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From M. C. Escher's Hexagonal Tiling to the Kiepert Hyperbola

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Abstract. Generalizing Fermat and Napoleon points of a triangle, we introduce the notion of complementary Jacobi points, showing their collinearity with the circumcenter of the given triangle. The coincidence of the associated perspective lines for complementary Jacobi points is also proved, together with the orthogonality of this line with the one joining the circumcenter and the Jacobi points. Involutions on the Kiepert hyperbola naturally arise, allowing a geometric insight on the relationship between Jacobi points, their associated perspective lines and Kiepert conics of a triangle.

Key Words: triangle geometry, M. C. Escher, Napoleon Point, Fermat Point, Kiepert hyperbola

MSC 2020: 51M04 (primary), 00A66

1 A Historical Introduction: Escher, Napoleon, Fermat

The theme of plane tessellations plays a role of outstanding relevance in the artistic production of the Dutch graphic artist M. C. Escher. It is perhaps since his second journey to the Alhambra in May/June 1936 that the idea of regularly organizing both Euclidean and hyperbolic plane surfaces by geometric tiles becomes an organic working plan. In 1937, Escher got from his stepbrother Beer, geologist at the university of Leiden, an article [3] by the German crystallographer F. Haag describing the properties of a particular hexagon built from an equilateral triangle and an arbitrary point on the plane. Escher sketched Haag's construction in one of his notebooks [5]. Figure 1 reproduces the original picture (on the right) and the related text (on the left), which can be translated from Dutch in the following way.

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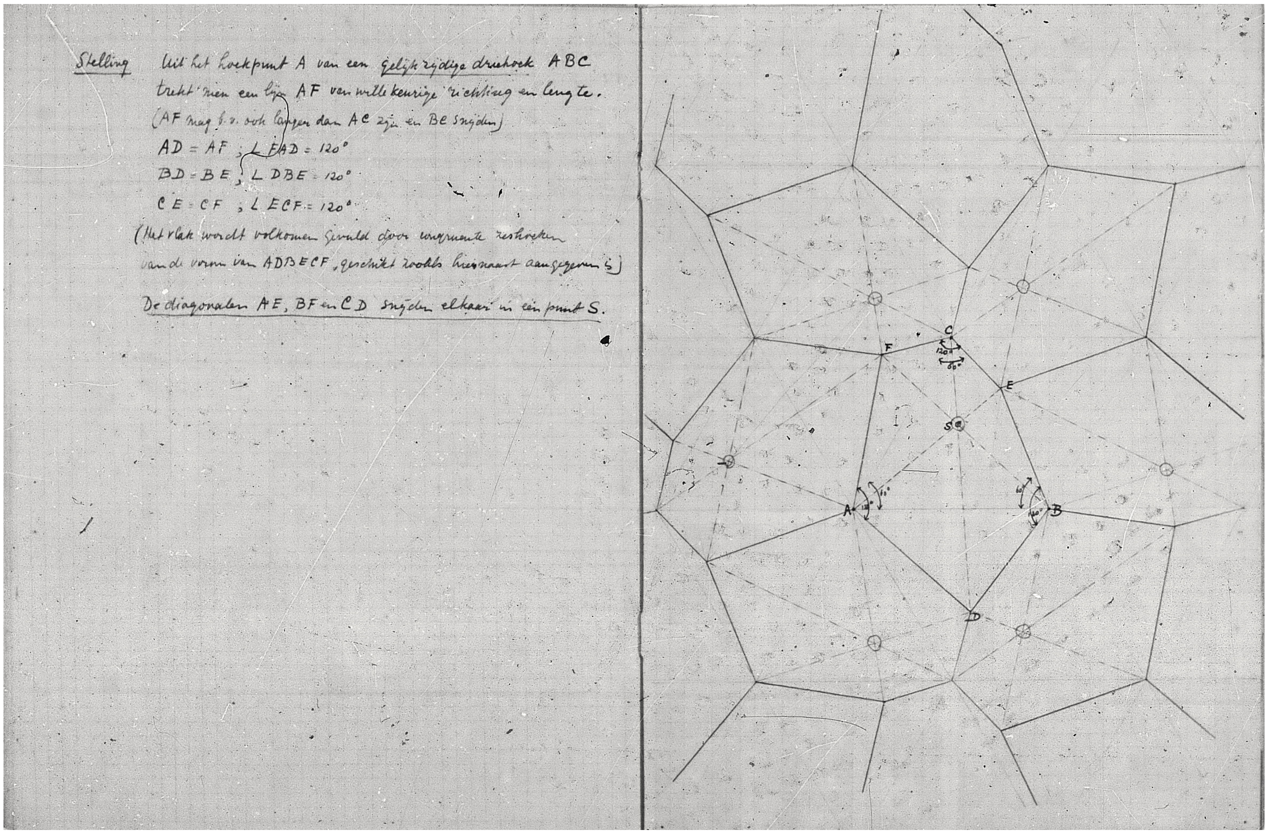

Figure 1: Escher's sketchpad (© M. C. Escher Company, Baarn NL)

From the vertex A of an equilateral triangle ABC, one draws a line AF of arbitrary length and direction. *(AF could, for example, be longer than AC and intersect BC.)*

$$AD = AF \quad \text{and} \quad \angle FAD = 120^\circ$$
$$BD = BE \quad \text{and} \quad \angle DBE = 120^\circ$$
$$CE = CF \quad \text{and} \quad \angle ECF = 120^\circ$$

(The plane is completely filled by congruent hexagons of the form $ADBECF$ arranged as in picture). The diagonals AE, BF and CD intersect in a single point S.

Haag thus proved that the construction gives rise to a hexagon $ADBECF$ and that this hexagon, precisely because it is built on a lattice of equilateral triangles, produces a monohedral tessellation of the Euclidean plane.

Studying Haag's hexagons, which we will later call *Escher's hexagons*, Escher noticed that the diagonals joining opposite pairs of vertices are concurrent in one point. Escher stated his result in the formal manner of mathematicians, that is in the form of a theorem: In fact, in stating it, he uses the term "stelling", the Dutch word to define the thesis of a theorem. Escher's statement is correct, even if he only verified it graphically, testing his thesis on different hexagons built in this way. Escher's son George had an engineering degree and had an extensive correspondence with his father on the mathematical questions surrounding planar tessellations. Asked for a proof of this "theorem", George pointed out that the hexagons could also be constructed in a different way. In fact, the centroids of the equilateral triangles externally erected on the sides of any given triangle DEF are the points A, B, C respectively, since they are the 120° vertices in the isosceles triangles built on these sides. Moreover, they are themselves vertices of an equilateral triangle (Figure 1): this result is known in geometry as Napoleon's theorem, since tradition attributes it to Napoleon Bonaparte, who seems to have been a talented mathematical amateur.

It is not known whether Escher understood that his construction was actually linked to this result. However, in 1942 he painted a watercolour in which, in addition to a multihedral hexagonal tessellation of the plane, the equilateral triangles of Napoleon's theorem are clearly recognizable (Figure 2).

The concurrency of the diagonals of a hexagon obtained by building similar isosceles triangles on the sides of an arbitrary given triangle is known as the Kiepert's theorem; a proof based on Ceva's Theorem, can be found in [1]. A generalization of this result, obtained by isogonal lines[1] from the vertices, was seemingly discovered and published in 1825 by Carl Friedrich Andreas Jacobi (see Section 2).

Later the theorem, in the special case described by Escher, was also proved by J. F. Rigby in 1973 [4], by making use of rotation and translation symmetries of Escher's tessellation. This argument, based on the hexagon property of tessellating the plane, would certainly have been liked by Escher.

The concurrent point in the general case of the Jacobi's construction is called the *Jacobi point* of the given triangle. In the special case of Escher's hexagon, where the base angles of the isosceles triangles are $\pi/6$, the point is often called in literature the *Napoleon point* E (X_{17}) of the triangle.[2]

Another notable hexagon is obtained when the angles at the base of the isosceles triangles are $\pi/3$, and therefore, the triangles become equilateral. In this case, if the given triangle does not contain angles greater than $2\pi/3$, the point of concurrency of the diagonals is the point of minimum distance from the vertices of the triangle. The point is known as the *Fermat point* F (X_{13}), as it was Pierre de Fermat who identified it in response to a question posed by Evangelista Torricelli, giving rise to a close correspondence between the two great mathematicians of the XVI century.

Drawing the two hexagons, starting from the same triangle, the collinearity of the corresponding Napoleon and Fermat points with its circumcenter catches the eye.

It is worth pointing out that the construction of the isosceles triangles may be also performed internally to the sides of the given triangle or, equivalently, by making use of negative base angles. This gives rise to the *second isogonic center* (X_{14}), also called *second Fermat point,* and to the *second Napoleon point* (X_{18}).

