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Similar Trapezoids on the Sides of a Triangle

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Abstract. Throughout the history configurations obtained by constructing three similar figures on the sides of a triangle were studied from several different viewpoints. The Pythagorean theorem shows a relation between their area, the case of equilateral triangles is related to the Napoleon triangle and the existence of Toricelli (or Fermat) point, while the case of squares is connected to the existence of the Vecten point of the triangle. Moreover the Kiepert perspectors are obtained by constructing similar isosceles triangles to the sides of a triangle. In this paper we study the case of similar isosceles trapezoids. This is a generalization of all the previously mentioned cases, so the obtained results are natural generalizations of several well known classical geometry properties. We emphasize several pairs of perspective triangles and we prove that some of them are also orthologic pairs (Theorem 2). Moreover we give a characterization for some of the orthology centers and in the special case when regular pentagons are constructed we give a characterization for a center of perspectivity. The necessary calculations are made using complex numbers and matrices.

Key Words: centroid, orthocenter, Euler-line, perspective triangles, orthologic triangles

MSC 2020: 51M04 (primary), 51N20

1 Introduction

Configurations obtained by constructing three similar figures on the sides of a triangle were studied from ancient times. Despite of the fact that the Pythagorean theorem (see [5]) gives a relation between the area of three constructed arbitrary but similar figures, the known geometric properties depend on the shape of the constructed figures. Some general properties of such figures can be found in [3]. The centers of the equilateral triangles $A'BC$, $B'CA$ and $C'AB$ constructed on the sides of a triangle form a new equilateral triangle, the so called Napoleon triangle (see [1]), which is perspective to the reference triangle and the center of perspectivity is the first Napoleon point ($X(17)$ in Kimberling's Encyclopedia of triangle centers, see [4] or [8]). Moreover in this case the triangles ABC and $A'B'C'$ are perspective

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and the center of perspectivity is the Toricelli (or Fermat) point of the triangle ($X(13)$ in Kimberling's notation) and this point is also a Kiepert type perspector (see [10]). The centroids of other regular polygons constructed on the sides of the triangle do not determine an equilateral triangle, but the perspectivity property remains true as a special case of Kiepert's perspectivity property (see [10]). A special case of this property can be obtained if in the exterior (or towards the interior) of a triangle we construct squares on the sides of a triangle. In this case the triangle determined by the center of these squares is perspective to the reference triangle and the center of perspectivity is the Vecten point ($X(485)$ in Kimberling's notation). Some further properties of this configuration are established in [9]. By constructing similar isosceles trapezoids on the sides of a triangle we obtain a configuration which includes both the case of equilateral triangles (as a degenerated case), the case of squares and also the case of isosceles triangles. Our aim is to establish some properties of these configurations and to emphasize the connections with the above mentioned properties. Moreover we study not only the perspectivity of the triangles, but also the orthogonality of them and we emphasize some connections with well known results.

2 Main results

Our main results are formulated in Theorem 1 and Theorem 2. In Theorem 3 we included some results concerning the special case when regular pentagons are constructed on the sides of a triangle. However the pentagon is not a trapezoid, this case can be considered as a special case because four adjacent vertices of a regular pentagon determine an isosceles trapezoid, so the obtained figure can be viewed as containing the trapezoids on the sides of a triangle and three more points (the fifth vertex in each pentagon).

Theorem 1. *On the sides of the triangle ABC as bases we construct the similar isosceles trapezoids $BCMN$, $CAPQ$ and $ABRS$ (see Figure 1).*

a) *The perpendiculars from A, B, C to the segments SP, RN and QM are concurrent in a point K_1.*
b) *The perpendicular bisectors of SP, RN and QM are concurrent in K_2.*
c) *The centroid of the triangle ABC belongs to the line K_1K_2.*
d) *If $AP \cap CQ = \{Y\}$, $AS \cap BR = \{Z\}$, $BN \cap CM = \{X\}$, then the line K_1K_2 is the Euler-line of the triangle XYZ and centroids of the triangles ABC and XYZ are the same.*
e) *Moreover if $m(\widehat{PAC}) = \alpha$ and $\frac{AP}{AC} = \rho$, then*

$$\frac{\overline{OK_1}}{\overline{OH}} = -\frac{\cos\alpha}{\cos 3\alpha} \quad \text{and} \quad \frac{\overline{OK_2}}{\overline{OH}} = \frac{\cos\alpha}{\cos 3\alpha}(2\rho\cos\alpha - 1),$$

where O and H are the circumcenter and the orthocentre of the triangle XYZ and $O \neq H$.

