Journal for Geometry and Graphics
Volume 25 (2021), No. 1, 45–52

Tubular Surfaces of Finite Type Gauss map

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Abstract. In this study, we continue the classification of finite type Gauss map surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. To do this, we investigate an important family of surfaces, namely, tubular surfaces in $\mathbb{E}^3$. We show that the Gauss map of a tubular surface is of an infinite type regarding the second fundamental form.

Key Words: tubular surface in Euclidean 3-space, surface of finite II-type, Laplace operator

MSC 2020: 53A05

1 Introduction

The theory of surfaces of finite Chen type was introduced by B.-Y. Chen in the late 1970s. From that moment on, interest in this field became widespread by many differential geometers, and it remains until this moment.

Let $\boldsymbol{x}$ be an isometric immersion of a surface S in the 3-dimensional Euclidean space $\mathbb{E}^3$. We represent by Δ^I the Laplacian operator of S acting on the space of smooth functions $C^\infty(S)$. Then S is said to be of finite I-type, if the position vector $\boldsymbol{x}$ of S can be decomposed as a finite sum of eigenvectors of Δ^I of S, that is

$$\boldsymbol{x} = \boldsymbol{c} + \boldsymbol{x}_1 + \boldsymbol{x}_2 + \cdots + \boldsymbol{x}_k,$$

where

$$\Delta^I \boldsymbol{x}_i = \xi_i \, \boldsymbol{x}_i, \quad i = 1, \ldots, k,$$

$\boldsymbol{c}$ is a fixed vector and $\xi_1, \xi_2, \ldots, \xi_k$ are eigenvalues of the operator Δ^I.

In the framework of surfaces of finite type S. Stamatakis and H. Al-Zoubi in [22] defined the notion of surfaces of finite type associated with the second or third fundamental form.

ISSN 1433-8157/\$ 2.50

For results concerning surfaces of finite type associated with the second or third fundamental form (see [1, 3, 5, 6, 8, 9, 12]).

Another generalization can be made by studying surfaces in $\mathbb{E}^3$ of coordinate finite type, that is, their position vector $\boldsymbol{x}$ satisfying the relation

$$\Delta^J \boldsymbol{x} = A\boldsymbol{x}, \quad J = II, III, \tag{1}$$

where $A \in \mathbb{R}^{3\times3}$.

H. Al-Zoubi and S. Stamatakis in [13] studied ruled and quadric surfaces of coordinate finite type with respect to the third fundamental form satisfying

$$\Delta^{III} \boldsymbol{x} = A\boldsymbol{x}. \tag{2}$$

The same authors in [23] classified the class of surfaces of revolution satisfying condition (2). Later in [14], the translation surfaces were studied, and it was proved that Sherk's surface is the only translation surface satisfying (2). B. Senoussi and M. Bekkar, in [21] classified the Helicoidal surfaces with $\triangle^J r = Ar, J = I, II, III$.

On the other hand, H. Al-Zoubi, T. Hamadneh in [11] classified the class of surfaces of revolution satisfying

$$\Delta^{II} \boldsymbol{x} = A\boldsymbol{x}.$$

where $A \in \mathbb{R}^{3\times3}$.

Similarly, we can apply the notion of surfaces of finite type to a smooth map, for example, the Gauss map of a surface. Here in this kind of research, many results can be found in ([2, 4, 7, 10, 15–18, 20]).

In this research, we will focus on surfaces of finite II-type. Firstly, we will define the second differential parameter of Beltrami with respect to the second fundamental form of M^2. Further, we continue our study by proving infinite type Gauss map for an important class of surfaces, namely, tubes in $\mathbb{E}^3$.

2 Preliminaries

Let $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$ be a regular parametric representation of a surface M^2 in the Euclidean 3-space $\mathbb{E}^3$ referred to any coordinate system, which does not contain parabolic points. We denote by b_{ij} the components of the second fundamental form $II = b_{ij}du^i du^j$ of S. Let $\varphi(u^1, u^2)$ be a sufficient differentiable function on M^2. Then the second differential parameter of Beltrami with respect to the second fundamental form of M^2 is defined by [20]

$$\Delta^{II}\varphi := -\frac{1}{\sqrt{|b|}}(\sqrt{|b|}b^{ij}\varphi_{/i})_{/j}, \tag{3}$$

where (b^{ij}) denotes the inverse tensor of (b_{ij}) and $b := \det(b_{ij})$.

