F_2N_{21}T_{131}$ (49) [black, dashed] bisects the following angles sharing the focus F_2 as vertex: (i) $\angle F_1F_2P$ [blue], (ii) $\angle IF_2E$ [red], (iii) $\angle N_{22}F_2T_{112}$ [red].*

Proof. As regards the item (i), observe that, by virtue of a well known Theorem, the normal to the ellipse H at P bisects the angle $\angle F_1PF_2$, formed by the focal radii of P; on the other hand, we know (Theorem 2.3, item i) that the line $F_1N_{21}T_{112}$ bisects $\angle PF_1F_2$; since these two bisectors – namely, the normal and the line $F_1N_{21}T_{112}$ – meet in N_{21} (18), such point is the incenter of the triangle PF_1F_2; accordingly, the bisector of the 3rd angle of such triangle, that is $\angle F_1F_2P$, passes through N_{21}, too; therefore, the line $F_2N_{21}T_{131}$ bisects $\angle F_1F_2P$. As regards the item (ii), the arcs IT_{131} and $T_{131}E$ are congruent because the point T_{131} belongs to the tangent to the ellipse H at P, which is the perpendicular bisector of the segment EI ([4], Theorem 2); it follows that $\measuredangle IF_2T_{131} = \measuredangle T_{131}F_2E$ because both angles are inscribed in the circle Φ_1 and are subtended by the afore mentioned, congruent arcs. As regards the item (iii), it is enough to remember ([8], Theorem 2.10) that the line $F_2N_{21}T_{131}$ (49) orthogonally meets $T_{112}F_2N_{22}$ (50). $\square$

Knowing the coordinates of the points T_{131} (14), Φ_{1p} (24), N_y (10) and $O(0,0)$, we can write the following equations of the lines $T_{131}O$ (51), $T_{131}\Phi_{1p}$ (52) and $T_{131}N_y$ (53):

$$T_{131}O: \qquad y = -\frac{b(1+\cos\varepsilon)}{a\sin\varepsilon}x \qquad (51)$$

$$T_{131}\Phi_{1p}: \qquad y = \frac{b - \sqrt{a^2\sin^2\varepsilon + b^2\cos^2\varepsilon}}{\sin\varepsilon} - \frac{b\cos\varepsilon + \sqrt{a^2\sin^2\varepsilon + b^2\cos^2\varepsilon}}{a\sin\varepsilon}x \tag{52}$$

$$T_{131}N_y: \qquad y = -\frac{c^2}{b}\sin\varepsilon - \frac{a^2\sin^2\varepsilon + b^2\cos^2\varepsilon + b^2\cos\varepsilon}{ab\sin\varepsilon}x \tag{53}$$

Theorem 2.5. *[Fig. 2] The line $T_{131}\Phi_{1p}$ (52) [black, dashed] bisects the following angles sharing the point T_{131} (14) as vertex: (i) $\angle N_yT_{131}O$ [blue], (ii) $F_1T_{131}F_2$ [F_1T_{131}: red; $T_{131}F_2$: black, dashed line] and (iii) $V_3T_{131}P$ [its sides are the tangents (green) to the ellipse H at the vertex $V_3(-a,0)$, and at P].*

Proof. item (i): the lines $T_{131}O$ and $T_{131}N_y$ meet the circle Φ_1 in points symmetrically lying about the minor axis, at the $\Delta x = ac^2\sin^2\varepsilon/\left(a^2\sin^2\varepsilon + b^2(1+\cos\varepsilon)^2\right)$ distance from the minor axis; accordingly, the Φ_1 arcs joining Φ_{1p} with either intersection are congruent and, therefore, the inscribed angles $\angle OT_{131}\Phi_{1p}$ and $\angle\Phi_{1p}T_{131}N_y$, subtended by these arcs, are congruent, too; the item (ii) is equivalent to: $\measuredangle F_1T_{131}\Phi_{1p} = \measuredangle\Phi_{1p}T_{131}F_2$; indeed, such angles are inscribed in the circle Φ_1 and are subtended by the congruent arcs $F_1\Phi_{1p}$ and $\Phi_{1p}F_2$, respectively; the item (iii) is equivalent to $\measuredangle V_3T_{131}\Phi_{1p} = \measuredangle\Phi_{1p}T_{131}O$; indeed, such angles are inscribed in the circle Φ_1 and are subtended by congruent arcs. □