Proof. In order to prove the previous statements we use complex numbers. We consider O as the origin and the radius of the circumcircle of the triangle XYZ to be 1. The points of the complex plane are denoted by capital letters ($A, B, ...$) and the corresponding complex numbers with the same small letters ($a, b, ...$). So if G is the centroid of the triangle XYZ, g denotes the corresponding complex number to G. Due to our notations we have

$$x \cdot \overline{x} = y \cdot \overline{y} = z \cdot \overline{z} = 1.$$

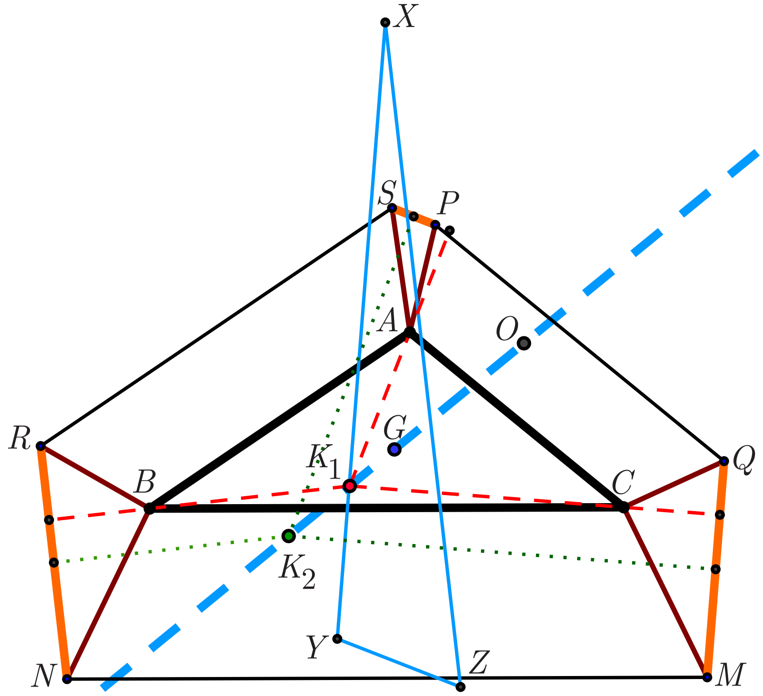

Figure 1: K_1 and K_2 are on the Euler-line of the triangle XYZ

It is well-known (see [2]) that $g = \frac{x+y+z}{3}$ and $h = x+y+z$, so a point T is on the Euler-line of the triangle XYZ if and only if there is a real number λ such that

$$t = \lambda(x+y+z).$$

In what follows we use the notation $u = \cos\alpha + i\sin\alpha$ and in the following four lemmas we suppose $\alpha \notin \{30°, 90°\}$.

Lemma 1. *If $z_0 = \frac{1}{\cos\alpha} \cdot u$ and x, y, z, a, b, c are the corresponding complex numbers to the points X, Y, Z, A, B, C constructed as in Theorem 1, then*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & z_0 & \overline{z_0} \\ \overline{z_0} & 0 & z_0 \\ z_0 & \overline{z_0} & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Proof of Lemma 1. If we denote by A_1 the midpoint of the side BC, then X can be obtained from C by applying a rotation of $90°$ and a dilation from A_1 and scaling factor $\frac{XA_1}{CA_1} = \tan(\pi - \alpha) = -\tan\alpha$, so

$$x - a_1 = i \cdot (-\tan\alpha) \cdot (c - a_1).$$

Since $a_1 = \frac{b+c}{2}$, we obtain

$$x = \frac{b+c}{2} + i \cdot \tan\alpha \cdot \frac{b-c}{2} = z_0 \cdot b + \overline{z_0} \cdot c.$$

In a similar way we deduce

$$y = z_0 \cdot c + \overline{z_0} \cdot a \quad \text{and} \quad z = z_0 \cdot a + \overline{z_0} \cdot b,$$

which gives Lemma 1. $\square$

Lemma 2. *Using the previous notations we have*

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{\cos\alpha}{\cos 3\alpha} \begin{bmatrix} -1 & \overline{u}^2 & u^2 \\ u^2 & -1 & \overline{u}^2 \\ \overline{u}^2 & u^2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Proof of Lemma 2. We calculate the inverse of the matrix

$$M = \begin{bmatrix} 0 & z_0 & \overline{z_0} \\ \overline{z_0} & 0 & z_0 \\ z_0 & \overline{z_0} & 0 \end{bmatrix}.$$

$\det(M) = z_0^3 + \overline{z_0}^3 = \dfrac{\cos 3\alpha}{4\cos^3\alpha}$, so

$$M^{-1} = \frac{1}{\det(M)} \cdot \begin{bmatrix} -|z_0|^2 & \overline{z_0}^2 & z_0^2 \\ z_0^2 & -|z_0|^2 & \overline{z_0}^2 \\ \overline{z_0}^2 & z_0^2 & -|z_0|^2 \end{bmatrix} = \frac{\cos\alpha}{\cos 3\alpha} \begin{bmatrix} -1 & \overline{u}^2 & u^2 \\ u^2 & -1 & \overline{u}^2 \\ \overline{u}^2 & u^2 & -1 \end{bmatrix},$$

hence Lemma 2 follows from Lemma 1. □

Lemma 3. *Using the previous notations we have*

$$\begin{bmatrix} p \\ r \\ m \end{bmatrix} = \begin{bmatrix} 1-\rho\cdot u & 0 & \rho\cdot u \\ \rho\cdot u & 1-\rho\cdot u & 0 \\ 0 & \rho\cdot u & 1-\rho\cdot u \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \textit{and}$$