Up to now, spheres are the only known surfaces of finite II-type in $\mathbb{E}^3$. So one can ask:

Problem. *Other than the spheres, which surfaces in $\mathbb{E}^3$ are of finite II-type?*

This paper provides the first attempt at the study of finite type Gauss map of surfaces in $\mathbb{E}^3$ corresponding to the second fundamental form. In general when the Gauss map of M^2

is of finite type k, then there exists a polynomial $R(x) \neq 0$, such that $R(\Delta^{II})(\boldsymbol{n} - \boldsymbol{c}) = \mathbf{0}$. If $R(x) = x^k + \rho_1 x^{k-1} + \cdots + \rho_{k-1}x + \rho_k$, then coefficients ρ_i are given by

$$\begin{aligned}
\rho_1 =& - (\xi_1 + \xi_2 + \cdots + \xi_k),\\
\rho_2 =&(\xi_1\xi_2 + \xi_1\xi_3 + \cdots + \xi_1\xi_k + \xi_2\xi_3 + \cdots + \xi_2\xi_k + \cdots + \xi_{k-1}\xi_k),\\
\rho_3 =& - (\xi_1\xi_2\xi_3 + \cdots + \xi_{k-2}\xi_{k-1}\xi_k),\\
&\cdots\\
\rho_k =&(-1)^k \xi_1\xi_2\cdots\xi_k.
\end{aligned}$$

Hence the Gauss map $\boldsymbol{n}$ satisfies, (see [19])

$$(\Delta^{II})^k \boldsymbol{n} + \rho_1 (\Delta^{II})^{k-1}\boldsymbol{n} + \cdots + \rho_k(\boldsymbol{n} - \boldsymbol{c}) = \boldsymbol{0}. \tag{4}$$

Our main result is the following

Theorem 1. *All tubes in $\mathbb{E}^3$ are of infinite type Gauss map corresponding to the second fundamental form.*

Our discussion is local, which means that we show in fact that any open part of the Gauss map of a tube is of infinite Chen type.

3 Tubes in $\mathbb{E}^3$

Let $\ell : \boldsymbol{q} = \boldsymbol{q}(u),\ u\epsilon(a, b)$ be a regular unit speed curve of finite length. Suppose that $\boldsymbol{t}, \boldsymbol{h}, \boldsymbol{b}$ is the Frenet frame and $\kappa > 0$ the curvature of the unit speed curve $\boldsymbol{q}$. Then a regular parametric representation of a tubular surface $\mathfrak{F}$ of radius r satisfies $0 < r < min\frac{1}{|\kappa|}$ is given by

$$\mathfrak{F} : \boldsymbol{x}(u, \psi) = \boldsymbol{q} + r\cos\psi\boldsymbol{h} + r\sin\psi\boldsymbol{b}.$$

One can find that fundamental forms I and II of $\mathfrak{F}$ are given by

$$\begin{aligned}
I &= \Big(\delta^2 + r^2\tau^2\Big)du^2 + 2r^2\tau du d\psi + r^2 d\psi^2,\\
II &= \Big(-\kappa\delta\cos\psi + r\tau^2\Big)du^2 + 2r\tau du d\psi + r d\psi^2,
\end{aligned}$$

where $\delta := (1 - r\kappa\cos\psi)$ and τ is the torsion of the curve $\boldsymbol{q}$. The Gauss curvature of $\mathfrak{F}$ is given by

$$K = -\frac{\kappa\cos\psi}{r\delta}.$$

In the following we consider an open portion of the tube surface where $K \neq 0 (\Longleftrightarrow \cos\psi \neq 0)$. From (3) the Laplace operator Δ^{II} of $\mathfrak{F}$ is (see [1], formula (9))

$$\begin{aligned}
\Delta^{II} = \frac{1}{\kappa\delta\cos\psi}\Bigg[&\frac{\partial^2}{\partial u^2} - 2\tau\frac{\partial^2}{\partial u\partial\psi} + \left(\tau^2 - \frac{\kappa\delta\cos\psi}{r}\right)\frac{\partial^2}{\partial\psi^2}\\
&+ \frac{(1-2\delta)\beta}{2\kappa\delta\cos\psi}\frac{\partial}{\partial u} + \left(-\tau' + \frac{\tau\beta(2\delta-1)}{2\kappa\delta\cos\psi} + \frac{\kappa(2\delta-1)\sin\psi}{2r}\right)\frac{\partial}{\partial\psi}\Bigg],
\end{aligned} \tag{5}$$

where $\beta := \kappa'\cos\psi + \kappa\tau\sin\psi$ and $' := \frac{d}{du}$.