Now, let the tangents to the ellipse H (1) be drawn from the point E; by means of the previously introduced symbol $(asbc)$ (22) and the following, compact notation:

$$(a4b4) = a^4\sin^2\varepsilon + b^4\cos^2\varepsilon, \tag{54}$$

the tangency points – denoted as H_1, H_2 – can be given the following representation:

$$H_i\left(a^2\frac{b^2\cos\varepsilon \pm \sin\varepsilon\sqrt{(a4b4)+2ab(asbc)}}{(a+b)(asbc)};\ b^2\frac{a^2\sin\varepsilon \mp \cos\varepsilon\sqrt{(a4b4)+2ab(asbc)}}{(a+b)(asbc)}\right)(i=1,2) \tag{55}$$

We may invoke the Theorems 2.3 and 2.4 for the ellipse H_Σ and its point O, so to state the following Theorems 2.6 and 2.7, respectively (any object entering the original statements has been replaced by its homologous, according to the Table I):

Theorem 2.6. *[Fig. 2] The line $E\Phi_{3p}\Phi_{1p}$ [black, dashed] bisects the following angles, sharing the vertex E: (i) $\angle OEN_y$ [it is formed by the eccentric line OE [magenta] (3) of P and the normal $EPIN_y$ (8) [blue] to the ellipse H (1) at P], (ii) $\angle F_1EF_2$ [red], (iii) $\angle H_1EH_2$ [green; it is formed by the tangents drawn to H from E] and (iv) $\angle\Phi_{1d}E\Phi_{3d}$.*

Theorem 2.7. *[Fig. 2] The line $I\Phi_{3p}\Phi_{1d}$ [black, dashed] bisects the following angles, sharing the vertex I: (i) $\angle OIP$ [it is the angle formed by the symm-eccentric line OI of P [magenta] (4) and the normal IP [blue] to the ellipse H at P]; (ii) $\angle F_1IF_2$ [red].*

Similar facts, we overlook for the sake of brevity, could be stated for some angles sharing as vertex the points T_{112}, T_{321} and T_{342}.

3 The "Bridge" Circles

The locus of points from which the ellipse (1) can be seen under a right angle is ([2], Theorem 9.2.1) the circle (Fig. 3):