$$\begin{bmatrix} s \\ n \\ q \end{bmatrix} = \begin{bmatrix} 1-\rho\cdot \overline{u} & \rho\cdot \overline{u} & 0 \\ 0 & 1-\rho\cdot \overline{u} & \rho\cdot \overline{u} \\ \rho\cdot \overline{u} & 0 & 1-\rho\cdot \overline{u} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Proof of Lemma 3. Since the constructed trapezoids are similar to each other, P is obtained from C by applying a counterclockwise rotation of angle α around A and a dilation with scaling factor ρ. Using complex numbers this can be written as

$$p - a = \rho\cdot u\cdot(c-a)$$

so we have $p = (1-\rho\cdot u)\cdot a + \rho\cdot u\cdot c$. Using a similar argument we have $r = b + \rho\cdot u\cdot(a-b) = (1-\rho\cdot u)\cdot b + \rho\cdot u\cdot a$ and $m = c + \rho\cdot u\cdot(b-c) = (1-\rho\cdot u)\cdot c + \rho\cdot u\cdot b$, which gives the first equality from Lemma 3. For the points S, N, Q the rotation is a clockwise rotation, so if we replace α by $-\alpha$ in the previous relations, we obtain the affixes of the points S, N, Q. This gives the second relation from Lemma 3. □

Combining the previous two lemmas and calculating the product of the corresponding matrices we obtain

Lemma 4. *With the previous notations we have*

$$\begin{bmatrix} p \\ r \\ m \end{bmatrix} = \frac{\cos\alpha}{\cos 3\alpha} \begin{bmatrix} 2\rho\cos\alpha - 1 & \overline{u}^2 - \rho\cdot\overline{u} + \rho\cdot u^3 & u^2 - \rho\cdot u^3 - \rho\cdot u \\ u^2 - \rho\cdot u^3 - \rho\cdot u & 2\rho\cos\alpha - 1 & \overline{u}^2 - \rho\cdot\overline{u} + \rho\cdot u^3 \\ \overline{u}^2 - \rho\cdot\overline{u} + \rho\cdot u^3 & u^2 - \rho\cdot u^3 - \rho\cdot u & 2\rho\cos\alpha - 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \textit{and}$$

$$\begin{bmatrix} s \\ n \\ q \end{bmatrix} = \frac{\cos\alpha}{\cos 3\alpha} \begin{bmatrix} 2\rho\cos\alpha - 1 & \overline{u}^2 - \rho\cdot\overline{u} - \rho\cdot\overline{u}^3 & u^2 - \rho\cdot u + \rho\cdot\overline{u}^3 \\ u^2 - \rho\cdot u + \rho\cdot\overline{u}^3 & 2\rho\cos\alpha - 1 & \overline{u}^2 - \rho\cdot\overline{u} - \rho\cdot\overline{u}^3 \\ \overline{u}^2 - \rho\cdot\overline{u} - \rho\cdot\overline{u}^3 & u^2 - \rho\cdot u + \rho\cdot\overline{u}^3 & 2\rho\cos\alpha - 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

In order to prove a) and the first relation of e) it is sufficient to prove that if $k_1 = -\frac{\cos\alpha}{\cos 3\alpha}(x+y+z)$, then $AK_1 \perp SP$ and similarly $BK_1 \perp RN$, $CK_1 \perp QM$. But $AK_1 \perp SP$ is equivalent to the fact that $k_1 - a$ is obtained from $s - p$ by a multiplication with a purely imaginary number. From Lemma 3 we deduce

$$s - p = -2\rho \cdot \cos\alpha \cdot (y - z)$$

and due to Lemma 2

$$\begin{aligned} k_1 - a =& \frac{\cos\alpha}{\cos 3\alpha}(-x - y - z + x - \overline{u}^2 \cdot y - u^2 \cdot z) \\ =& -\frac{\cos\alpha}{\cos 3\alpha}\left[(1+\cos 2\alpha - i\cdot\sin 2\alpha)y + (1+\cos 2\alpha + i\cdot \sin 2\alpha)z\right] \\ =& -\frac{\cos\alpha}{\cos 3\alpha}\cdot 2\cos\alpha\cdot(\overline{u}\cdot y + u\cdot z), \end{aligned}$$

so it is sufficient to prove that $E = \dfrac{y - z}{\overline{u}\cdot y + u \cdot z}$ is purely imaginary. On the other hand

$$\overline{E} = \frac{\overline{y} - \overline{z}}{u \cdot \overline{y} + \overline{u}\cdot\overline{z}} = \frac{\frac{1}{y} - \frac{1}{z}}{u \cdot \frac{1}{y} + \overline{u}\cdot \frac{1}{z}} = -E,$$

so E is purely imaginary. By a similar calculation we deduce $BK_1 \perp RN$ and $CK_1 \perp QM$, so we have a) and the first relation from e).

