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Some Properties of the García Reflection Triangles

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Abstract. We prove some results related to the two García reflection triangles of a given triangle.

Key Words: triangle geometry, triangle center, barycentric coordinates, García reflection triangle, perspective

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1 Introduction

The *excentral triangle*, also called the *tritangent triangle*, of a reference triangle ABC is the triangle with vertices corresponding to the excenters of ABC. Denote with D, E, F the $A-, B-, C-$ excenters, respectively (Figure 1). The excentral triangle is the anticevian triangle with respect to the incenter I of ABC [2]. The circumcircle of the excentral triangle is called the *Bevan circle*.

The lengths of sides of reference triangle ABC will be noted by $a = |BC|$, $b = |CA|$, $c = |AB|$. We denote with Δ the area of triangle ABC, with S the double of its area ($S = 2\Delta$), with s the semiperimeter of ABC ($2s = a + b + c$).
It is well-known that the incenter I of the reference triangle ABC is the orthocenter of the excentral triangle DEF and its circumcenter O_e is the point of concurrence of the perpendiculars from the excenters D, E, F to the sides of ABC (see [2, p. 73]). If P', Q', R' denote the touchpoints of the excircles of triangle ABC with its sides BC, CA, AB respectively, then $Oe = DP' \cap EQ' \cap FR'$. The triangle $P'Q'R'$ is called the *extouch triangle* of ABC (Figure 1).
Henceforth we will use the Conway triangle notations:

$$S_A = bc \cdot \cos A = \frac{1}{2}(-a^2 + b^2 + c^2), \quad S_B = ca \cdot \cos B = \frac{1}{2}(a^2 - b^2 + c^2),$$
$$S_C = ab \cdot \cos C = \frac{1}{2}(a^2 + b^2 - c^2), \quad S_\omega = \frac{1}{2}(a^2 + b^2 + c^2),$$

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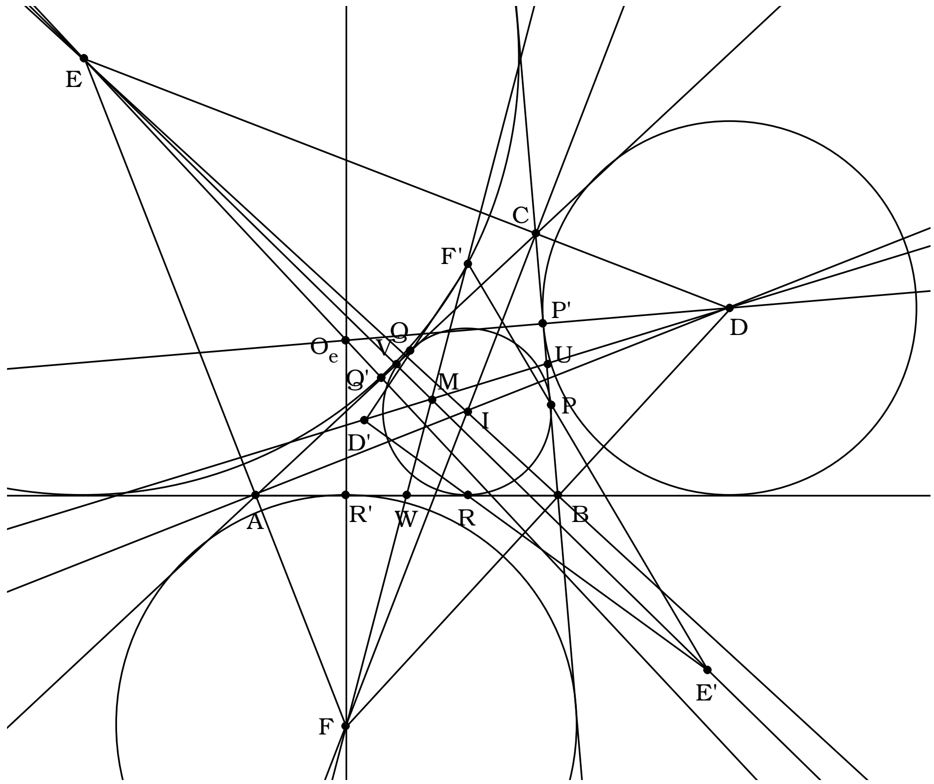

Figure 1: Reflections of the points D, E, F about midpoints U, V, W of the sides BC, CA, AB, respectively

where ω is the Brocard angle.