2 Centered hexagons and Jacobi points

Escher and Fermat hexagons, together with their associated points, are special cases of a general situation given by "centered" hexagons. A hexagon $AC'BA'CB'$ is said to be *centered* if its opposite diagonals AA', BB', CC' (joining opposite pairs of vertices) are concurrent at a point P. As a direct consequence of Desargues's theorem applied to the triangles ABC

[1] Isogonal lines through a triangle vertex are obtained by reflecting an initial line about the corresponding angle bisector.

[2] In order to identify central points of a triangle. $\mathcal{T} = ABC$, we shall recall their notations, as listed by Clark Kimberling in [8]; for example: Incenter $I = X_1$, Centroid $G = X_2$, Circumcenter $O = X_3$, Ortocenter $H = X_4$.

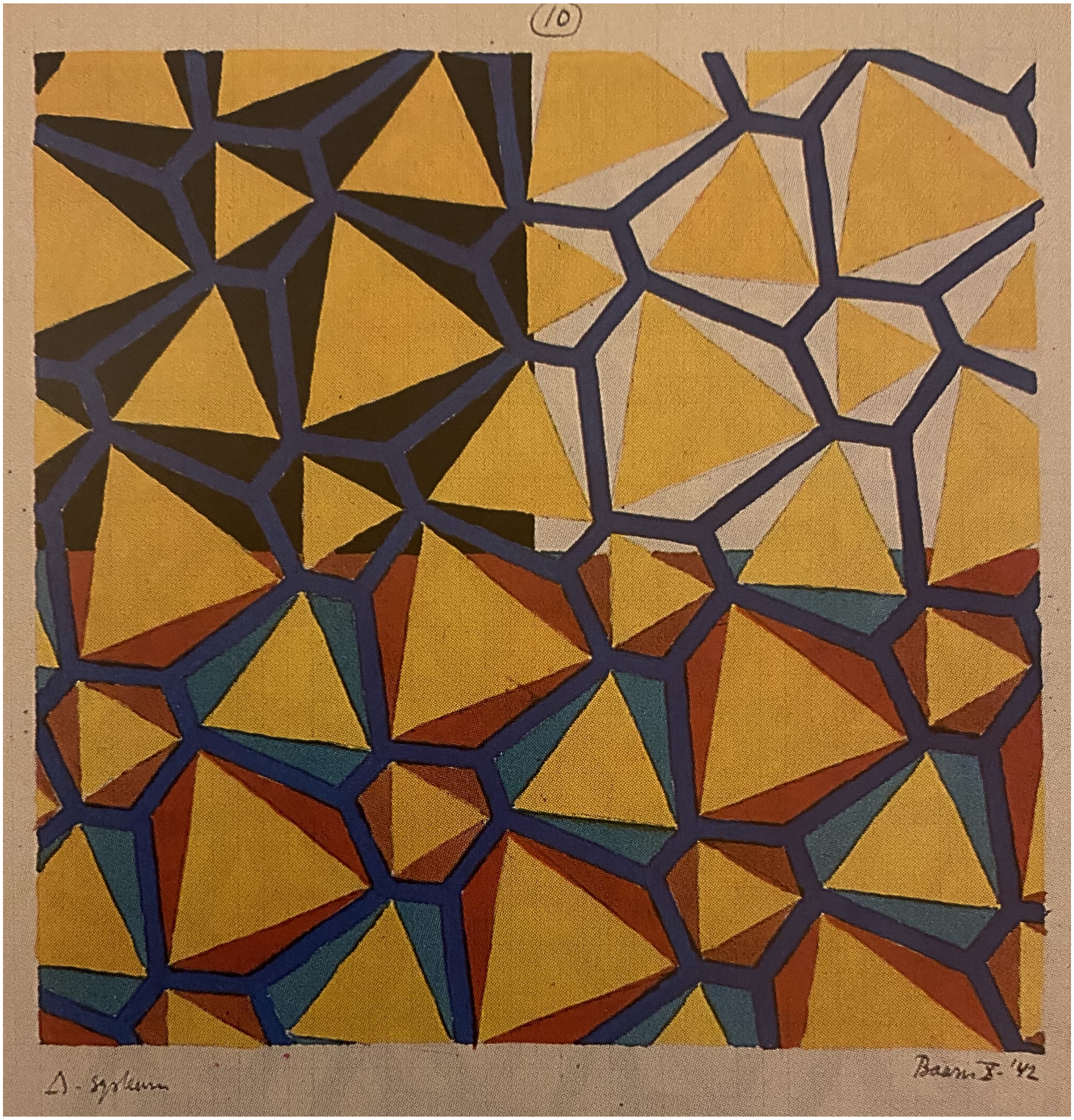

Figure 2: Escher's watercolor Nr 10 (© M. C. Escher Company, Baarn NL) with tessellating motive based on Haag's hexagon

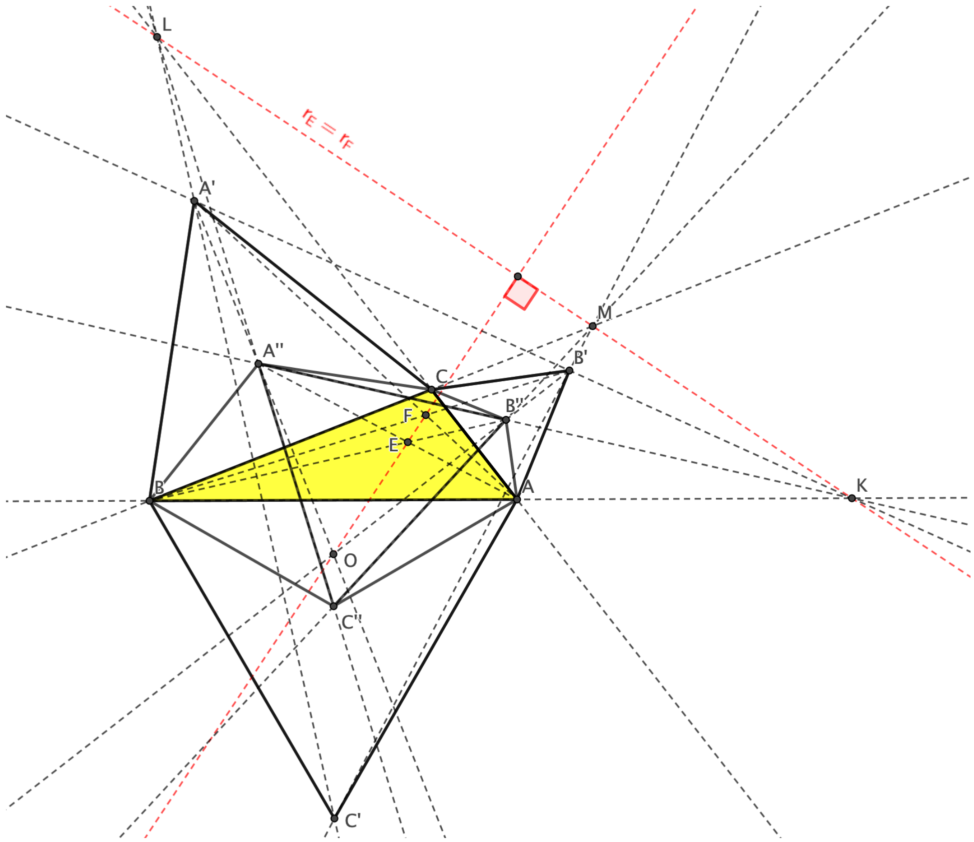

Figure 3: Circumcenter O, Napoleon point E and Fermat point F of the base triangle ABC are collinear and this line is orthogonal to the perspectrix $r_E = r_F$

and $A'B'C'$, the intersection points $K = AB \cap A'B'$, $L = AC \cap A'C'$, $M = BC \cap B'C'$ of the "twin" lines are collinear in the perspectrix r_P of P, which we will later refer to as its *Desargues axis.*

The following statement, regarding the special case of Escher's and Fermat's hexagons (Figure 3), is a collection of more general results which will be proved in the next sections.

Theorem 1. *Let $\mathcal{T}$ be a triangle. Then,*

(a) The circumcenter O, the Napoleon point E, and the Fermat point F of $\mathcal{T}$ are collinear.

(b) The Desargues axis r_E of E coincides with the Desargues axis r_F of F.

(c) The line OEF is orthogonal to the Desargues axis $r_E = r_F$.

Since the line $r_E = r_F$ arises from Escher's and Fermat's hexagons, we call it *Escher-Fermat line* of $\mathcal{T}$. Of course, an analogous theorem may be stated for the second Napoleon and Fermat points, giving rise to the *second Escher-Fermat line* of $\mathcal{T}$.