To prove b) and the second relation from e) it is sufficient to prove that the point K_2 whose affix is $k_2 = \frac{\cos\alpha}{\cos 3\alpha}(2\rho\cdot\cos\alpha - 1)(x+y+z)$ belongs to the three perpendicular bisector. But if z_1 and z_2 corresponds to two arbitrary points Z_1, Z_2 in the plane, the perpendicular bisector of Z_1Z_2 is the loci of points T for which $TZ_1 = TZ_2$. Using complex numbers this relation can be written as $|t - z_1|^2 = |t - z_2|^2$ or $(t - z_1)\cdot(\overline{t} - \overline{z_1}) = (t - z_2)\cdot(\overline{t} - \overline{z_2})$, which gives the equation of the perpendicular bisector in complex numbers:

$$t \cdot (\overline{z_1} - \overline{z_2}) + \overline{t}\cdot(z_1 - z_2) + z_2 \cdot \overline{z_2} - z_1 \cdot \overline{z_2} = 0.$$

So we have to check that $t = k_2$ satisfies the previous relation for $z_1 = p$ and $z_2 = s$ (and two more corresponding relations). First we calculate $s\cdot\overline{s} - p\cdot\overline{p}$. Using Lemma 3. and the notation $c_0 = \frac{2\cos^2\alpha}{\cos 3\alpha}$, we have

$$\begin{aligned} &s\cdot\overline{s} - p\cdot\overline{p} = s\cdot(\overline{s} - \overline{p}) + \overline{p}\cdot(s - p) = \\ &= c_0 \cdot \Big[((2\rho\cos\alpha - 1)\cdot x + (\overline{u}^2 - \rho\cdot\overline{u} - \rho\cdot\overline{u}^3)\cdot y + (u^2 - \rho\cdot u + \rho\cdot\overline{u}^3)\cdot z)\cdot(\overline{z} - \overline{y}) \\ &+((2\rho\cos\alpha - 1)\cdot x + (u^2 - \rho\cdot u + \rho\cdot\overline{u}^3)\cdot y + (\overline{u}^2 - \rho\cdot\overline{u} - \rho\cdot\overline{u}^3)\cdot z)\cdot(z - y)\Big] \\ &= \rho\cdot c_0\cdot(2\rho\cdot\cos\alpha - 1)\cdot[x\cdot(\overline{z} - \overline{y}) + \overline{x}\cdot(z - y)]. \end{aligned}$$

Based on this result and Lemma 3. the relation

$$k_2 \cdot (\overline{p} - \overline{s}) + \overline{k_2}\cdot(p - s) + s\cdot\overline{s} - p\cdot\overline{p} = 0$$

holds if and only if

$$2\rho\cdot\cos\alpha\cdot\left[k_2\cdot(\overline{y} - \overline{z}) + \overline{k_2}\cdot(y - z) + \frac{\cos\alpha}{\cos 3\alpha}(2\rho\cdot\cos\alpha - 1)(x\cdot(\overline{z} - \overline{y}) + \overline{x}\cdot(z - y))\right] = 0.$$

Replacing k_2 with $\frac{\cos\alpha}{\cos 3\alpha}(2\rho\cdot\cos\alpha-1)(x+y+z)$ this relation is equivalent to

$$\frac{2\rho\cos^2\alpha}{\cos 3\alpha}(2\rho\cdot\cos\alpha-1)((x+y+z)(\overline{y}-\overline{z})+(\overline{x}+\overline{y}+\overline{z})(y-z)+x\cdot(\overline{z}-\overline{y})+\overline{x}\cdot(z-y))=0$$

and this is true.

To prove c) and d) it is sufficient to prove that the centroid of the triangle ABC is the same as the centroid of the triangle XYZ. But this is equivalent to $a+b+c=x+y+z$. But using Lemma 2, we have

$$a+b+c=\frac{\cos\alpha}{\cos 3\alpha}(u^2+\overline{u}^2-1)(x+y+z)=(x+y+z)$$

since

$$u^2+\overline{u}^2-1=2\cdot\cos 2\alpha-1 \quad \text{and} \quad \cos\alpha\cdot(2\cdot\cos 2\alpha-1)=\cos 3\alpha.$$

This concludes the proof. □

Remark. 1. For $\alpha=60°$ and $\rho=1$ the trapezoids degenerate into triangle and we obtain an additional property of the figure constructed for emphasizing the Napoleon and the Torriceli points. Similarly if $\alpha=60°$ and $\rho=1$ the trapezoids are squares and we obtain an additional property of the figure constructed for emphasizing the Vecten point. Moreover in this case a) and b) remain true without the similarity assumptions, the triangle XYZ vanishes and we have $K_1=G$. The existence of the point K_2 for squares appeared in a selection test for the Romanian IMO team in the '90s.