$$x^2 + y^2 = a^2 + b^2 \tag{56}$$

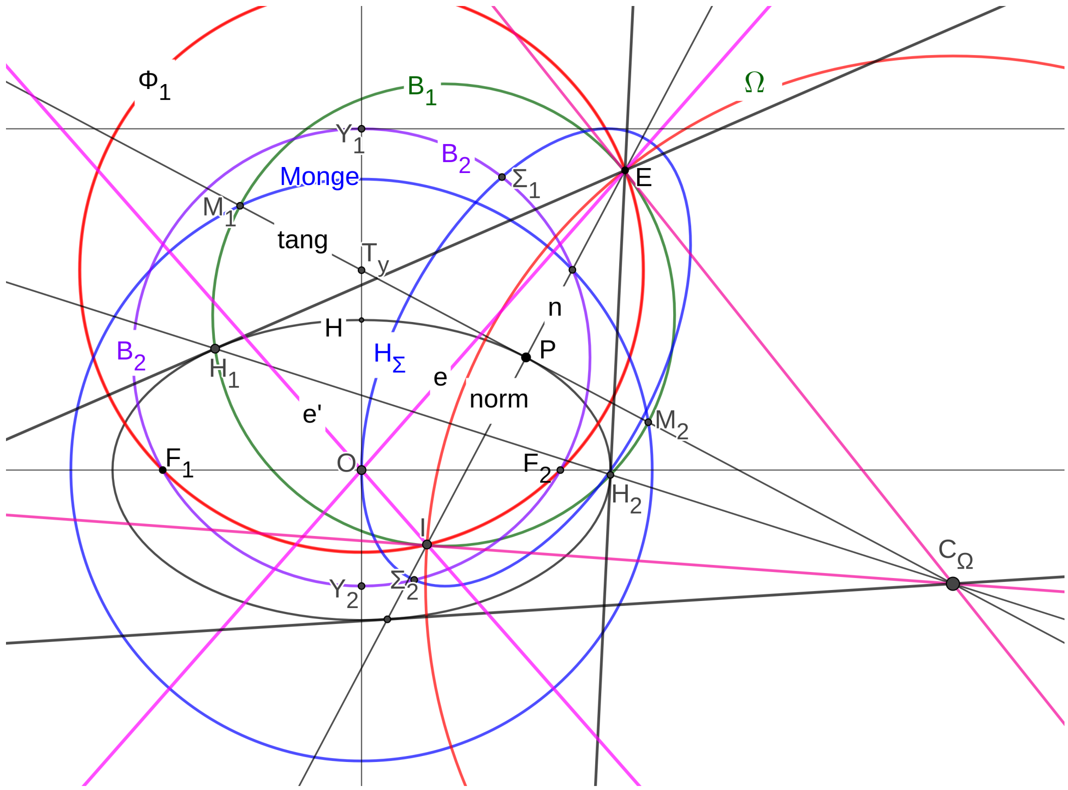

Figure 3: The ellipse H is drawn with its point P, Monge's circle [blue], the circle Φ_1 [red], the symbiotic ellipse H_Σ [blue] the bridge circle B_1 [green] with its tangents at E and I [magenta] and the bridge circle B_2 [violet]

.

which is referred to as Monge's circle.

The tangent (5) to the ellipse H at P meets Monge's circle in two points M_1, M_2, synthetically represented as follows:

$$M_i\Big(\frac{b^2\cos\varepsilon \mp \sin\varepsilon\sqrt{(a4b4)}}{(asbc)}a;\ \frac{a^2\sin\varepsilon \pm \cos\varepsilon\sqrt{(a4b4)}}{(asbc)}b\Big);\ (i=1,2) \tag{57}$$

In this Section, I will deal with circles [B_1 (58), B_2 (61), B_3 (63)] which link couples of noticeable points belonging to the ellipse H (1), the symbiotic ellipse H_Σ (37), the circle Φ_1 (13) and Monge's circle (56); such property accounts for their name.

Theorem 3.1. *[The 1st bridge-circle Theorem] [Fig. 3] Let the following six points be taken: (i) E (11) and I (12)* [where the normal to the ellipse H at P meets the circle Φ_1 (13)], *(ii) M_1, M_2 (57)* [where the tangent to the ellipse H at P meets Monge's circle]; *(iii) H_1 and H_2 (55)* [where the tangents drawn to the ellipse H from E (11) touch H]; *such points are concyclic on the following circle B_1:*

$$\Big(x - \frac{ab^2\cos\varepsilon}{a^2\sin^2\varepsilon + b^2\cos^2\varepsilon}\Big)^2 + \Big(y - \frac{a^2b\sin\varepsilon}{a^2\sin^2\varepsilon + b^2\cos^2\varepsilon}\Big)^2 = \frac{a^4\sin^2\varepsilon + b^4\cos^2\varepsilon}{a^2\sin^2\varepsilon + b^2\cos^2\varepsilon} \tag{58}$$

Proof. The Theorem 3.1 can be proved by checking that the coordinates of the afore mentioned six points fulfill (58); for this purpose, we shall begin by rewriting (58) as follows:

$$x^2(asbc) - 2xab^2\cos\varepsilon + y^2(asbc) - 2ya^2b\sin\varepsilon = (a^2-b^2)(a^2\sin^2\varepsilon - b^2\cos^2\varepsilon) \tag{59}$$

Replacing either the E (11) or I (12) coordinates in (59), we get two relationships, summarized as follows:

$$(a \pm b)(a^2 \sin^2 \varepsilon + b^2 \cos^2 \varepsilon) - 2ab(\pm a \sin^2 \varepsilon + b \cos^2 \varepsilon) = (a \mp b)(a^2 \sin^2 \varepsilon - b^2 \cos^2 \varepsilon)$$

One can easily see that both expressions are identically fulfilled. Now, replacing either the M_1 or M_2 (57) coordinates in (59), we get two relationships which, after clearing and simplifying, can be written as follows:

$$\begin{gathered} -a^2 b^4 \sin^2 \varepsilon \cos^2 \varepsilon + a^2 (a^4 \sin^2 \varepsilon + b^4 \cos^2 \varepsilon) \sin^2 \varepsilon - a^4 b^2 \sin^2 \varepsilon \cos^2 \varepsilon \\ + b^2 (a^4 \sin^2 \varepsilon + b^4 \cos^2 \varepsilon) \cos^2 \varepsilon - a^6 \sin^4 \varepsilon - b^6 \cos^4 \varepsilon = 0 \end{gathered}$$

this expression is an identity, too.