The normalized barycentric coordinates of the midpoints U, V, W of sides BC, CA, AB and of vertices of the excentral triangle DEF are well-known:

$$U = \frac{1}{2}(0,1,1), \quad V = \frac{1}{2}(1,0,1), \quad W = \frac{1}{2}(1,1,0);$$
$$D = \frac{1}{2}\left(-\frac{a}{s-a}, \frac{b}{s-a}, \frac{c}{s-a}\right), \quad E = \frac{1}{2}\left(\frac{a}{s-b}, -\frac{b}{s-b}, \frac{c}{s-b}\right),$$
$$F = \frac{1}{2}\left(\frac{a}{s-c}, \frac{b}{s-c}, -\frac{c}{s-c}\right).$$

Let D', E', F' be the reflections of the points D, E, F with respect to the midpoints U, V, W of sides BC, CA, AB respectively (Figure 1):

$$D' = 2U - D = \frac{1}{2}\left(\frac{a}{s-a}, \frac{c-a}{s-a}, \frac{b-a}{s-a}\right),$$
$$E' = 2V - E = \frac{1}{2}\left(\frac{c-b}{s-b}, \frac{b}{s-b}, \frac{a-b}{s-b}\right),$$
$$F' = 2W - F = \frac{1}{2}\left(\frac{b-c}{s-c}, \frac{a-c}{s-c}, \frac{c}{s-c}\right).$$

The triangle $D'E'F'$ is named the *García reflection triangle of the excentral triangle* DEF [3].

2 García Reflection Triangle

Let P, Q, R be the touchpoints of the incircle with the sides BC, CA, AB respectively. The triangle PQR is called the *intouch triangle* of ABC (Figure 1). Barycentric coordinates of the points P, Q, R are $P = 0 : s-c : s-b$, $Q = s-c : 0 : s-a$, $R = s-b : s-a : 0$.

Proposition 1. *The points D', E', F' are the orthocenters of the triangles BCI, CAI, ABI respectively, where I is the incenter of the reference triangle ABC.*

Proof. The equation of the line perpendicular to the line $lx + my + nz = 0$ which passes by the point $u : v : w$ is

$$\begin{vmatrix} x & y & z \\ u & v & w \\ \mu_a & \mu_b & \mu_c \end{vmatrix} = 0,$$

where $\mu_a = a^2 l - S_C m - S_B n$, $\mu_b = -S_C l + b^2 m - S_A n$, $\mu_c = -S_B l - S_A m + c^2 n$. Denote with the symbol L^M_{PQ} the perpendicular from M to the line PQ. Now we write the equation of the altitude L^C_{BI}. Since the equation of BI is $cx - az = 0$, the equation of this altitude will be

$$\begin{vmatrix} x & y & z \\ 0 & 0 & 1 \\ \mu_a & \mu_b & \mu_c \end{vmatrix} = 0 \iff \mu_b x - \mu_a y = 0,$$

where $\mu_a = a^2 c + aS_B = a(ca + S_B)$ and $\mu_b = -cS_C + aS_A = (c-a)(ca + S_B)$. So, the equation of L^C_{BI} is $(c-a)(ca+S_B)x - a(ca+S_B)y = 0 \iff (c-a)x - ay = 0$. Similarly, we obtain the equation of the altitude $L^B_{CI} : (b-a)x - az = 0$. Since $D' \in L^C_{BI}$ and $D' \in L^B_{CI}$, it follows that the point D' is the orthocenter of triangle BIC. □

Remark 1. From Proposition 1 we see that the triplets of points (I, P, D'), (I, Q, E'), (I, R, F') are collinear, i.e. the triangles $D'E'F'$ and PQR are perspective. Their perspector is the incenter I of the triangle ABC.

Proposition 2. *The points in each of the sets $\{E', P, F'\}$, $\{F', Q, D'\}$, $\{D', R, E'\}$ are collinear.*

Proof. For example, the points E', P, F' are collinear if and only if

$$\begin{vmatrix} a & c-a & b-a \\ s-b & s-a & 0 \\ c-b & b & a-b \end{vmatrix} = 0 \iff \begin{vmatrix} a & c-a & b-a \\ s-b & s-a & 0 \\ s-b & s-a & 0 \end{vmatrix} = 0,$$

which is true, since two lines of determinant are equal. □

Proposition 3. *The lines DD', EE', FF' are concurrent in the Mittenpunkt $X(9)$ of ABC, which is the symmedian point of the excentral triangle DEF.*