2. The case $\alpha=108°$ and $\rho=1$ (regular pentagons are constructed on the sides of the triangle ABC) was proposed by Puiu Braica in a Geometry Research forum. In this case it is also easy to prove from Theorem 1 that K_1 and K_2 are symmetric with respect to the center of the nine-point circle. Indeed in this case $\frac{OK_1}{OH}=\frac{\cos 72°}{\cos 36°}$ and $\frac{OK_2}{OH}=\frac{\cos 72°}{\cos 36°}\cdot(2\cos 72°+1)$, so $\frac{OK_1}{OH}+\frac{OK_2}{OH}=1$ because of $4\cdot\sin 18°\cdot\cos 36°=1$.
3. If $\alpha=120°$ and $\rho=1$ (regular hexagons are constructed outside the triangle ABC), then $K_2=H$ and K_1 is the center of the nine-point circle in the triangle XYZ.
4. If $\alpha=30°$ or $\alpha=150°$, the triangle XYZ is equilateral, so it has no Euler-line, but K_1, K_2 and G are still collinear.

Theorem 2. *If in the previous theorem A_1, B_1 and C_1 are the midpoints of the segments SP, RN respectively MQ and O_a, O_b, O_c are the circumcenters of the triangles ASP, BRN and CMQ respectively, then*

a) AA_1, BB_1 and CC_1 are concurrent in a point K_3 (see Figure 2).
b) AX, BY and CZ are concurrent in a point K_4 (see Figure 2).
c) A_1X, B_1Y and C_1Z are concurrent in a point K_5 (see Figure 3).
d) AO_a, BO_b and CO_c are concurrent in a point K_6 (see Figure 3).

In other words the triplet of triangles $(ABC, A_1B_1C_1, O_aO_bO_c)$ and $(XYZ, ABC, A_1B_1C_1)$ are formed by pairwise perspective triangles. Moreover the triangles from the second triplet are pairwise orthologic.

Proof. a) The equation of the line passing through the points Z_1 and Z_2 is

$$z\cdot(\overline{z_1}-\overline{z_2})-\overline{z}\cdot(z_1-z_2)+z_1\cdot\overline{z_2}-\overline{z_1}\cdot z_2=0,$$