Centered hexagons can be obtained by starting from a given triangle $\mathcal{T} = ABC$ and considering isogonal lines arising from its vertices A, B, C with angles α, β, γ respectively. In fact, the intersection points A', B', C' of the isogonal lines produce the hexagon $AC'BA'CB'$ and Jacobi's theorem states that the lines AA', BB', CC' are concurrent at a point $J_{\alpha,\beta,\gamma}$, the *Jacobi* $\{\alpha,\beta,\gamma\}$*-point* of $\mathcal{T}$. Among other proofs of Jacobi's theorem, the one given in [7] is based on Ceva's theorem and allows us to obtain the areal coordinates of $J_{\alpha,\beta,\gamma}$[3]

$$\left(\frac{1}{\cot A + \cot\alpha}, \frac{1}{\cot B + \cot\beta}, \frac{1}{\cot C + \cot\gamma}\right)$$

[3]For the sake of conciseness, we identify each angle of $\mathcal{T}$ with the name of the corresponding vertex.

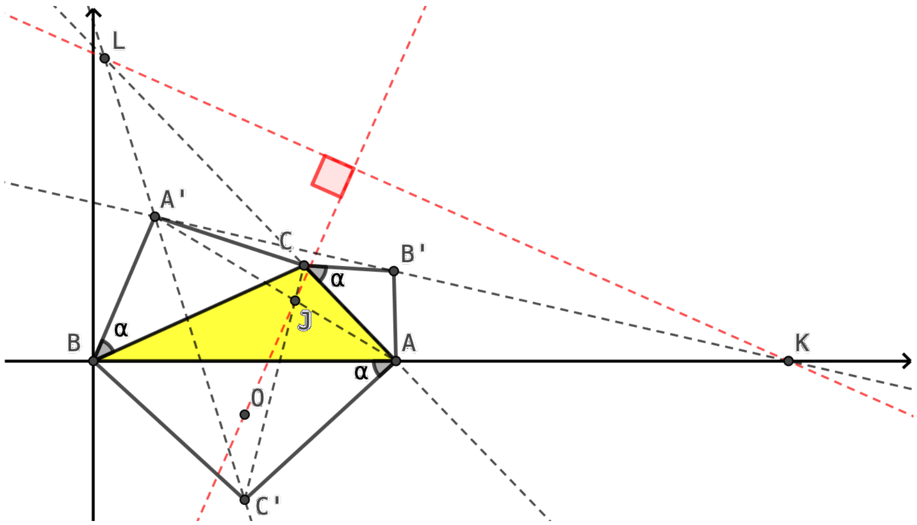

Figure 4: Jacobi's construction in Cartesian coordinates

If, in particular, we set $\alpha = \beta = \gamma$, then $A'BC$, $B'AC$, $C'AB$ are similar isosceles triangles with base angle α ($-\pi/2 \leq \alpha \leq \pi/2$): The corresponding point of concurrency J_α is called the *Jacobi α-point* of $\mathcal{T}$[4]. Note that the Jacobi $\pi/6$-point (resp. $\pi/3$-point) of $\mathcal{T}$ is its Napoleon (resp. Fermat) point.

It is also worth noting that the isosceles triangles are erected externally (resp. internally) on the sides of the given triangle if, and only if, α is positive (resp. negative). Given a positive α, we will call J_α (resp. $J_{-\alpha}$) the *first* (resp. *second*) *Jacobi α-point* of $\mathcal{T}$, in analogy with the terminology used for Fermat or Napoleon first and second point.

Theorem 2. *If $\mathcal{T}$ is a given triangle, O be its circumcenter and J_α be its Jacobi α-point, then the line OJ_α is orthogonal to the Desargues axis r_α of J_α.*

Proof. In the special case in which $\mathcal{T}$ is isosceles[5] (based on AB, say), all Jacobi α-points J_α lie on the median line CO joining C with the circumcenter O. Moreover, the Desargues axis of each J_α are parallel to AB and hence orthogonal to CG.

If $\mathcal{T}$ is non isosceles, the result can be proved by simple, but rather lengthy computations in Cartesian coordinates. In fact, we may suppose that the coordinates of the vertices A, B, C of $\mathcal{T}$ are (Figure 4)

$$A = (a, 0), \quad B = (0, 0), \quad C = (x_C, y_C), \qquad \text{with} \quad a, y_C > 0,\ x_C \geq 0.$$

and the coordinates of its circumcenter O are

$$O = \left(\frac{a}{2}, \frac{x_C^2 + y_C^2 - x_C a}{2y_C}\right).$$

If $k := \tan\alpha$, then the vertices A', B', C' of the isosceles triangles with base angle α are

$$A' = \left(\frac{x_C - y_C k}{2}, \frac{y_C + x_C k}{2}\right), \quad B' = \left(\frac{x_C + y_C k + a}{2}, \frac{-x_C k + y_C + ak}{2}\right), \quad C' = \left(\frac{a}{2}, -\frac{a}{2}k\right).$$

[4] The *Jacobi α-point* is often called the *Kiepert α-perspector* of $\mathcal{T}$ and the triangle $A'B'C'$ its *Kiepert α-triangle.*

[5] The case of an equilateral triangle $\mathcal{T}$ is trivial, since all Jacobi α-points coincide with the centroid G, whose Desargues axis is the line at infinity. From now on, $\mathcal{T}$ is assumed to be non-equilateral.

Since

$$AA'\colon Y = \frac{(y_C + x_C k)(X - a)}{x_C - y_C k - 2a}$$

and

$$CC'\colon Y = \frac{(2y_C + ak)X - a(y_C + x_C k)}{2x_C - a}$$

we can obtain the following coordinates of the Jacobi point $J = J_\alpha = AA' \cap CC'$:

$$x_J = \frac{a[x_C y_C(1+k^2) + x_C^2 k + y_C^2 k + a y_C + a x_C k]}{2x_C^2 k + 2y_C^2 k - 2a x_C k + (3+k^2) a y_C + 2a^2 k},$$

$$y_J = \frac{a[2x_C y_C^2 - 2x_C^3 k^2 + 2a x_C y_C k - a y_C^2 + 3a x_C^2 k^2 - a^2 y_C k - a^2 x_C k^2]}{(2x_C - a)[2x_C^2 k + 2y_C^2 k - 2a x_C k + (3+k^2) a y_C + 2a^2 k]}.$$

In the same way, $K = AB \cap A'B'$:

$$AB\colon Y = 0, \quad A'B'\colon X = \frac{1}{-2x_C k + ak} Y + \frac{2x_C^2 k + 2y_C^2 k + a y_C(1+k^2)}{2k(2x_C - a)},$$

$$x_K = \frac{2x_C^2 k + 2y_C^2 k + a y_C(1+k^2)}{2k(2x_C - a)}, \quad y_K = 0,$$

and $L = AC \cap A'C'$

$$AC\colon X = \frac{x_C - a}{y_C} Y + a, \quad A'C'\colon X = \frac{x_C - y_C k - a}{y_C + x_C k + ak} Y + \frac{a y_C(1-k^2) + 2a x_C k}{2(y_C + x_C k + ak)},$$

$$x_L = \frac{a[2x_C^2 k + 2y_C^2 k - x_C y_C(1+k^2) - 2a x_C k + a y_C(1+k^2)]}{2k(x_C^2 + y_C^2 - a^2)},$$

$$y_L = -\frac{a y_C[(1+k^2) y_C + 2ak]}{2k(x_C^2 + y_C^2 - a^2)}.$$

If $\overrightarrow{LK} \equiv (\overrightarrow{LK}_x, \overrightarrow{LK}_y)$, with $\overrightarrow{LK}_x = x_K - x_L, \overrightarrow{LK}_y = y_K - y_L$, and $\overrightarrow{OJ} \equiv (\overrightarrow{OJ}_x, \overrightarrow{OJ}_y)$, with $\overrightarrow{OJ}_x = x_J - x_O$, $\overrightarrow{OJ}_y = y_J - y_O$, from the coordinates of O, J, L, K we obtain:

$$\overrightarrow{LK}_x = \frac{\overrightarrow{LK}_{xn}}{\overrightarrow{LK}_{xd}}, \quad \overrightarrow{LK}_y = \frac{\overrightarrow{LK}_{yn}}{\overrightarrow{LK}_{yd}}, \quad \overrightarrow{OJ}_x = \frac{\overrightarrow{OJ}_{xn}}{\overrightarrow{OJ}_{xd}}, \quad \overrightarrow{OJ}_y = \frac{\overrightarrow{OJ}_{yn}}{\overrightarrow{OJ}_{yd}},$$