The replacement of the H_1 coordinates (55) in (59) results in an expression where radicals are easily seen to form a vanishing set; the remnant terms can be written as follows:

$$\begin{gathered} a^4 b^4 + (a^4 \sin^2 \varepsilon + b^4 \cos^2 \varepsilon)(a^4 \sin^2 \varepsilon + b^4 \cos^2 \varepsilon) + 2ab(a^2 \sin^2 \varepsilon + b^2 \cos^2 \varepsilon)(a^4 \sin^2 \varepsilon + b^4 \cos^2 \varepsilon) \\ -2a^3 b^3 (a+b)(b \cos^2 \varepsilon + a \sin^2 \varepsilon) = (a^2 - b^2)(a^4 \sin^4 \varepsilon - b^4 \cos^4 \varepsilon)(a+b)^2 \end{gathered}$$

Even this expression, as one can see by trivial manipulations, is an identity. □

If we wish to invoke the Theorem 3.1 for the symbiotic ellipse H_Σ, we should previously determine the homologous points to the ones ($M_1,\ M_2$ (57)) the tangent to H at P shares with Monge's orthoptic circle (56). Remembering that Monge's circle is the locus of points from which the ellipse can be seen under a right angle and that the tangent to H_Σ at its point O is the y-axis, we conclude that the points we are looking for are the y-axis points from which a tangent to H_Σ *orthogonal to the y-axis* can be drawn. Easy calculations allow us to determine such points – hereinafter denoted $Y_1,\ Y_2$ – as follows:

$$Y_i\Big(0,\ b \sin \varepsilon \pm \sqrt{a^2 - b^2 \cos^2 \varepsilon}\Big)\ (i = 1, 2) \tag{60}$$

Therefore, we can state the following:

Theorem 3.2. *[The 2nd bridge-circle Theorem] [Fig. 3] Let the following six points be taken: (i) the foci $F_1,\ F_2$ of the ellipse H (1); (ii) the points $Y_1,\ Y_2$ (60) [where the tangent to the ellipse H_Σ at O ($x = 0$) meets H_Σ Monge's circle]; (iii) the points $\Sigma_1,\ \Sigma_2$ (39) [where the tangents $t_1,\ t_2$ (38) drawn to the symbiotic ellipse H_Σ from the focus F_1 of H touch H_Σ].*

Such six points are concyclic on the following circle B_2:

$$x^2 + (y - b \sin \varepsilon)^2 = b^2 \sin^2 \varepsilon + c^2 \tag{61}$$

Now, let us take (Fig. 4) the intersections of the tangent and normal to H at P with Φ_1 and Monge's circle, respectively; the formers are the well known points T_{131} (14) and T_{112} (15); the latters, – we denote $M_3,\ M_4$ – are synthetically represented as follows:

$$M_i\left(\frac{ac^2 \sin^2 \varepsilon \pm b\sqrt{(asbc)^2 + a^2 b^2}}{(asbc)} \cos \varepsilon;\ \frac{\pm a\sqrt{(asbc)^2 + a^2 b^2} - bc^2 \cos^2 \varepsilon}{(asbc)} \sin \varepsilon\right)\ (i = 3, 4) \tag{62}$$