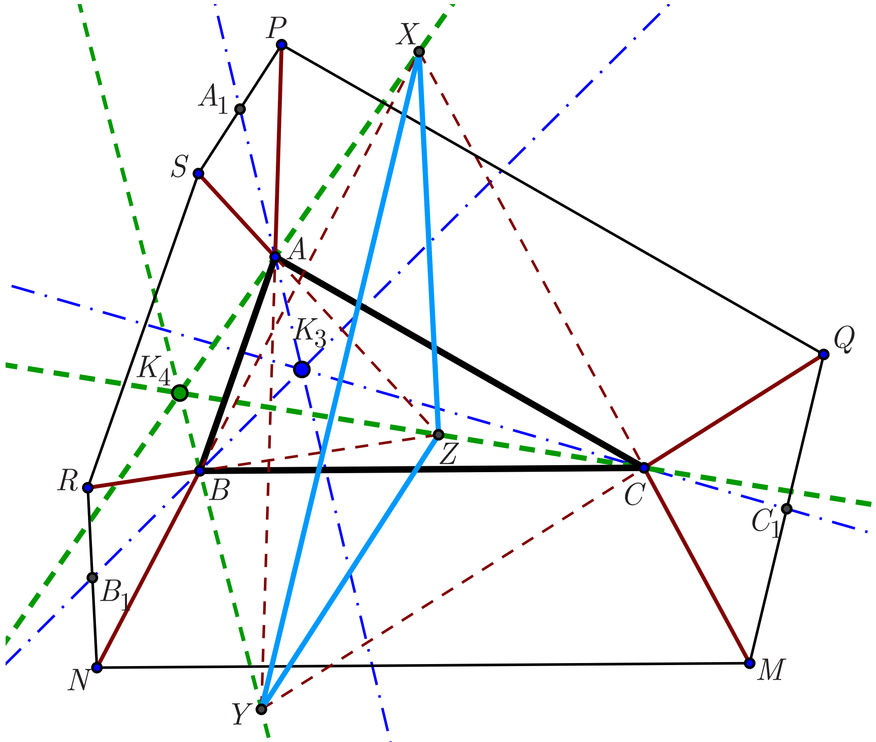

Figure 2: The perspectivity centers K_3 and K_4

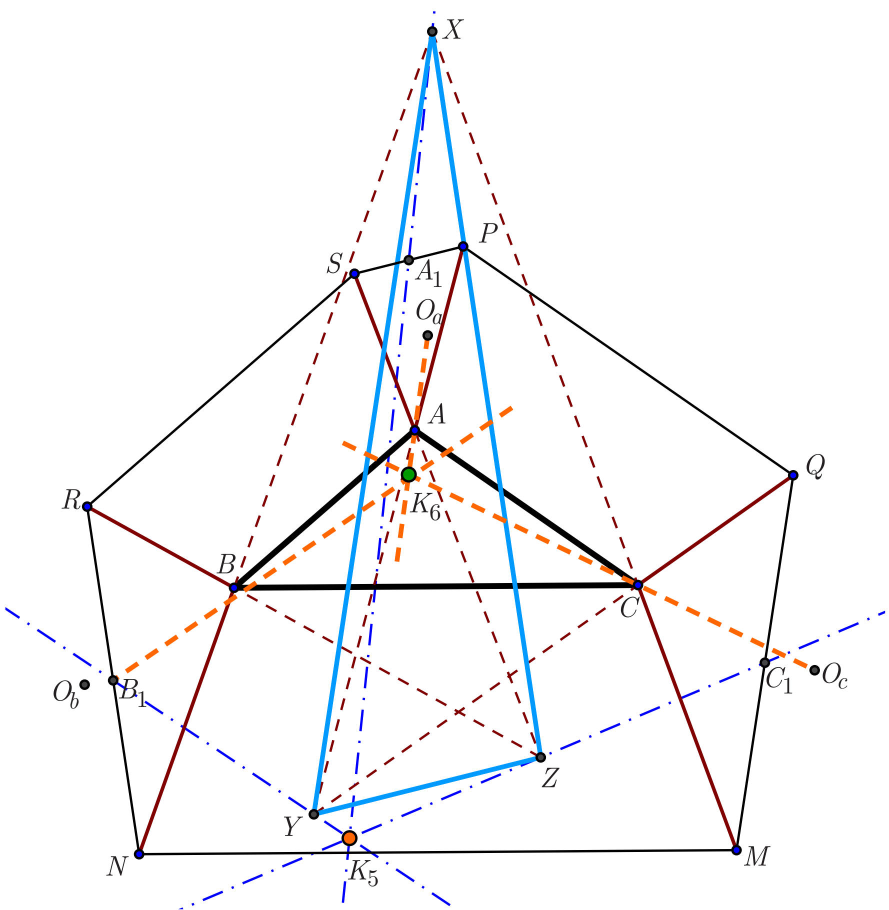

Figure 3: The perspectivity centers K_5 and K_6

so a necessary and sufficient condition for the lines AA_1, BB_1 and CC_1 to be concurrent is

$$\begin{vmatrix} \overline{a}-\overline{a_1} & a-a_1 & a\cdot\overline{a_1}-\overline{a}\cdot a_1 \\ \overline{b}-\overline{b_1} & b-b_1 & b\cdot\overline{b_1}-\overline{b}\cdot b_1 \\ \overline{c}-\overline{c_1} & c-c_1 & c\cdot\overline{c_1}-\overline{c}\cdot c_1 \end{vmatrix} = 0. \tag{1}$$

Based on Lemma 3 the triangles PRM and SNQ have the same centroid as ABC, so $A_1B_1C_1$ has also the same centroid. Hence the sum of elements in the first two columns of the determinant (1) is 0. This implies that it is sufficient to prove that

$$a\cdot\overline{a_1}-\overline{a}\cdot a_1+b\cdot\overline{b_1}-\overline{b}\cdot b_1+c\cdot\overline{c_1}-\overline{c}\cdot c_1=0.$$

This is equivalent to $S\in\mathbb{R}$, where $S=a\cdot\overline{a_1}+b\cdot\overline{b_1}+c\cdot\overline{c_1}$. For this reason we show that $S=\overline{S}$. But

$$\begin{aligned}\overline{2S} =&\overline{a}(p+s)+\overline{b}(r+n)+\overline{c}(m+q)\\ =&\overline{a}((2-\rho(u+\overline{u}))a+\rho uc+\rho\overline{u}b)+\overline{b}((2-\rho(u+\overline{u}))b+\rho ua+\rho\overline{u}c)\\ &+\overline{c}((2-\rho(u+\overline{u}))c+\rho ub+\rho\overline{u}a)\\ =&S_1+\rho\cdot[u\overline{a}c+u\overline{b}a+u\overline{c}b+\overline{ua}b+\overline{ub}c+\overline{uc}a],\end{aligned}$$

where $S_1=(\overline{a}a+\overline{b}b+\overline{c}c)(2-\rho(u+\overline{u}))$ is real and $S_2=u\overline{a}c+u\overline{b}a+u\overline{c}b+\overline{ua}b+\overline{ub}c+\overline{uc}a$ is also a real number, so $S\in\mathbb{R}$ and this completes the proof.

b) Since XYZ and ABC have the same centroid we can use the same technique as in a), so we need to prove that $S=a\overline{x}+b\overline{y}+c\overline{z}$ is real. Due to Lemma 3 we have

$$\begin{aligned}S =&(-x+\overline{u}^2y+u^2z)\overline{x}+(-y+\overline{u}^2z+u^2x)\overline{y}+(-z+\overline{u}^2x+u^2y)\overline{z}\\ =&S_3+\overline{u}^2(\overline{x}y+\overline{y}z+\overline{z}x)+u^2(x\overline{y}+y\overline{z}+z\overline{x}),\end{aligned}$$

where $S_3=-x\overline{x}-y\overline{y}-z\overline{z}$ is real and $S_4=\overline{u}^2(\overline{x}y+\overline{y}z+\overline{z}x)+u^2(x\overline{y}+y\overline{z}+z\overline{x})$ is also real. This completes the proof.

c) The triangles XYZ and $A_1B_1C_1$ have also the same centroid, so it is sufficient to prove $S_5\in\mathbb{R}$, where $S_5=a_1\cdot\overline{x}+b_1\cdot\overline{y}+c_1\cdot\overline{z}$.