The Gauss map $\boldsymbol{n}(u,\psi)$ of $\mathfrak{F}$ has parametric representation

$$\boldsymbol{n}(u,\psi) = -\cos\psi\boldsymbol{h} - \sin\psi\boldsymbol{b}.$$

Inserting $\boldsymbol{n}$ in (5), we obtain

$$\Delta^{II}\boldsymbol{n} = \frac{\beta}{2\kappa\delta^2\cos\psi}\boldsymbol{t} + \left(\frac{\sin^2\psi}{2r\delta\cos\psi} + \frac{\cos\psi}{r\delta} - \frac{2\cos\psi}{r}\right)\boldsymbol{h} + \frac{(1-4\delta)\sin\psi}{2r\delta}\boldsymbol{b}, \tag{6}$$

We distinguish two cases

Case I. $\beta \neq 0$. Relation (6) can be written

$$\Delta^{II}\boldsymbol{n} = \frac{\beta}{2\kappa\delta^2\cos\psi}\boldsymbol{t} + \frac{1}{\kappa\delta\cos\psi}\boldsymbol{P}_1(\cos\psi,\sin\psi), \tag{7}$$

where $\boldsymbol{P}_1(\cos\psi,\sin\psi)$ is a vector with components polynomials of the functions $\cos\psi$ and $\sin\psi$ with coefficients functions of the variable u.

After long calculations, $(\Delta^{II})^2\boldsymbol{n}$ can be computed as follows

$$(\Delta^{II})^2\boldsymbol{n} = \frac{(3\delta-2)(12\delta-7)\beta^3}{4\kappa^4\delta^5\cos^4\psi}\boldsymbol{t} + \frac{1}{(\kappa\delta\cos\psi)^4}\boldsymbol{P}_2(\cos\psi,\sin\psi), \tag{8}$$

where $\boldsymbol{P}_2(\cos\psi,\sin\psi)$ is a vector with components polynomials of the functions $\cos\psi$ and $\sin\psi$ with coefficients functions of the variable u. For later use we put

$$(3\delta-2)(12\delta-7) = h_2(\delta). \tag{9}$$

From (7), one can see that $h_1(\delta) = 1$.

We need the following lemma which can be proved directly by using (5).

Lemma 1. *For any natural numbers m and n, we have*

$$\Delta^{II}\left(\frac{h_k(\delta)\beta^m}{\delta^n(\kappa\cos\psi)^{n-1}}\right) = -\frac{h_{k+1}(\delta)\beta^{m+2}}{2\delta^{n+3}(\kappa\cos\psi)^{n+2}} + \frac{1}{(\kappa\delta\cos\psi)^{n+2}}Q(\cos\psi,\sin\psi), \tag{10}$$

where $h_k(\delta)$ is a polynomial in δ of degree d, Q is a polynomial in $\cos\psi, \sin\psi$ of degree $n+3$ with functions in u as coefficients, deg($h_{k+1}(\delta)$) $= d+2$ and

$$h_{k+1}(\delta) = ((2n-1)\delta - n)(4(n+1)\delta - (2n+3))h_k(\delta). \tag{11}$$

So, it is easily verified that

$$(\Delta^{II})^3\boldsymbol{n} = \frac{(3\delta-2)(12\delta-7)(9\delta-5)(24\delta-13)\beta^5}{8\kappa^7\delta^8\cos^7\psi}\boldsymbol{t} + \frac{1}{(\kappa\delta\cos\psi)^7}\boldsymbol{P}_3(\cos\psi,\sin\psi), \tag{12}$$

where $\boldsymbol{P}_3(\cos\psi,\sin\psi)$ is a vector with components polynomials of the functions $\cos\psi$ and $\sin\psi$ with coefficients functions of the variable u.

Moreover, by using Lemma 1, one can find

$$(\Delta^{II})^k\boldsymbol{n} = \frac{h_k(\delta)\beta^{2k-1}}{2^k\delta^{3k-1}(\kappa\cos\psi)^{3k-2}}\boldsymbol{t} + \frac{1}{(\kappa\delta\cos\psi)^{3k-2}}\boldsymbol{P}_k(\cos\psi,\sin\psi), \tag{13}$$

and

$$(\Delta^{II})^{k+1}\boldsymbol{n} = \frac{h_{k+1}(\delta)\beta^{2k+1}}{2^{k+1}\delta^{3k+2}(\kappa\cos\psi)^{3k+1}}\boldsymbol{t} + \frac{1}{(\kappa\delta\cos\psi)^{3k+1}}\boldsymbol{P}_{k+1}(\cos\psi,\sin\psi), \tag{14}$$