with

$$\begin{aligned}
\overrightarrow{LK}_{xn} &= 2a^3 x_c k - 4a^2 {x_C}^2 k + 3a^2 x_C y_C(1+k^2) + 4a {x_C}^3 k + 4a x_C {y_C}^2 k \\
&\quad - 3a {x_C}^2 y_C(1+k^2) - a {y_C}^3(1+k^2) - 2{x_C}^4 k - 4k {x_C}^2 {y_C}^2 k - 2{y_C}^4 k, \\
\overrightarrow{LK}_{xd} &= 2k(a - 2x_C)(-a^2 + {x_C}^2 + {y_C}^2), \\
\overrightarrow{LK}_{yn} &= a y_C(2ak + y_C(1+k^2)), \\
\overrightarrow{LK}_{yd} &= 2k(-a^2 + {x_C}^2 + {y_C}^2), \\
\overrightarrow{OJ}_{xn} &= -a(a - 2x_C)(2ak + y_C(1+k^2)), \\
\overrightarrow{OJ}_{xd} &= 2(2a^2 k - 2a x_C k + a y_C(3+k^2) + 2{x_C}^2 k + 2{y_C}^2 k), \\
\overrightarrow{OJ}_{yn} &= 2a^3 x_c k - 4a^2 {x_C}^2 k + 3a^2 x_C y_C(1+k^2) + 4a {x_C}^3 k + 4a x_C {y_C}^2 k \\
&\quad - 3a {x_C}^2 y_C(1+k^2) - a {y_C}^3(1+k^2) - 2{x_C}^4 k - 4k {x_C}^2 {y_C}^2 k - 2{y_C}^4 k, \\
\overrightarrow{OJ}_{yd} &= 2y_C(2a^2 k - 2a x_C k + a y_C(3+k^2) + 2{x_C}^2 k + 2{y_C}^2 k).
\end{aligned}$$

Since the orthogonality condition between $\overrightarrow{LK}$ and $\overrightarrow{OJ}$ is

$$\overrightarrow{LK}_x\overrightarrow{OJ}_x+\overrightarrow{LK}_y\overrightarrow{OJ}_y=\frac{\overrightarrow{LK}_{xn}\overrightarrow{OJ}_{xn}}{\overrightarrow{LK}_{xd}\overrightarrow{OJ}_{xd}}+\frac{\overrightarrow{LK}_{yn}\overrightarrow{OJ}_{yn}}{\overrightarrow{LK}_{yd}\overrightarrow{OJ}_{yd}}=0 \tag{1}$$

and

$$\overrightarrow{LK}_{xn}=\overrightarrow{OJ}_{yn},\quad \overrightarrow{OJ}_{yd}=y_C\overrightarrow{OJ}_{xd},\quad \overrightarrow{LK}_{yd}=\frac{\overrightarrow{LK}_{xd}}{(a-2x_C)},\quad \overrightarrow{OJ}_{xn}=-\frac{(a-2x_C)\overrightarrow{LK}_{yn}}{y_C},$$

by substituting into (1), we obtain

$$-\frac{\overrightarrow{OJ}_{yn}\overrightarrow{LK}_{yn}(a-2x_C)}{\overrightarrow{LK}_{xd}\overrightarrow{OJ}_{xd}\,y_C}+\frac{\overrightarrow{LK}_{yn}\overrightarrow{OJ}_{yn}(a-2x_C)}{\overrightarrow{LK}_{xd}\overrightarrow{OJ}_{xd}\,y_C}=0$$

which shows that (1) holds true. □

3 Jacobi complementary points

Let J_α (resp. $J_{\alpha'}$) be a Jacobi α-point (resp. α'-point) of $\mathcal{T}$: J_α and $J_{\alpha'}$ are said to be complementary if α and α' are complementary angles. Since the range of α and α' is the interval $[-\pi/2,\pi/2]$, the complementarity between α and α' means $\alpha+\alpha'=\pi/2$ (resp. $-\pi/2$) if α and α' belongs to $[0,\pi/2]$ (resp. to $[-\pi/2,0]$).

We call the line containing J_α and $J_{\alpha'}$ the *α-line* of $\mathcal{T}$.

Theorem 3. *Let O be the circumcenter of the triangle $\mathcal{T}=ABC$ and let J_α and $J_{\alpha'}$ be a pair of complementary Jacobi points. Then, O, J_α, and $J_{\alpha'}$, are collinear.*

Proof. It is sufficient to proof that the determinant of the matrix M whose rows are the areal coordinates of O, J_α, and $J_{\alpha'}$, equals 0.

By using the areal coordinates of the three points [7], since α and α' are complementary, we have

$$M=\begin{bmatrix}\sin(2A) & \sin(2B) & \sin(2C)\\ \dfrac{1}{\cot A+\cot\ \alpha} & \dfrac{1}{\cot B+\cot\ \alpha} & \dfrac{1}{\cot C+\cot\ \alpha}\\ \dfrac{1}{\cot A+\tan\ \alpha} & \dfrac{1}{\cot B+\tan\ \alpha} & \dfrac{1}{\cot C+\tan\ \alpha}\end{bmatrix}.$$

By applying double-angle formulae in the first row and by expressing cot, tan by sin, cos in the 2nd and 3rd row, we obtain

$$\det M=2\sin\alpha\cos\alpha\sin A\sin B\sin C\det M',$$

and hence

$$M'=\begin{bmatrix}\cos A & \cos B & \cos C\\ \dfrac{1}{\cos A\sin\alpha+\sin A\cos\alpha} & \dfrac{1}{\cos B\sin\alpha+\sin B\cos\alpha} & \dfrac{1}{\cos C\sin\alpha+\sin C\cos\alpha}\\ \dfrac{1}{\cos A\cos\alpha+\sin A\sin\alpha} & \dfrac{1}{\cos B\cos\alpha+\sin B\sin\alpha} & \dfrac{1}{\cos C\cos\alpha+\sin C\sin\alpha}\end{bmatrix}.$$

Therefore O, J_α, and $J_{\alpha'}$ are collinear if, and only if, $\det M' = 0$.

Note that the three denominators of the 2nd row can be expressed as $\sin(A+\alpha)$, $\sin(B+\alpha)$, $\sin(C+\alpha)$, while those of the 3rd row as $\cos(A-\alpha)$, $\cos(B-\alpha)$, $\cos(C-\alpha)$. Therefore, we have

$$M' = \begin{bmatrix} \cos A & \cos B & \cos C \\ \dfrac{1}{\sin(A+\alpha)} & \dfrac{1}{\sin(B+\alpha)} & \dfrac{1}{\sin(C+\alpha)} \\ \dfrac{1}{\cos(A-\alpha)} & \dfrac{1}{\cos(B-\alpha)} & \dfrac{1}{\cos(C-\alpha)} \end{bmatrix}$$

with $\det M' = \cos A \det M'_{11} + \cos B \det M'_{12} + \cos C \det M'_{13}$, where

$$M'_{11} = \begin{bmatrix} \dfrac{1}{\sin(B+\alpha)} & \dfrac{1}{\sin(C+\alpha)} \\ \dfrac{1}{\cos(B-\alpha)} & \dfrac{1}{\cos(C-\alpha)} \end{bmatrix}, \quad M'_{12} = \begin{bmatrix} \dfrac{1}{\sin(C+\alpha)} & \dfrac{1}{\sin(A+\alpha)} \\ \dfrac{1}{\cos(C-\alpha)} & \dfrac{1}{\cos(A-\alpha)} \end{bmatrix},$$

$$M'_{13} = \begin{bmatrix} \dfrac{1}{\sin(A+\alpha)} & \dfrac{1}{\sin(B+\alpha)} \\ \dfrac{1}{\cos(A-\alpha)} & \dfrac{1}{\cos(B-\alpha)} \end{bmatrix}.$$

After some simple computations and by defining $Y_1, Y_2, Y_3, X_1, X_2, X_3$ as

$$\begin{aligned} Y_1 &= \sin(C+\alpha)\cos(B-\alpha) - \sin(B+\alpha)\cos(C-\alpha), \\ Y_2 &= \sin(A+\alpha)\cos(C-\alpha) - \sin(C+\alpha)\cos(A-\alpha), \\ Y_3 &= \sin(B+\alpha) cos(A-\alpha) - \sin(A+\alpha)\cos(B-\alpha), \end{aligned}$$

$$\begin{aligned} X_1 &= \cos A \sin(A+\alpha)\cos(A-\alpha), \\ X_2 &= \cos B \sin(B+\alpha)\cos(B-\alpha), \\ X_3 &= \cos C \sin(C+\alpha)\cos(C-\alpha), \end{aligned}$$

we obtain the following expression for $\det M'$:

$$\det M' = \frac{X_1Y_1 + X_2Y_2 + X_3Y_3}{\sin(A+\alpha)\sin(B+\alpha)\sin(C+\alpha)\cos(A-\alpha)\cos(B-\alpha)\cos(C-\alpha)} \tag{2}$$

and then, $\det M' = 0$ if, and only if, $Y = X_1Y_1 + X_2Y_2 + X_3Y_3 = 0$.