Due to Lemma 3 we have

$$\begin{aligned}2S_5 =&(p+s)\overline{x}+(r+n)\overline{y}+(m+q)\overline{z}\\ =&(2-\rho(u+\overline{u})(a\overline{x}+b\overline{y}+c\overline{z})+\rho\overline{u}(b\overline{x}+c\overline{y}+a\overline{z})+\rho u(c\overline{x}+a\overline{y}+b\overline{z})\end{aligned}$$

The first term is real because of b), so we have to deal with the sum

$$S_6=\overline{u}(b\overline{x}+c\overline{y}+a\overline{z})+u(c\overline{x}+a\overline{y}+b\overline{z}).$$

Using Lemma 4 we obtain

$$S_6=|u|^2\cdot(u+\overline{u})(x\overline{x}+y\overline{y}+z\overline{z})+S_7,$$

where

$$S_7=(u^3-\overline{u})(y\overline{x}+z\overline{y}+x\overline{z})+(\overline{u}^3-u)(x\overline{y}+y\overline{z}+z\overline{x})\in\mathbb{R},$$

so S_5 is real and the proof is complete.

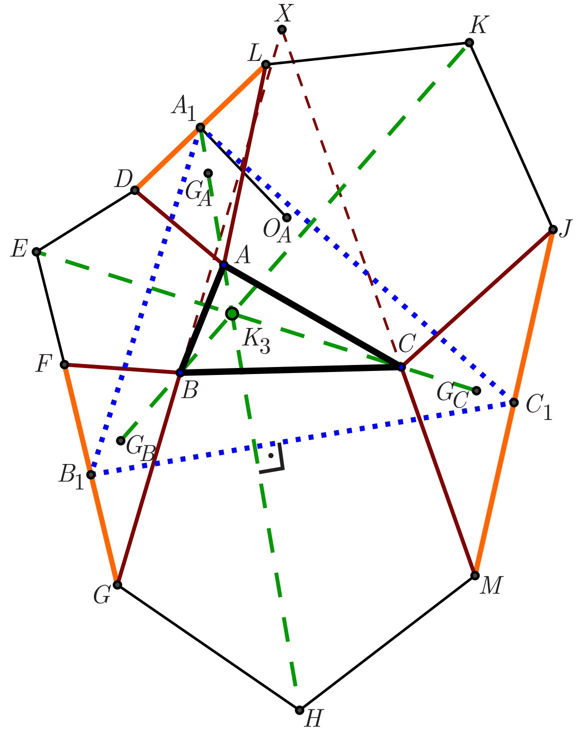

Figure 4: For pentagons K_3 is the orthocenter of the triangle $A_1B_1C_1$

d) Observe that according to the construction the lines AO_a, BO_b and CO_c are isogonals of the perpendiculars from A, B, C to SP, RN and MQ respectively. Due to Theorem 1, a) they are concurrent and the point K_6 is isogonal conjugate to K_1.

Since $SP \parallel YZ$, $RN \parallel XZ$ and $MQ \parallel XY$, the orthology of the pairs (XYZ, ABC) and $(XYZ, A_1B_1C_1)$ is proved in Theorem 1, a) and b).

The orthology of the triangles ABC and $A_1B_1C_1$ can be proved in a similar manner, the calculations are left to the reader. A special case (when $\alpha = 108°$ and $\rho = 1$) is proved in Theorem 3. □

Remark. The calculations can be made also by using the real product defined in [2].

Theorem 3. *On the sides of the triangle ABC we construct the regular pentagons $BCMHG$, $CALKJ$ and $ABFED$ in the exterior of the triangle (see Figure 4). Denote by G_A, G_B and G_C the centroid of the triangles ADL, BFG, respectively CMJ.*

a) The lines AG_A, BG_B and CG_C are concurent in a point K_3 and they pass through the points H, K, E. Moreover K_3 is the orthocenter of the triangle $A_1B_1C_1$, where A_1, B_1 and C_1 are the midpoints of the segments DL, FG and IJ respectively.

b) If $BG \cap CM = \{X\}$, $AL \cap CJ = \{Y\}$ and $DA \cap FB = \{Z\}$, O_A, O_B, O_C are the circumcenters of triangles BCX, CAY and ABZ, then O_A belongs to the perpendicular bisector of DL.

c) The triangles $O_AO_BO_C, XYZ$ and ABC have the same centroid, they are pairwise perspective and orthologic.