where $\boldsymbol{P}_k(\cos\psi,\sin\psi)$, $\boldsymbol{P}_{k+1}(\cos\psi,\sin\psi)$ are vectors with components polynomials of the functions $\cos\psi,\sin\psi$ with coefficients functions of the variable u, and

$$h_\lambda(\delta) = \prod 12j\delta(4j-5) \neq 0, \quad j = 1,2,\dots,\lambda-1 \tag{15}$$

for any positive integer λ.

Now, we suppose that the tube $\mathfrak{F}$ is of finite type, thus from (7)-(14), and (4) we find that

$$\begin{aligned}\frac{h_{k+1}(\delta)\beta^{2k+1}}{2^{k+1}\delta^{3k+2}(\kappa\cos\psi)^{3k+1}}\boldsymbol{t} + \frac{1}{(\kappa\delta\cos\psi)^{3k+1}}\boldsymbol{P}_{k+1} + \frac{\rho_1 h_k(\delta)\beta^{2k-1}}{2^k\delta^{3k-1}(\kappa\cos\psi)^{3k-2}}\boldsymbol{t} + \frac{\rho_1}{(\kappa\delta\cos\psi)^{3k-2}}\boldsymbol{P}_k \\ + \dots + \frac{\rho_{k-1}(3\delta-2)(12\delta-7)\beta^3}{4\kappa^4\delta^5\cos^4\psi}\boldsymbol{t} + \frac{\sigma_{k-1}}{(\kappa\delta\cos\psi)^4}\boldsymbol{P}_2 + \frac{\rho_k\beta}{2\kappa\delta^2\cos\psi}\boldsymbol{t} + \frac{\rho_k}{\kappa\delta\cos\psi}\boldsymbol{P}_1 = \boldsymbol{0}.\end{aligned}$$

The above equation can be simply written as follows

$$\frac{h_{k+1}(\delta)\beta^{2k+1}}{\delta}\boldsymbol{t} = Q_1(\cos\psi,\sin\psi)\boldsymbol{t} + Q_2(\cos\psi,\sin\psi)\boldsymbol{h} + Q_3(\cos\psi,\sin\psi)\boldsymbol{b}, \tag{16}$$

where $Q_i(\cos\psi,\sin\psi)$ are polynomials in $\cos\psi$ and $\sin\psi$.

Case I. $\beta \neq 0$. From (16), we have

$$\frac{h_{k+1}(\delta)\beta^{2k+1}}{\delta} = Q_1(\cos\psi,\sin\psi). \tag{17}$$

This is impossible for any $k \geq 1$ since $h_{k+1}(\delta) \neq 0$.

Case II. $\beta \equiv 0$. Then $\kappa' = 0$ and $\kappa\tau = 0$. Thus $\kappa = \text{const.} \neq 0$ and $\tau = 0$, therefore the curve $\boldsymbol{q}$ is a plane circle and so, $\mathfrak{F}$ is an anchor ring. In this case, the first fundamental form becomes

$$I = \delta^2 du^2 + r^2 d\psi^2,$$

while the second is

$$II = -\kappa\delta\cos\psi du^2 + r d\psi^2.$$

Hence, equation (5) reduces to

$$\Delta^{II} = \frac{1}{\kappa\delta\cos\psi}\left(\frac{\partial^2}{\partial u^2} - \frac{\kappa\delta\cos\psi}{r}\frac{\partial^2}{\partial\psi^2} + \frac{\kappa(2\delta-1)\sin\psi}{2r}\frac{\partial}{\partial\psi}\right). \tag{18}$$

Applying (18) for the position vector $\boldsymbol{n}$, one finds

$$\Delta^{II}\boldsymbol{n} = \left(\frac{\sin^2\psi}{2r\delta\cos\psi} + \frac{\cos\psi}{r\delta} - \frac{2\cos\psi}{r}\right)\boldsymbol{h} + \frac{(1-4\delta)\sin\psi}{2r\delta}\boldsymbol{b},$$

which can be written as follows

$$\Delta^{II}\boldsymbol{n} = \frac{\sin^2\psi}{2r\delta\cos\psi}\boldsymbol{h} + \frac{1}{\delta}\boldsymbol{F}_1(\cos\psi,\sin\psi). \tag{19}$$

Consequently, we get

$$(\Delta^{II})^2 \boldsymbol{n} = -\frac{3\sin^4\psi}{4r^2\delta^3\cos^3\psi}\boldsymbol{h} + \frac{1}{\delta^2\cos^3\psi}\boldsymbol{F}_2(\cos\psi,\sin\psi), \tag{20}$$

where $\boldsymbol{F}_1(\cos\psi,\sin\psi)$, $\boldsymbol{F}_2(\cos\psi,\sin\psi)$ are vectors with components polynomials of the functions $\cos\psi,\sin\psi$ with coefficients functions of the variable u.