The application of the prostaphaeresis formulae $\sin(\theta)\cos(\varphi) = \frac{1}{2}(\sin(\theta+\varphi) + \sin(\theta-\varphi))$ to Y_1, Y_2, Y_3 yields

$$Y_1 = -\sin(B-C)\cos(2\alpha), \quad Y_2 = -\sin(C-A)\cos(2\alpha), \quad Y_3 = -\sin(A-B)\cos(2\alpha),$$

while the application to X_1 ,X_2 ,X_3 yields

$$\begin{aligned} X_1 &= \cos A(\sin A \cos A + \sin\alpha\cos\alpha), \\ X_2 &= \cos B(\sin B \cos B + \sin\alpha\cos\alpha), \\ X_3 &= \cos C(\sin C \cos C + \sin\alpha\cos\alpha). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} Y = &-\cos A(\sin A\cos A + \sin\alpha\cos\alpha)\sin(B-C)\cos(2\alpha) \\ &-\cos B(\sin B\cos B + \sin\alpha\cos\alpha)\sin(C-A)\cos(2\alpha) \\ &-\cos C(\sin C\cos C + \sin\alpha\cos\alpha)\sin(A-B)\cos(2\alpha). \end{aligned}$$

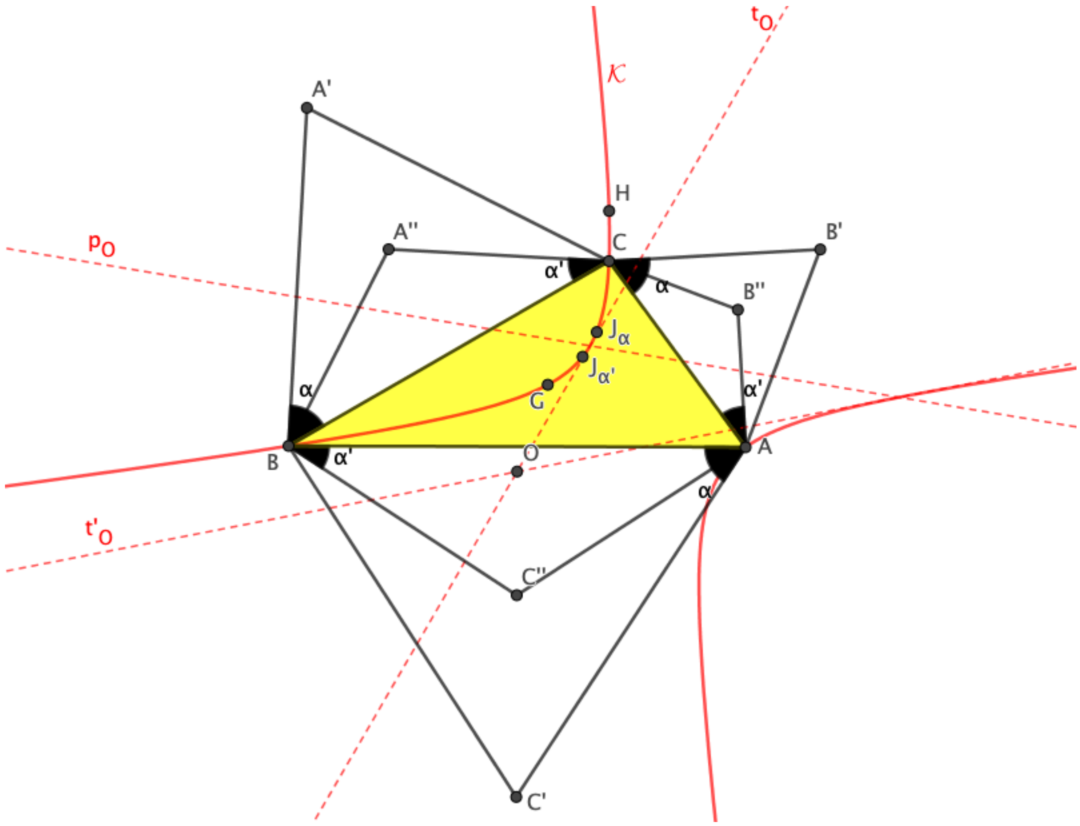

Figure 5: Kiepert hyperbola $\mathcal{K}$ and complementary Jacobi points

Now, we write Y in the following way

$$Y = -\cos(2\alpha)(Y' + Y'')$$

where

$$Y' = \sin A \cos^2 A \sin(B - C) + \sin B \cos^2 B \sin(C - A) + \sin C \cos^2 C \sin(A - B)$$

and

$$Y'' = \cos A \sin\alpha \cos\alpha \sin(B - C) + \cos B \sin\alpha \cos\alpha \sin(C - A) + \cos C \sin\alpha \cos\alpha \sin(A - B).$$

A simple computation gives $Y'' = 0$. Hence $Y = 0$ if, and only if, $Y' = 0$.

Finally, recall that $C = \pi - (A + B)$, and hence,

$$\sin C = \sin(A + B) = \sin A \cos B + \sin B \cos A,$$
$$\cos C = -\cos(A + B) = -\cos A \cos B + \sin A \sin B.$$

By substituting these formulas, a straightforward computation gives $Y' = 0$.

Summarizing, O, J_α, and $J_{\alpha'}$ are collinear because $\det M = 0$. In fact, by reducing $\det M$ to (2) and decomposing the numerator $Y = X_1Y_1 + X_2Y_2 + X_3Y_3$ via the prostaphaeresis formulae into the additive terms $Y' + Y''$, one can prove that both Y' and Y'' evaluate identically equal to zero (regardless of α). □

Note that a direct consequence of Theorem 3 is statement (a) of Theorem 1: In this way, the Escher-Fermat line of $\mathcal{T}$ is its $\pi/6$-line. It is also worth noting that the degenerate situations given by $\alpha = 0$ and $\alpha = \pi/2$ respectively produces the complementary Jacobi points $J_0 = G$ and $J_{\pi/2} = H$, where G denotes the centroid of $\mathcal{T}$ and H its orthocenter: Thus, the Euler-line of $\mathcal{T}$ is its 0-line.

Remark 3.1. By a slight modification in the proof of Theorem 3, it can be shown that the Lemoine or symmedian point U (X_6) of $\mathcal{T}$ is the intersection of all lines joining the first and second Jacobi α-points of $\mathcal{T}$. This result appears as Exercise 2, Page 48 in [9].

4 An Involution on the Kiepert hyperbola

Theorem 3, together with Theorem 2, puts in evidence the role of the circumcenter in the geometry of the triangle related to Kiepert conics.

The Kiepert hyperbola $\mathcal{K}$ of a (non-equilateral) triangle $\mathcal{T}$ is the (unique) rectangular hyperbola containing the vertices A, B, C of $\mathcal{T}$ and its centroid G (Figure 5). Note that $\mathcal{K}$ is degenerate if, and only if, $\mathcal{T}$ is isosceles (whose base is AB, say). In fact, $\mathcal{K}$ reduces to a pair of perpendicular lines, namely to the side AB and the opposite median CG. Moreover, the circumcenter O, together with any Jacobi point, lies on CG. Thus, suppose $\mathcal{T}$ is non-isosceles, i.e., $\mathcal{K}$ is non-degenerate. It is well-known (see for example [2] or [6]) that many remarkable points of $\mathcal{T}$ are contained in $\mathcal{K}$, among them the orthocenter H, the Fermat points and the Napoleon points.

Each point X in the plane induces an involution ω_X on $\mathcal{K}$, centered in X, defined as follows. For each $P \in \mathcal{K}$, if the line XP is a secant of $\mathcal{K}$, then $\omega_X(P)$ is the other intersection point of XP with $\mathcal{K}$. If XP is tangent to $\mathcal{K}$ at P, then set $\omega_X(P) = P$. The point X and its polar line p_X are respectively center and axis of the involution ω_X. Recall that p_X is the locus of all intersection points of line pairs which are associated in ω_X.

The perspective collineation φ_X on the plane inducing the involution ω_X on $\mathcal{K}$ is a harmonic (involutory) homology with center X and axis p_X. Its center X and all points of p_X are fixed points in φ_X, while p_X and any line containing X is mapped into itself. Moreover, since φ_X is harmonic, if P, P' are corresponding points (collinear with X) and P^* denotes the intersection point of the line PP' with the axis p_X of φ_X, then P^* is the harmonic conjugate of X with respect to P and P'. In other words: The cross ratio of the ordered points P, P', X, P^* is $(P, P', X, P^*) = -1$.