Proof. We write the equation of the line DD' (Figure 1):

$$\begin{vmatrix} x & y & z \\ -a & b & c \\ a & c-a & b-a \end{vmatrix} = 0 \iff \begin{vmatrix} x & y & z \\ -a & b & c \\ 0 & 1 & 1 \end{vmatrix} = 0 \iff (b-c)x + ay - az = 0.$$

Similarly, we obtain equations for EE' and FF':

$$-bx + (c-a)y + bz = 0 \quad \text{and} \quad cx - cy + (a-b)z = 0$$

Each equation is a linear combination of other two, so that the lines DD', EE', FF' are concurrent. It is easy to verify that the barycentric coordinates of the Mittenpunkt

$$M = X(9) = a(s-a) : b(s-b) : c(s-c)$$

satisfy the equations of the lines DD', EE', FF'. □

Remark 2. From Proposition 3 it follows that each pairs of triangles

$$\{DEF, UVW\}, \quad \{D'E'F', UVW\} \quad \text{and} \quad \{DEF, D'E'F'\}$$

are perspectives and their perspector is the Mittenpunkt of reference triangle ABC.

We introduce the notation $\Delta(T)$ for the area of a triangle T.

Proposition 4. *The García reflection triangle $D'E'F'$ has the same area as the reference triangle ABC*

Proof. We apply the area formula for the triangle $D'E'F'$:

$$\begin{aligned}
\Delta(D'E'F') &= \frac{\Delta}{8(s-a)(s-b)(s-c)}\begin{vmatrix} a & c-a & b-a \\ c-b & b & a-b \\ b-c & a-c & c \end{vmatrix} = \frac{s}{8\Delta}\begin{vmatrix} a & b & c \\ c-b & b & a-b \\ b-c & a-c & c \end{vmatrix} \\
&= \frac{s}{8\Delta}\begin{vmatrix} a & b & c \\ -2(s-c) & 0 & -2(s-a) \\ -2(s-b) & -2(s-a) & 0 \end{vmatrix} = \frac{s}{2\Delta}\begin{vmatrix} a & b & c \\ s-c & 0 & s-a \\ s-b & s-a & 0 \end{vmatrix} \\
&= \frac{s}{2\Delta}[-a(s-a)^2 + b(s-a)(s-b) + c(s-a)(s-c)] \\
&= \frac{s(s-a)}{\Delta}[s(s-a) - S_A] = \frac{s(s-a)(s-b)(s-c)}{\Delta} = \frac{\Delta^2}{\Delta} = \Delta = \Delta(ABC). \quad \square
\end{aligned}$$

Proposition 5. *The area of the García reflection triangle $D'E'F'$ is the geometric mean of the area of intouch triangle PQR and the area of excentral triangle DEF of ABC.*

Proof. Using the absolute barycentric coordinates of the touchpoints

$$P = \left(0, \frac{s-c}{a}, \frac{s-b}{a}\right); \quad Q = \left(\frac{s-c}{b}, 0, \frac{s-a}{b}\right); \quad R = \left(\frac{s-b}{c}, \frac{s-a}{c}, 0\right)$$

we calculate the area of PQR:

$$\begin{aligned}
\Delta(PQR) &= \frac{\Delta}{abc}\begin{vmatrix} 0 & s-c & s-b \\ s-c & 0 & s-a \\ s-b & s-a & 0 \end{vmatrix} \\
&= \frac{2\Delta(s-a)(s-b)(s-c)}{abc} = \frac{(s-a)(s-b)(s-c)}{2R} = \frac{r}{2R}\Delta,
\end{aligned}$$

where r is the inradius, R the circumradius of the triangle ABC. Applying the area formula for the excentral triangle DEF we obtain:

$$\Delta(DEF) = \frac{\Delta}{8(s-a)(s-b)(s-c)} \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} = \frac{abc \cdot s}{2\Delta} = 2R \cdot s = \frac{2R}{r}\Delta.$$

Since $\Delta(PQR) \cdot \Delta(DEF) = \dfrac{r}{2R}\Delta \cdot \dfrac{2R}{r}\Delta = \Delta^2$, from Proposition 4, we have

$$\Delta(PQR) \cdot \Delta(DEF) = \Delta(D'E'F')^2. \qquad \square$$

Denote with G' the centroid of triangle $D'E'F'$. Let G_e be the centroid of the excentral triangle DEF, and G the centroid of ABC.