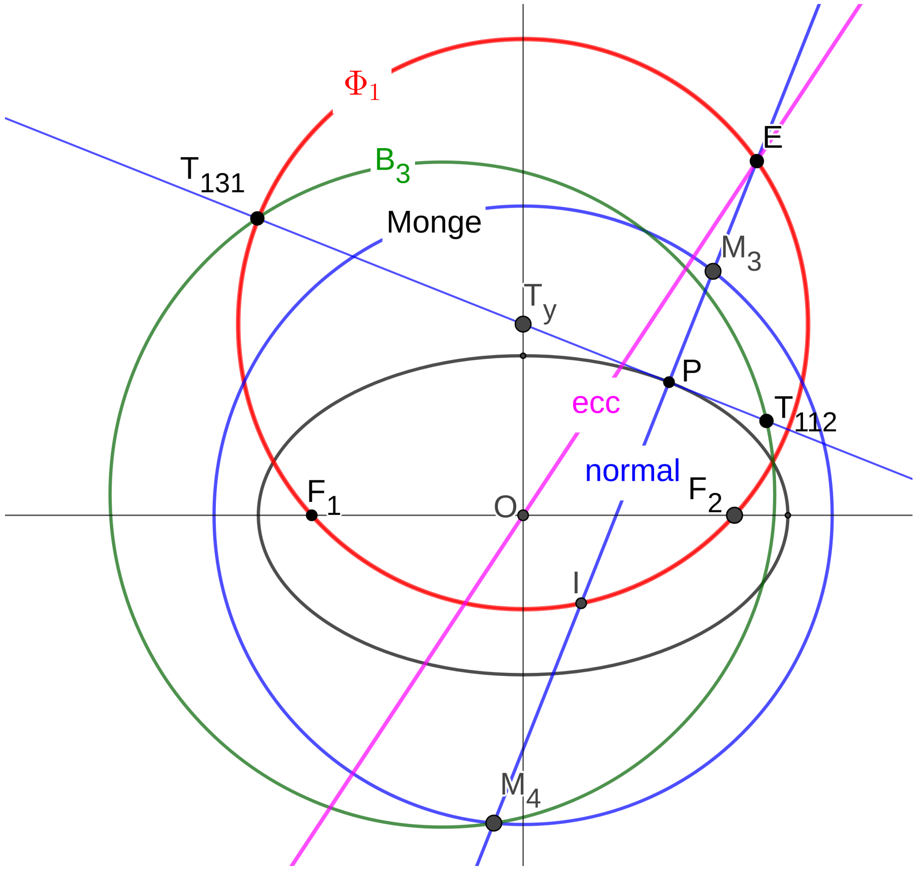

Figure 4: The tangent to the ellipse H at P meet the circle Φ_1 [red] at T_{131} and T_{112}; the normal to H at P meets Monge's circle [blue] at M_3, M_4. The circle B_3 [green] passes through T_{131}, T_{112}, M_3 and M_4 (Theorem 3.3).

Theorem 3.3. *[The 3rd bridge-circle Theorem] [Fig. 4]. Let the following four points be taken: (i)* T_{131} (14), T_{112} (15) *[where the tangent* (5) *to the ellipse* H *at* P *meets the circle* Φ_1 (13) *[red] and (ii)* M_3, M_4 (62) *[where the normal* (8) *to the ellipse* H *at* P *meets Monge's circle (blue)]; such points are concyclic on the following circle* B_3 *[green]:*

$$\left(x+\frac{ab^2\cos\varepsilon}{(asbc)}\right)^2+\left(y-\frac{b^3\cos^2\varepsilon}{(asbc)\sin\varepsilon}\right)^2=\frac{(asbc)^2+a^2b^2\sin^2\varepsilon}{(asbc)\sin^2\varepsilon} \tag{63}$$

The Theorem (3.3) can be demonstrated by showing that the coordinates of the four points T_{131} (14), T_{112} (15), M_3, M_4 (62) fulfil the circle B_3 equation (63).

Another Theorem, omitted for the sake of brevity, could be stated invoking the Theorem 3.3 for the symbiotic ellipse H_Σ.

The Theorems 3.4 and 3.5 describe further relationships linking the bridge circles with the ellipse H.