One can easily prove that

$$\Delta^{II}\frac{\sin^m\psi}{(\delta\cos\psi)^n} = -\frac{3\sin^{m+2}\psi}{2r(\delta\cos\psi)^{n+2}} + \frac{1}{(\delta\cos\psi)^{n+1}}Q(\cos\psi,\sin\psi),$$

where $Q(\cos\psi,\sin\psi)$ is a polynomial in $\cos\psi,\sin\psi$ with coefficients functions of the variable u. Therefore, we find that

$$(\Delta^{II})^k \boldsymbol{n} = (-1)^{k-1}\frac{3^{k-1}\sin^{2k}\psi}{(2r)^k(\delta\cos\psi)^{2k-1}}\boldsymbol{h} + \frac{1}{\delta^{2k-1}\cos^{2k-2}\psi}\boldsymbol{F}_k(\cos\psi,\sin\psi) \tag{21}$$

and

$$(\Delta^{II})^{k+1} \boldsymbol{n} = (-1)^{k}\frac{3^{k}\sin^{2k+2}\psi}{(2r)^{k+1}(\delta\cos\psi)^{2k+1}}\boldsymbol{h} + \frac{1}{\delta^{2k+1}\cos^{2k}\psi}\boldsymbol{F}_{k+1}(\cos\psi,\sin\psi), \tag{22}$$

where $\boldsymbol{F}_k(\cos\psi,\sin\psi)$, $\boldsymbol{F}_{k+1}(\cos\psi,\sin\psi)$ are vectors with components polynomials of the functions $\cos\psi,\sin\psi$ with coefficients functions of the variable u.

On account of (19), (20), (21), (22) and (4), we conclude that

$$\begin{aligned}
&(-1)^k\frac{3^k\sin^{2k+2}\psi}{(2r)^{k+1}(\delta\cos\psi)^{2k+1}}\boldsymbol{h} + \frac{1}{\delta^{2k+1}\cos^{2k}\psi}\boldsymbol{F}_{k+1}(\cos\psi,\sin\psi)\\
&+(-1)^{k-1}\rho_1\frac{3^{k-1}\sin^{2k}\psi}{(2r)^k(\delta\cos\psi)^{2k-1}}\boldsymbol{h} + \rho_1\frac{1}{\delta^{2k-1}\cos^{2k-2}\psi}\boldsymbol{F}_k(\cos\psi,\sin\psi) + \cdots - \rho_{k-1}\frac{3\sin^4\psi}{4r^2\delta^3\cos^3\psi}\boldsymbol{h}\\
&+\rho_{k-1}\frac{1}{\delta^3\cos^2\psi}\boldsymbol{F}_2(\cos\psi,\sin\psi) + \rho_k\frac{\sin^2\psi}{2r\delta\cos\psi}\boldsymbol{h} + \rho_k\frac{1}{\delta}\boldsymbol{F}_1(\cos\psi,\sin\psi) = \boldsymbol{0}.
\end{aligned} \tag{23}$$

For simplicity, equation (23) can be written as follows

$$\frac{3^k\sin^{2k+2}\psi}{(2r)^{k+1}\cos\psi}\boldsymbol{h} = Q_1(\cos\psi,\sin\psi)\boldsymbol{h} + Q_2(\cos\psi,\sin\psi)\boldsymbol{b}, \tag{24}$$

where $Q_i(\cos\psi,\sin\psi)$, $i=1,2$ are polynomials in $\cos\psi,\sin\psi$ with coefficients functions of the variable u. Therefore, we find that

$$\frac{3^k\sin^{2k+2}\psi}{(2r)^{k+1}\cos\psi} = Q_1(\cos\psi,\sin\psi). \tag{25}$$

This is impossible, since the right side hand Q_1 is a polynomial in $\cos\psi,\sin\psi$ while the left side hand is not. Thus, our theorem is proved.

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Received January 15, 2021; final form February 26, 2021.