Proof. a) First we prove that $H \in AA_1$, where A_1 is the midpoint of DL. For this observe that H is the symmetric of X with respect to BC, hence $h = b + c - x$. But from Lemma 1 we have $x = \dfrac{1}{2\cos\alpha}(b \cdot u + c \cdot \overline{u})$, so $h - a = b + c - x - a$. But $a - a_1 = a - \frac{d+\ell}{2}$, and using Lemma 2, we obtain

$$\begin{aligned}2(a - a_1) =& 2a - (1 - \rho \cdot u) \cdot a - \rho \cdot u \cdot c - (1 - \rho \cdot \overline{u}) \cdot a - \rho \cdot \overline{u} \cdot b\\ =& \rho \cdot (u + \overline{u}) \cdot a - \rho(uc + \overline{u}b)\\ =& 2\rho \cdot \cos\alpha \cdot a - \rho \cdot (uc + \overline{u}b)\end{aligned}$$

From these relations, we deduce

$$2(a - a_1) = (-2\rho \cdot \cos\alpha)(h - a),$$

so A, A_1 and H are on the same line. To finish the proof it is sufficient to prove that $AA_1 \perp B_1C_1$. This is equivalent to $\frac{b_1 - c_1}{a - a_1} \in i \cdot \mathbb{R}$. From the above relations we have

$$2\cos\alpha \cdot (h - a) = 2\cos\alpha \cdot (b + c) - (b \cdot u + c \cdot \overline{u}) - 2\cos\alpha \cdot a$$

and multiplying with $-i \cdot \tan\alpha$ we obtain

$$-2i \cdot \sin\alpha \cdot (h - a) = 2i \sin\alpha \cdot a + b \cdot \overline{u} - c \cdot u + \frac{c - b}{\cos\alpha}.$$

On the other hand using Lemma 3, we have

$$\begin{aligned}2(b_1 - c_1) =& u \cdot a + (1 - u) \cdot b + (1 - \overline{u}) \cdot b + \overline{u} \cdot c - \overline{u} \cdot a - (1 - \overline{u}) \cdot c - u \cdot b - (1 - u) \cdot c\\ =& 2i \cdot \sin\alpha \cdot a + b \cdot \overline{u} - c \cdot u + (2 - 4\cos\alpha)(b - c).\end{aligned}$$

To complete the proof of this part we need to show that

$$2 - 4\cos\alpha = -\frac{1}{\cos\alpha}$$

if $\alpha = 108°$. This is equivalent to

$$4\sin^2\beta + 2\sin\beta - 1 = 0$$

for $\beta = 18°$ and this is a consequence of the relation $\sin 2\beta = \cos 3\beta$, which is true for $\beta = 18°$. From the proved relation we deduce that AA_1 is an altitude in the triangle $A_1B_1C_1$. Using a similar reasoning we obtain that BB_1 and CC_1 are also altitudes, so K_3 is the orthocenter of the triangle $A_1B_1C_1$.

b) As in Lemma 3 we have

$$o_a = \frac{b + c}{2} + i \cdot \frac{c - b}{2} \cdot \tan 54°,$$

so by using Lemma 3 for d and l we obtain

$$\begin{aligned}2(o_1 - a_1) =& -2a - b \cdot \overline{u} - c \cdot u + a(u + \overline{u})\\ =& -4a\sin^2 54° - b \cdot \overline{u} - c\overline{u} + b \cdot (1 - i\tan 54°) + c \cdot (1 + i\tan 54°)\end{aligned}$$

On the other hand we have

$$\begin{aligned}d - l =& b \cdot \overline{u} - c \cdot u - a \cdot \overline{u} + a \cdot u\\ =& 4a \cdot i \cdot \sin 54° \cdot \cos 54° + b \cdot \overline{u} - c \cdot u.\end{aligned}$$

These equalities suggest (by identifying the terms containing a) that

$$2(o_a - a_1) = (d - l) \cdot i \cdot \tan 54^\circ.$$

This can be proved with a straightforward calculation since $\alpha = 108^\circ = 2 \cdot 54^\circ$. This relation shows that $O_A A_1 \perp DL$, so O_A belongs to the perpendicular bisector of DL.

c) In the previous proof we saw that

$$o_a = \frac{b+c}{2} + i \cdot \tan 54^\circ \cdot \frac{c-b}{2}.$$

By adding up this relation with the corresponding relations for o_b and o_c we obtain $o_a + o_b + o_c = a + b + c$, so the triangles $O_A O_B O_C$ and ABC have the same centroid. Using that A_1, B_1 and C_1 are midpoints and Lemma 3, we have

$$2a_1 + 2b_1 + 2c_1 = (d + g + j) + (l + m + f) = 2(a + b + c),$$

so the triangle $A_1 B_1 C_1$ has the same centroid as the triangle ABC. □

Remark.

1. We can apply Sondat's second theorem (see [6] or [7]) for the pairs of bilogic triangles to obtain several pairs of perpendicular lines determined by the mutual intersections of the sides and the centers of orthology and also several triplets of centers.
2. As a further research it would be interesting to establish relations between the possible perspectivity and orthology centers, their coordinates and geometric loci if the parameters α and ρ vary.

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