The line d_1 (resp. d_2) from X which is parallel to the asymptote a_1 (resp. a_2) of $\mathcal{K}$ intersects the hyperbola in the point at infinity D_1^∞ of a_1 (resp. D_2^∞ of a_2) and in a proper point D_1 (resp. D_2). Therefore, the intersection point of D_1D_2 with $D_1^\infty D_2^\infty$, i.e., the point at infinity of D_1D_2, lies on the axis p_X of ω_X. In other words: D_1D_2 and p_X are parallel. Note that D_1D_2 and the line at infinity are corresponding elements in φ_X: this proves that D_1D_2 is equidistant from X and the axis p_X. Moreover, the fourth point D in the rectangle XD_1D_2D, i.e., the intersection point of the corresponding lines $D_1D_2^\infty$ and $D_2D_1^\infty$, lies on the axis p_X. On the other side, D lies on the diameter of $\mathcal{K}$ containing X. In fact, with respect to the canonical Cartesian coordinate system whose axes are the asymptotes of $\mathcal{K}$, the coordinates of D and X are reciprocal each other and hence these points are collinear with the origin of the reference system, i.e., the center of the hyperbola. Since this diameter is the polar line of the point at infinity of the axis p_X (or of D_1D_2), the intersection points V_1, V_2 of the hyperbola with p_X are symmetric with respect to the diameter XD along p_X, and then, D is the midpoint between V_1 and V_2.

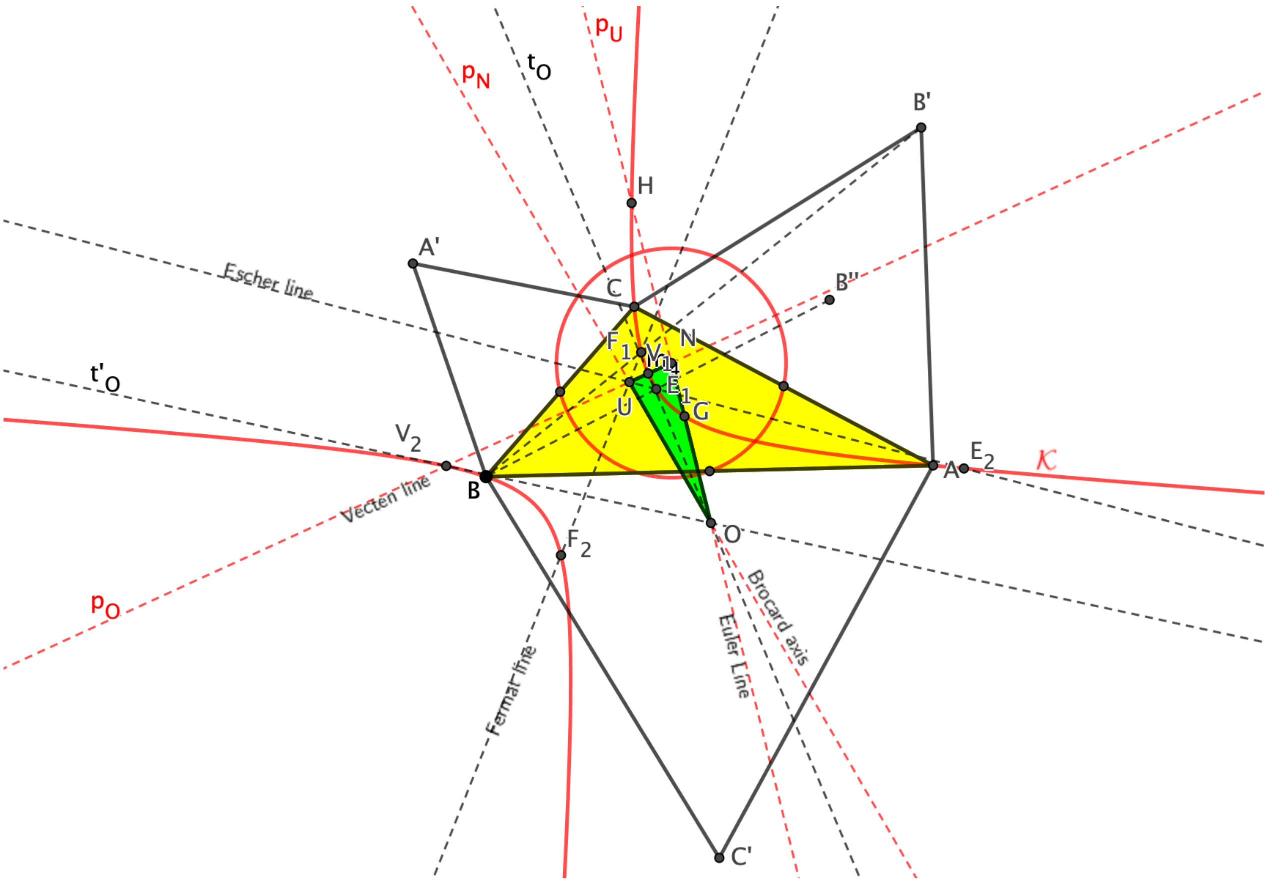

Figure 6: Autopolar triangle OUN with respect to Kiepert hyperbola

If, in particular, we take the circumcenter O of $\mathcal{T}$ as the center of the involution, we can reformulate Theorem 3 in the following way: *Complementary Jacobi points of $\mathcal{T}$ are pairs of corresponding points in the involution ω_O on $\mathcal{K}$.*

The following remarks are direct consequences of the above result:

Remark 4.1. If E_1, E_2 (resp. F_1, F_2) are the first and the second Napoleon (resp. Fermat) point, the pairs (G, H), (E_1, F_1), (E_2, F_2) are corresponding points in the involution ω_O, via the Euler line and the Escher-Fermat lines, respectively.

Remark 4.2. If t_O and t'_O denote the tangent lines from O to $\mathcal{K}$, then the tangent points V_1 and V_2 are the first and second Jacobi $\pi/4$-point[6]. Therefore, t_O (resp. t'_O) is the polar line of V_1 (resp. V_2) and the line joining these two tangent points is the polar line p_O of O, with respect to $\mathcal{K}$, i.e., the axis of ω_O and φ_O: We will call it the *Vecten line of $\mathcal{T}$*. Note that, if N is the intersection point of p_O with the Euler line, the cross ratio (G, H, O, N) is -1. This proves that N is the midpoint of OH and hence N is the nine-point center of $\mathcal{T}$ (see X_5 properties in [8]).

Remark 4.3. Let d_1 (resp. d_2) denote the line from O which is parallel to the asymptote a_1 (resp. a_2) of $\mathcal{K}$ and let D_1 (resp. D_2) be the proper intersection point of the hyperbola with d_1 (resp. with d_2)[7]. Then, the line D_1D_2 is parallel to the axis p_O and is equidistant from O

[6]These points are also known as Vecten Points X_{485} and X_{486}.

[7]By using Peter J. C. Moses's "6, 9, 13 *Search Function*" in [8], we identify D_1, D_2 as the triangle centers X_{6177} and X_{6178}, the Ceva points of the real (resp. imaginary) foci of the Steiner inellipse.

and p_O. As a consequence, the fourth point[8] D in the rectangle OD_1D_2D lies on the Vecten line p_O and is the midpoint between the Vecten points.

Remark 4.4. Recall (see Remark 3.1) that the Lemoine point U belongs to the Fermat line f (resp. to the Escher line e) joining the Fermat points F_1, F_2 (resp. the Napoleon points E_1, E_2). In other words, U is the intersection point of f and e. Thus, U belongs to the axis p_O of ω_O, and hence, its polar line p_U contains O. On another side, U belongs to the diameter $f = F_1F_2$ of the hyperbola. Therefore, the tangent lines to $\mathcal{K}$ at F_1 and F_2 are parallel and p_U is the line through O which is parallel to these tangent lines. In fact, later we will see that it is the Euler line.

Besides the circumcenter O of $\mathcal{T}$, other remarkable points P may be considered as centers of the involution ω_P on $\mathcal{K}$ defined in the beginning of this section (Figure 6). In particular, since the Lemoine point U is the intersection point of all lines joining the first and second Jacobi $\alpha -$*points* J_α and $J_{-\alpha}$ of $\mathcal{T}$ (Remark 3.1), then J_α and $J_{-\alpha}$ are pairs of corresponding points in the involution ω_U.

Another interesting involution arises if we consider the nine-point center N of $\mathcal{T}$, as we may see in what follows.

First of all, recall that four distinct points on a conic define an autopolar triangle, whose vertices are the diagonal points of the quadrilateral. Now, if E_1, E_2 (resp. F_1, F_2) are the first and the second Escher (resp. Fermat) points of $\mathcal{T}$, then the vertices of the autopolar triangle defined by these four points on Kiepert hyperbola $\mathcal{K}$ are the circumcenter O, the Lemoine point U, and the nine-point center N.

This property which brings us back to the classical context of the historical introduction, may be obtained as a special case of the following theorem, summarizing, in some sense, the results obtained in the present section.