Proposition 6. *The centroids G, G' and G_e are collinear and G is the midpoint of segment $G'G_e$.*

Proof. We will prove that $G' + G_e = 2G$. Indeed:

$$\begin{aligned} G' + G_e &= \frac{1}{3}\left(D' + E' + F' + D + E + F\right) \\ &= \frac{1}{6}\Bigg[\left(\frac{a}{s-a}, \frac{c-a}{s-a}, \frac{b-a}{s-a}\right) + \left(\frac{c-b}{s-b}, \frac{b}{s-b}, \frac{a-b}{s-b}\right) + \left(\frac{b-c}{s-c}, \frac{a-c}{s-c}, \frac{c}{s-c}\right) \\ &\quad + \left(-\frac{a}{s-a}, \frac{b}{s-a}, \frac{c}{s-a}\right) + \left(\frac{a}{s-b}, -\frac{b}{s-b}, \frac{c}{s-b}\right) + \left(\frac{a}{s-c}, \frac{b}{s-c}, -\frac{c}{s-c}\right)\Bigg] \\ &= \frac{1}{6}\left(\frac{a-b+c}{s-b} + \frac{a+b-c}{s-c}, \frac{a+b-c}{s-c} + \frac{-a+b+c}{s-a}, \frac{-a+b+c}{s-a} + \frac{a-b+c}{s-b}\right) \\ &= \frac{1}{6}\left(4, 4, 4\right) = 2G. \end{aligned} \qquad \square$$

3 Reflection Triangle of the Incenter

Let D'', E'', F'' be the reflections of the incenter I of ABC with respect to the midpoints U, V, W of sides BC, CA, AB, respectively (Figure 2):

$$D'' = 2U - I = \frac{1}{2}\left(-\frac{a}{s}, \frac{c+a}{s}, \frac{b+a}{s}\right),$$

$$E'' = 2V - I = \frac{1}{2}\left(\frac{c+b}{s}, -\frac{b}{s}, \frac{a+b}{s}\right), \quad F'' = 2W - I = \frac{1}{2}\left(\frac{b+c}{s}, \frac{a+c}{s}, -\frac{c}{s}\right).$$

The triangle $D''E''F''$ is called the *outer García triangle of the incenter I*.

Proposition 7. *The points D'', E'', F'' are the orthocenters of the triangles BCD, CAE, ABF respectively.*

Proof. A proof is similar to that of the demonstration of Proposition 1. $\square$

Remark 3. The triangle $D''E''F''$ is congruent and homothetic to the reference triangle ABC. Their homothetic center is the Spieker center $X(10)$ of the triangle ABC. (See [1] Proposition 1).

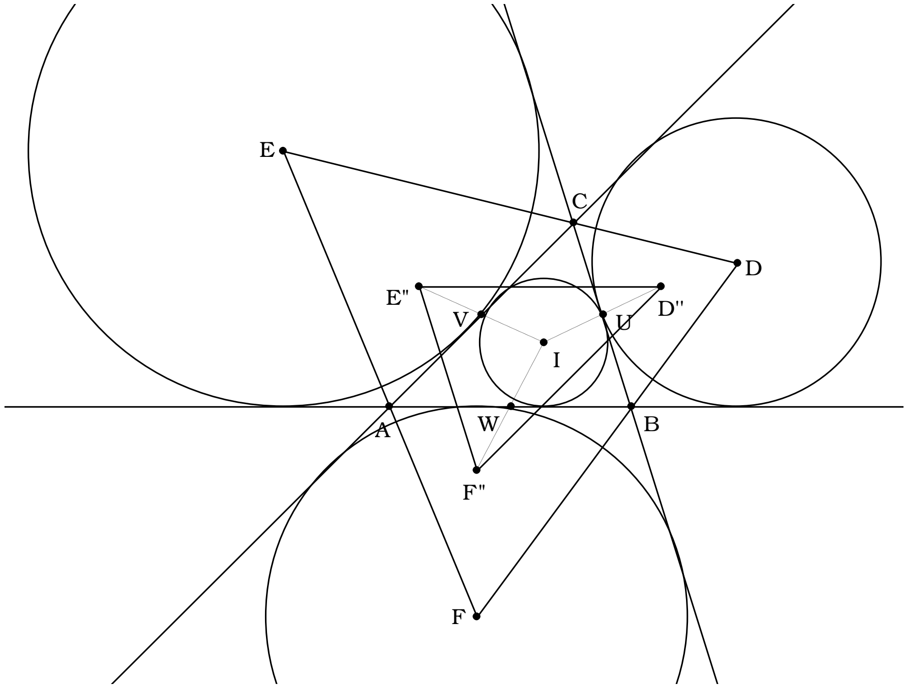

Figure 2: Reflections of incenter I about midpoints U, V, W of the sides BC, CA, AB, respectively.