Theorem 3.4. *The centers of the circles* B_1 (58) *and* B_2 (61) *lie on the circle constructed on the segment* OP *as diameter, whose equation follows:*

$$\left(x-\frac{a\cos\varepsilon}{2}\right)^2+\left(y-\frac{b\sin\varepsilon}{2}\right)^2=\left(\frac{a\cos\varepsilon}{2}\right)^2+\left(\frac{b\sin\varepsilon}{2}\right)^2 \tag{64}$$

The Theorem 3.4 can be proved by checking that the coordinates of the mentioned points fulfill the equation (64); analogously, one can do for the next statement (Theorem 3.5):

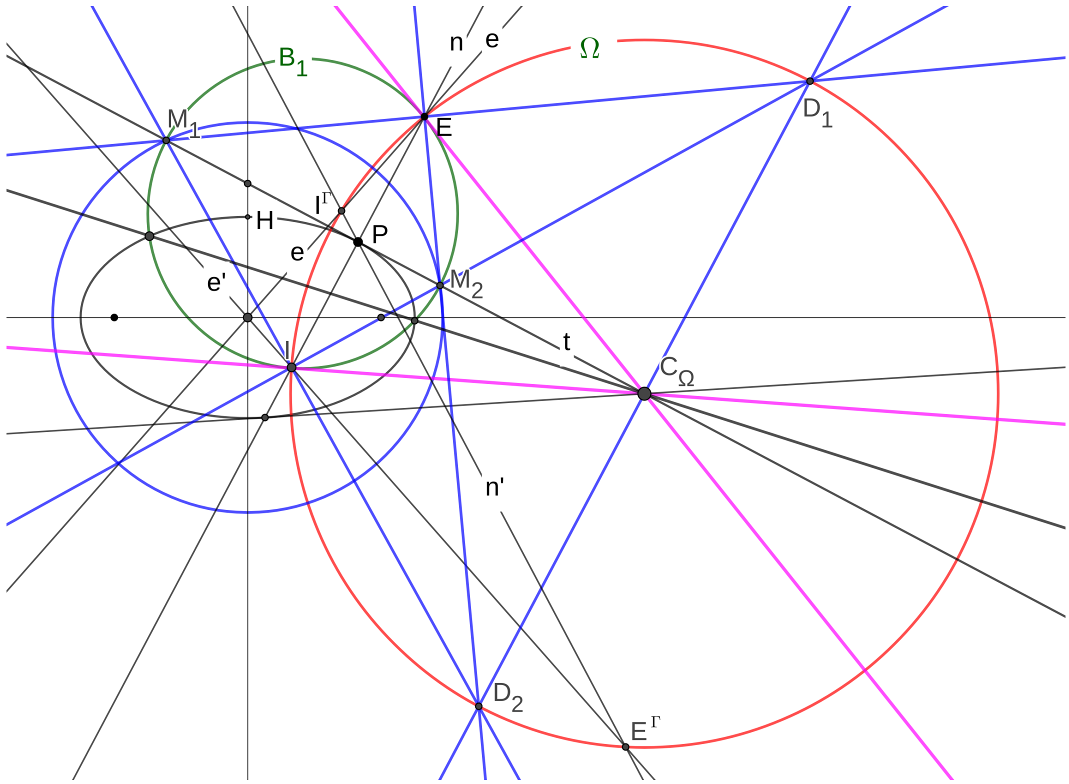

Figure 5: The same as Fig. 3 (smaller and simplified), with the addition of the symm-normal line n' [black], the points E^Γ and I^Γ, the circle Ω [red], the sides [blue] of the complete quadrangle M_1IM_2E and the line [blue] joining the diagonal points D_1 and D_2.

Theorem 3.5. *The centers of the circles B_1 (58) and B_3 (63) lie on the circle constructed on the segment OT_y as diameter, whose equation follows:*

$$x^2 + \left(y - \frac{b}{2\sin\varepsilon}\right)^2 = \left(\frac{b}{2\sin\varepsilon}\right)^2 \tag{65}$$

In 2007, Ternullo [4] introduced the following line through P, denoted *symm-normal*:

$$y = -x\frac{a}{b}\tan\varepsilon + \frac{a^2+b^2}{b}\sin\varepsilon; \tag{66}$$

In 2009, the same author [5] introduced the following circle Ω:

$$\left(x - \frac{a^3}{(a^2-b^2)\cos\varepsilon}\right)^2 + \left(y + \frac{b^3}{(a^2-b^2)\sin\varepsilon}\right)^2 = \frac{a^6\sin^2\varepsilon + b^6\cos^2\varepsilon}{(a^2-b^2)^2\sin^2\varepsilon\cos^2\varepsilon} - (a^2+b^2) \tag{67}$$