Theorem 4. *Let $\mathcal{T}$ be a triangle and let α, α' be complementary angles.*

(a) *The intersection point of the lines $J_\alpha J_{\alpha'}$ and $J_{-\alpha}J_{-\alpha'}$ is the circumcenter O of $\mathcal{T}$.*
(b) *The intersection point of the lines $J_\alpha J_{-\alpha}$ and $J_{\alpha'}J_{-\alpha'}$ is the Lemoine point U of $\mathcal{T}$.*
(c) *The intersection point of the lines $J_\alpha J_{-\alpha'}$ and $J_{-\alpha}J_{\alpha'}$ is the nine-point center N of $\mathcal{T}$.*

Moreover, the triangle OUN is autopolar with respect to Kiepert hyperbola $\mathcal{K}$ and its sides are the Brocard axis OU, the Euler line ON, and the Vecten line UN.

Proof. The statement (a) (resp. (b)) is a direct consequence of Theorem 3 (resp. of Remark 3.1). Recall that the polar line p_O of the circumcenter O is the Vecten line, containing N and U (Remarks 4.2 and 4.4). The fixed points of the involution ω_U are the centroid $G = J_0$ and the orthocenter $H = J_{-\pi/2}$ of $\mathcal{T}$. Hence, the tangent lines to $\mathcal{K}$ from U are the lines UG and UH: Equivalently, the polar line p_U, i.e., the axis of the involution ω_U, is the Euler line (containing O). Remark 3.1 states that the intersection point of the Euler line p_U with the polar line p_O of the circumcenter O is the nine-point center N (Figure 6): Therefore, the polar line p_N of N is the line OU, the Brocard axis of $\mathcal{T}$ (cf. [2, p. 195]). This proves the last part of the theorem. Finally, if N' identifies the intersection point of the lines $J_\alpha J_{-\alpha'}$ and $J_{-\alpha}J_{\alpha'}$, the triangle OUN' is autopolar and then N' is the pole of the Brocard axis OU. This proves $N = N'$ and hence statement (c). □

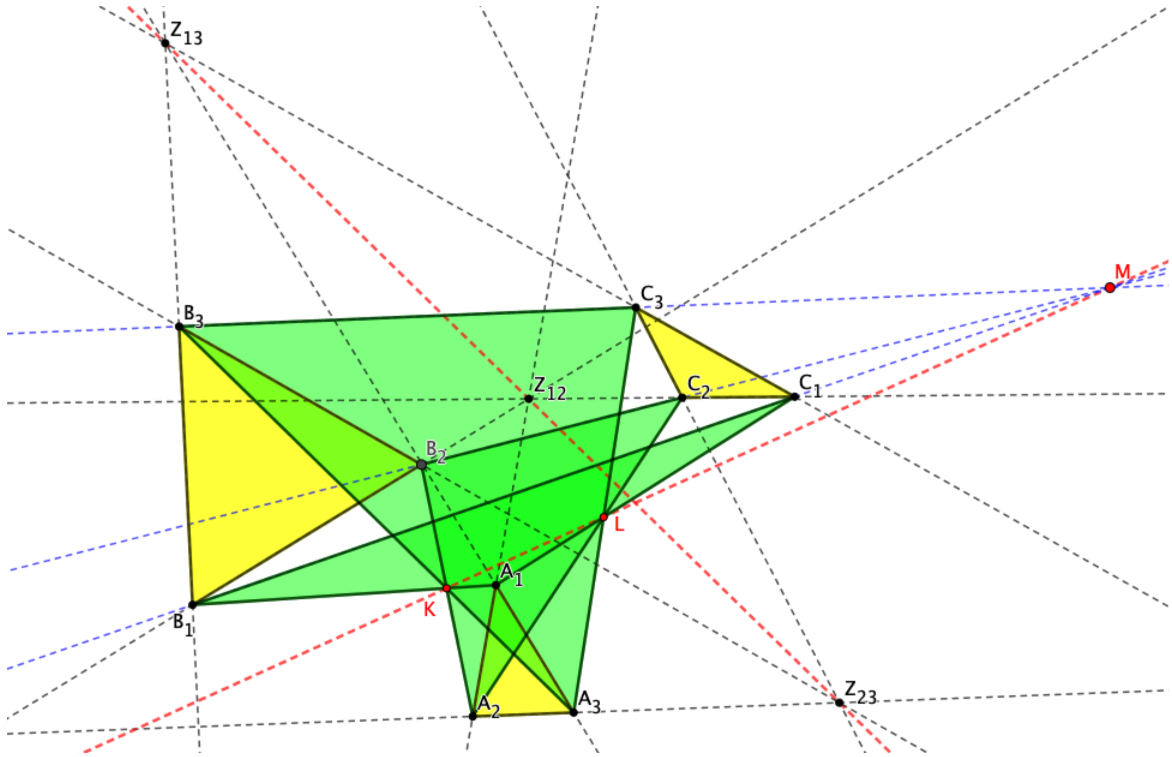

Figure 7: Desarguesian configuration

5 Desarguesian configurations and Kiepert parabola

Now we describe a configuration which allows to obtain an interesting consequence of the collinearity of O with complementary Jacobi points given in Theorem 3.

Let be given in the plane an *extended Desargues configuration*, i.e., nine distinct points $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$ such that all the triads of points (A_1, A_2, A_3), (B_1, B_2, B_3), (C_1, C_2, C_3) and, for each $i = 1, 2, 3$, (A_i, B_i, C_i), are not collinear.

In what follows, $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$. Then, for each (i, j) set $K_{i,j} = A_iB_i \cap A_jB_j$, $L_{i,j} = A_iC_i \cap A_jC_j$, $M_{i,j} = B_iC_i \cap B_jC_j$ and $R_{i,j} = A_iA_j \cap B_iB_j$, $S_{i,j} = A_iA_j \cap C_iC_j$, $T_{i,j} = B_iB_j \cap C_iC_j$ (Figure 7).

Lemma 1. *For each (i, j), we have $R_{i,j} = S_{i,j} = T_{i,j}$ if, and only if, $K_{i,j}$, $L_{i,j}$, $M_{i,j}$ are collinear.*

Proof. It is Desargues's theorem applied to the triangles A_i, B_i, C_i and A_j, B_j, C_j. □

Lemma 2. *With the above notations, we have:*

(a) $K_{12} = K_{13} = K_{23}$ if, and only if, R_{12}, R_{13}, R_{23} are collinear.
(b) $L_{12} = L_{13} = L_{23}$ if, and only if, S_{12}, S_{13}, S_{23} are collinear.
(c) $M_{12} = M_{13} = M_{23}$ if, and only if, T_{12}, T_{13}, T_{23} are collinear.

Proof. It is sufficient to apply Desargues's theorem to the triangle pairs $(A_1A_2A_3,\ B_1B_2B_3)$, $(A_1A_2A_3,\ C_1C_2C_3)$, $(B_1B_2B_3,\ C_1C_2C_3)$ respectively (Figure 7). □

[8] This point is the intersection point of the Vecten line and the line joining the circumcenter with the center X_{115} of the Kiepert hyperbola and could not be found in [8].

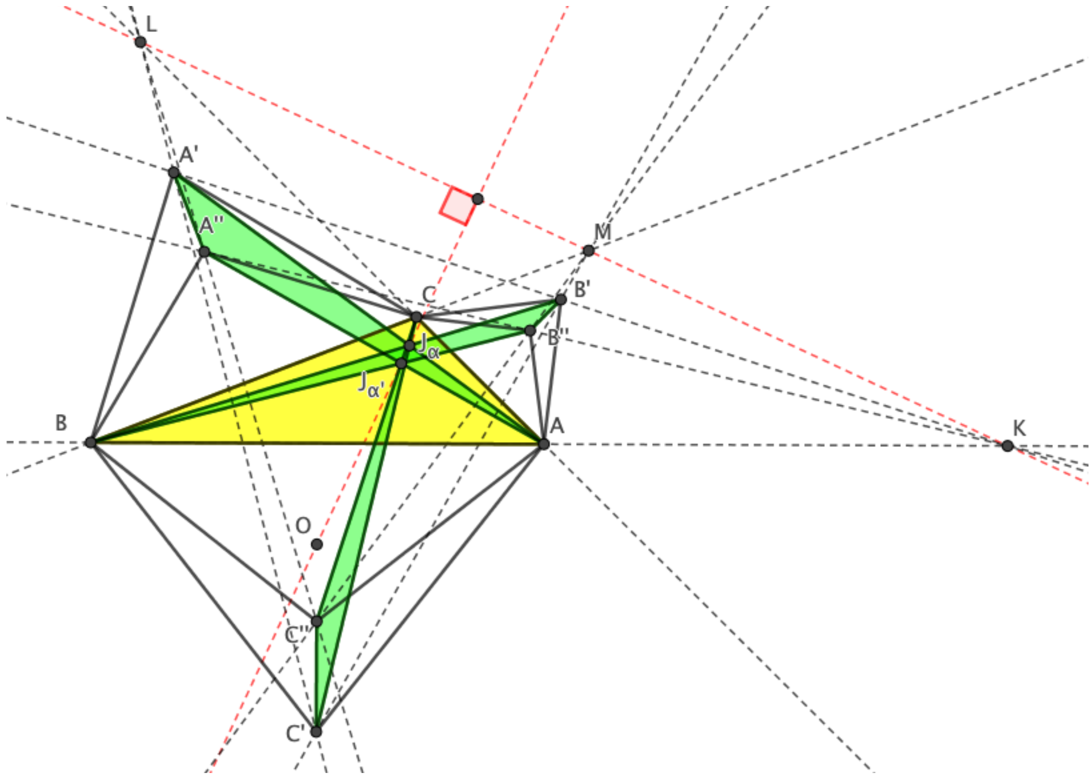

Figure 8: Desargues's theorem applied to triangles (A, A', A''), (B, B', B''), and (C, C', C'')

.