The Spieker center (Sp) of triangle ABC is the incenter of the medial triangle UVW with barycentric coordinates:

$$Sp = b+c : c+a : a+b = \left(\frac{b+c}{4s}, \frac{c+a}{4s}, \frac{a+b}{4s}\right) = \frac{1}{2}(3G - I).$$

Each set $\{A, Sp, D''\}$, $\{B, Sp, E''\}$, $\{C, Sp, F''\}$ are collinear (Figure 3). Sp is the midpoint of segments AD'', BE'', CF'', so the triangle $D''E''F''$ is the reflection of the triangle ABC with respect to Sp. Therefore, the centers of $D''E''F''$ we get from the centers of ABC by the geometric transformation $f(T) = 2Sp - T$, where T is an arbitrary point in the plane of ABC (the points Sp and T are given by their normalized barycentric coordinates). For example, the orthocenter of $D''E''F''$ is the transformation by f of the orthocenter H of ABC:

$$\begin{aligned} H'' &= f(H) = 2Sp - H = 3G - I - H = 2O + H - I - H = 2O - I \\ &= \left(\frac{a^2S_A}{S^2}, \frac{b^2S_B}{S^2}, \frac{c^2S_C}{S^2}\right) - \left(\frac{a}{2s}, \frac{b}{2s}, \frac{c}{2s}\right) = 2sa^2S_A - aS^2 : 2sb^2S_B - bS^2 : 2sc^2S_C - cS^2. \end{aligned}$$

Proposition 8. *The García reflection triangle $D'E'F'$ and the outer García triangle $D''E''F''$ are perspective. Their perspector is the Nagel point $X(8)$ of the triangle ABC.*

Proof. The equation of the line $D'D''$ is

$$\begin{vmatrix} x & y & z \\ a & c-a & b-a \\ -a & c+a & b+a \end{vmatrix} = 0 \iff \begin{vmatrix} x & y & z \\ a & c-a & b-a \\ 0 & c & b \end{vmatrix} = 0 \iff (b-c)x + by - cz = 0.$$

Similarly, we obtain equations of $E'E''$ and $F'F''$: $-ax + (c-a)y + cz = 0$ and $ax - by + (a-b)z = 0$ (Figure 3). It is easy to verify that the barycentric coordinates of the Nagel point $N = X(8) = (s-a : s-b : s-c)$ satisfy the equations of these lines. □