The circle Ω (67) passes ([5], Theorem 2.11) through the points E (11) and I (12) (where the normal to H at P meets the eccentric (3) and symm-eccentric line of P (4), respectively), as well as through the following points E^Γ, I^Γ:

$$E^\Gamma\left(\frac{a^2+b^2}{a-b}\cos\varepsilon;\ -\frac{a^2+b^2}{a-b}\sin\varepsilon\right); \qquad I^\Gamma\left(\frac{a^2+b^2}{a+b}\cos\varepsilon;\ \frac{a^2+b^2}{a+b}\sin\varepsilon\right)$$

where the symm-normal (66) meets the symm-ecc (4) and the eccentric line (3), respectively.

The center C_Ω of the circle Ω is the pole of the normal (8) to H at P wrt the ellipse H ([5], Theorem 2.2); therefore, the normal contains the poles – wrt the ellipse H – of all lines through C_Ω. On the other hand, we have determined the points H_1, H_2 where the tangents drawn to the ellipse H from E touch H; the line linking H_1 and H_2 is, therefore, the polar of E wrt the ellipse H. Remembering that E belongs to the normal and that, accordingly, the polar of E passes through C_Ω, we conclude as follows:

Theorem 3.6. *The points* H_1, H_2 (55)*, where the tangents drawn to the ellipse* H (1) *from* E (11) *touch* H*, are collinear with the center* C_Ω *of the circle* Ω (67)*.*

Theorem 3.7. *The tangents drawn to the circle* B_1 (58) *at* E *and* I *concur in* C_Ω

Proof. To demonstrate the Theorem 3.7, let us begin by writing the matrix of coefficients of the circle B_1 equation (58) as follows:

$$\begin{vmatrix} (asbc) & 0 & -ab^2\cos\varepsilon \\ 0 & (asbc) & -a^2b\sin\varepsilon \\ -ab^2\cos\varepsilon & -a^2b\sin\varepsilon & a^2b^2-a^4\sin^2\varepsilon-b^4\cos^2\varepsilon \end{vmatrix}$$

Afterwards, we can write the equation of the polar of C_Ω w.r.t the circle B_1; few manipulations are enough to check that such polar coincides with the normal (8) to H at P. On the other hand, we know that (i) the normal to H at P contains the points E and I and (ii) the normal shares such points with the circle B_1 (58); therefore, we conclude that the tangents drawn to the circle B_1 from the point C_Ω touch B_1 at E and I. □

Remembering that the circle Ω shares the points E and I with the circle B_1, the afore mentioned conclusion – namely, that the tangents drawn to the circle B_1 from the center C_Ω of the circle Ω touch B_1 at E and I – implies that the circles B_1 and Ω are orthogonal. On the other hand, let us consider the complete quadrangle determined by the points M_1, I, M_2, E; the special symmetry with which such points are disposed – namely, M_1, M_2 on a diameter of B_1 and E, I on a chord orthogonal to M_1M_2 – implies that, if the opposite sides of the quadrangle meet in points we denote D_1 and D_2, then the circle constructed taking the segment D_1D_2 as diameter orthogonally meets the circle B_1 at E and I. Conversely, as the circle Ω orthogonally meets the circle B_1 at E and I (and, of course, there is precisely one circle which ortogonally meets a given circle at two given points), we conclude that there is a diameter of the circle Ω joining the points D_1 and D_2 where the opposite sides M_1E, M_2I and M_1I, M_2E of the quadrangle meet. Obvious reasons of symmetry require that such diameter parallels the line EI (namely, the normal to H at P). The following statement represents the conclusion of this reasoning:

Theorem 3.8. *Taken the complete quadrangle* M_1IM_2E*, its diagonal points* D_1, D_2 *– representing the intersections of the opposite sides* M_1E, M_2I *and* M_1I, M_2E *– belong to the circle* Ω*, where they are diametrically opposed on a line paralleling the normal to* H *at* P*.*

It is worth mentioning that the point C_Ω is the pole of a unique line – that is, the normal to H at P – under the polarity relationships defined by two conics, namely the ellipse H and the circle B_1.

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