The following result is a direct consequence of the previous lemmas:

Theorem 5. *For each (i, j), we have $R_{ij} = S_{ij} = T_{ij} = Z_{ij}$ with Z_{12}, Z_{13}, Z_{23} collinear if, and only if, $K_{12} = K_{13} = K_{23} = K$, $L_{12} = L_{13} = L_{23} = L$, $M_{12} = M_{13} = M_{23} = M$ with K, L, M collinear.*

Let now $\mathcal{T} = ABC$ be a given triangle and let A', B', C' (resp. A'', B'', C'') denote the vertices opposite to the three vertices of the similar isosceles triangles with base BC, AC, AB and base angle α (resp. α'), with α and α' complementary (Figure 8). We refer to the pair $(\mathcal{T}, \alpha)$ as the Jacobi α-triangle ABC. Then, we have:

$R_{12} = S_{12} = T_{12} = A'A'' \cap B'B'' \cap C'C'' =$ circumcenter O of $\mathcal{T}$, $R_{13} = S_{13} = T_{13} = AA' \cap BB' \cap CC' =$ Jacobi α-point J_α of $\mathcal{T}$, $R_{23} = S_{23} = T_{23} = AA'' \cap BB'' \cap CC'' =$ Jacobi α'-point $J_{\alpha'}$ of $\mathcal{T}$.

Note that J_α and $J_{\alpha'}$ are complementary Jacobi points.

Theorem 6. *In each Jacobi α-triangle ABC, the lines $AB, A'B', A''B''$ are concurrent in a point K, the lines $AC, A'C', A''C''$ are concurrent in a point L, the lines $BC, B'C', B''C''$ are concurrent in a point M.*

Moreover, K, L, and M are collinear on a line r_α, which is the common Desargues axis of the complementary Jacobi points J_α and $J_{\alpha'}$ of $\mathcal{T}$ and is orthogonal to the α-line $J_\alpha J_{\alpha'}$ of $\mathcal{T}$.

Proof. Since O, J_α, $J_{\alpha'}$ are collinear (Theorem 3), Lemma 2 proves $K_{12} = K_{13} = K_{23} = K$, $L_{12} = L_{13} = L_{23} = L$ and $M_{12} = M_{13} = M_{23} = M$, with K, L, M collinear. This implies, in particular, that the line containing the collinear points K_{13}, L_{13}, M_{13} (the Desargues axis of J_α) and the line containing the collinear points K_{23}, L_{23}, M_{23} (the Desargues axis of $J_{\alpha'}$) actually coincide. The last statement of the Theorem follows from Theorem 2. □

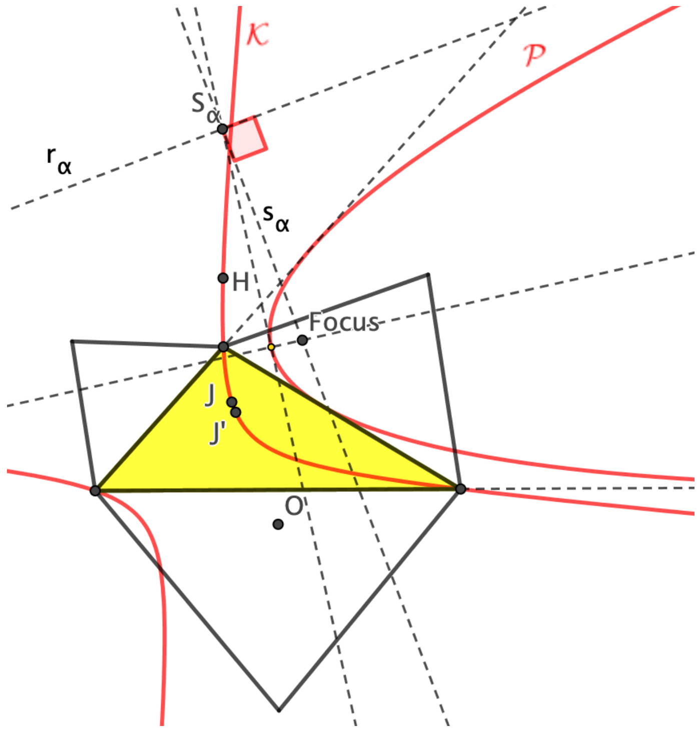

Figure 9: Kiepert hyberbola $\mathcal{K}$ and Kiepert parabola $\mathcal{P}$

Note that direct consequences of Theorem 6 are statements (b) and (c) of Theorem 1.

Theorem 6 leads to an alternative construction of the Desargues axis r_α of the complementary Jacobi points J_α and $J_{\alpha'}$ and to a deeper insight on the relations between the Kiepert hyperbola and the Kiepert parabola of the given triangle. Following [2], we recall that the Kiepert parabola $\mathcal{P}$ of a triangle $\mathcal{T}$ is the envelope of the Desargues axes of all α-Jacobi points of $\mathcal{T}$. Moreover, $\mathcal{P}$ is inscribed into $\mathcal{T}$ and the Euler line is the directrix of $\mathcal{P}$ (Figure 9).

Corollary 6.1. *Let $\mathcal{T}$ be a triangle and let J_α, $J_{\alpha'}$ denote complementary Jacobi points of $\mathcal{T}$. Then, the common Desargues axis r_α of J_α and $J_{\alpha'}$ is the (unique) line which is orthogonal to the α-line $J_\alpha J_{\alpha'}$ and tangent to the Kiepert parabola $\mathcal{P}$.*

Proof. The statement is a direct consequence of Theorem 6. In fact, recall that the foot of the perpendicular from the focus of a parabola to any tangent belongs to the tangent at the vertex. So, if s_α is the line from the focus of $\mathcal{P}$ which is parallel to the α-line $J_\alpha J_{\alpha'}$ and if S_α denotes the intersection point of s_α with the tangent to $\mathcal{P}$ at its vertex, then the common Desargues axis r_α is the tangent to $\mathcal{P}$ simply obtained as the line from S_α which is orthogonal to s_α (or to $J_\alpha J_{\alpha'}$). □

In fact, the construction of r_α described in the proof of Corollary 6.1 induces a correspondence Φ which associates to each α-line $J_\alpha J_{\alpha'}$ of $\mathcal{T}$ the Desargues axis r_α of J_α and $J_{\alpha'}$.

Reversing this construction, given a line t enveloping the Kiepert parabola $\mathcal{P}$, we may consider the line from O which is orthogonal to t. In this way, besides the Euler line whose associated Desargues axis is the line at infinity, we can deduce, for example, that the line of the pencil centered in O associated to the Lemoine axis of the triangle, the so-called fifth tangent to $\mathcal{P}$ [2], is just its Brocard axis OU.

Another remarkable example may be considered by recalling that the axes s_1, s_2 of the Steiner ellipse of a triangle $\mathcal{T}$ are parallel to the asymptotes of its Kiepert hyperbola and tangent to its Kiepert parabola (see [6], Theorem 3). In this case, if d_1 (resp. d_2) is the α-line (resp. β-line) from the circumcenter O of $\mathcal{T}$ which is parallel to the asymptote a_1 (resp. a_2) of $\mathcal{K}$ (recall Remark 4.3), we have $\Phi(d_1) = s_2$ and $\Phi(d_2) = s_1$. Therefore, the intersections D_1, D_1^∞ (resp. D_2, D_2^∞) of d_1 (resp. d_2) with $\mathcal{K}$ are complementary Jacobi α-points (resp. β-points) of $\mathcal{T}$, for a suitable base angle α (resp. β), whose common Desargues axis is just the Steiner axis s_1 (resp. s_2). Note that, if Q_i denotes the tangent point of the Steiner axis s_i with $\mathcal{P}$, the polar line of the centroid G is the line Q_1Q_2, which, for the polarity reciprocity law, also contains the focus of $\mathcal{P}$.

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