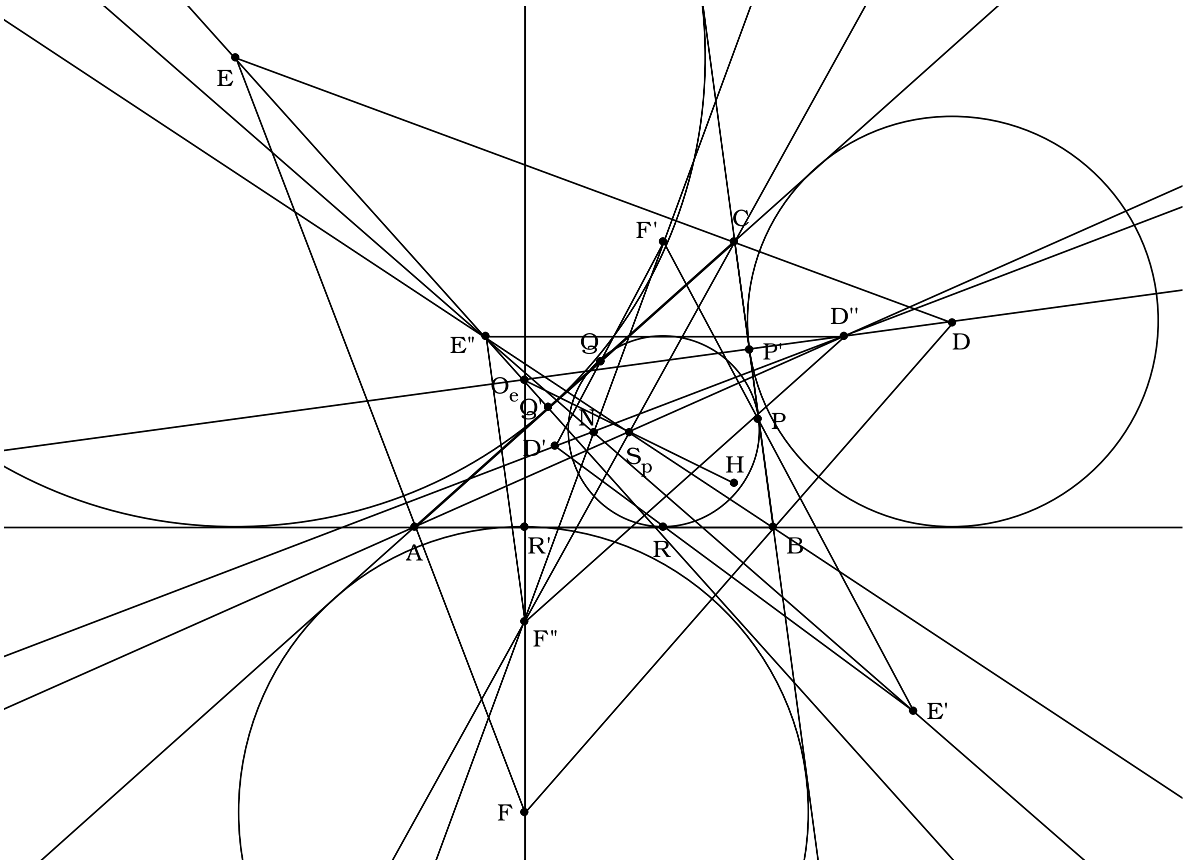

Figure 3: Perspectivity of the García reflection triangles $D'E'F'$ and $D''E''F''$

Barycentric coordinates of the points P', Q', R' are $P' = 0 : s-b : s-c$, $Q' = s-a : 0 : s-c$, $R' = s-a : s-b : 0$.

Proposition 9. *The pairs of triangles* $\{DEF, P'Q'R'\}$, $\{D''E''F'', P'Q'R'\}$, $\{DEF, D''E''F''\}$ *are perspective. The perspectors in all three cases is the orthocenter* H'' *of the outer García triangle* $D''E''F''$.

Proof. The sets of points $\{D, D'', P'\}$, $\{E, E'', Q'\}$, $\{F, F'', R'\}$ are collinear. For example, the points D, D'', P' are collinear if and only if

$$\begin{vmatrix} -a & b & c \\ -a & c+a & b+a \\ 0 & s-b & s-c \end{vmatrix} = 0 \iff \begin{vmatrix} -a & b & c \\ 0 & s-b & s-c \\ 0 & s-b & s-c \end{vmatrix} = 0,$$

which is true. But, from Proposition 7, we have that $DD'' \perp BC$, and from Remark 3 we have $BC \| E''F''$. Consequently $DD'' \perp E''F''$, so the line DD'' is altitude in the triangle $D''E''F''$ (Figure 3). Similarly, the lines EE'' and FF'' are altitudes of $D''E''F''$. Denote with H'' the orthocenter of triangle $D''E''F''$, so that $H'' = DD'' \cap EE'' \cap FF''$. □

Proposition 10. *The circumcenter* O_e *of the excentral triangle* DEF *(i.e. the Bevan point* $X(40)$ *of* ABC*) coincide with the orthocenter* H'' *of triangle* $D''E''F''$*. The points* H, Sp, O_e *are collinear and the Spieker center* Sp *is the midpoint of segment* HO_e.

Proof. The circumcenter O of ABC is however the center of the Feuerbach circle of the excentral triangle DEF and the Bevan point $X(40) \equiv O_e$ is the reflection of the incenter I

in the circumcenter O of ABC (Figure 3). Consequently, the circumcenter O is the midpoint of segment IO_e, therefore $O_e = 2O - I = H''$. Since $2O + H = 3G = 2Sp + I$, result that

$$2Sp = H + 2O - I = H + O_e. \qquad \square$$

Remark 4. Each of these pairs of triangles

$$\{ABC, D'E'F'\}, \quad \{D'E'F', P'Q'F'\} \quad \text{and} \quad \{PQR, D''E''F''\}$$

are not perspective.

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