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Quadrigon Geometry: Circumscribed Squares and Van Aubel Points

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Abstract. After a brief introduction on the quadrigon formal definition and the Van Aubel configuration, we present the main and original result of this work. The theorem estabilishes a connection between the Van Aubel configuration of a given quadrigon and the squares circumscribing the quadrigon. In particular, it states that the centers of the circumscribed squares coincide with the Van Aubel points. The proof is developed synthetically.

Two different solutions to the problem *to circumscribe a square to a given quadrigon* are then given.

Finally, a curious self-evident corollary regarding the six-point circle and the circumscribed rectangles of the quadrigon is presented.

Key Words: quadrigon, circumscribed squares, Van Aubel points, six-point circle

MSC 2020: 51F20, 51G05, 51M04, 51M15

1 Introduction

1.1 Quadrigon definition and circumscribed squares

A quadrigon is defined as a configuration of four random points in a plane and four lines joining these points in a cycle, where a cycle and its reverse cycle are considered the same [9](item QA-3QG1). A quadrangle is defined as a configuration of four random points that can be cyclically ordered in three ways (1-2-3-4, 1-2-4-3, 1-3-2-4). A quadrigon can be seen as the representation of one of these permutations.

A quadrilateral instead is defined as a configuration of four random lines that can also be cyclically ordered in 3 ways (1-2-3-4, 1-2-4-3, 1-3-2-4). Again a quadrigon can be seen as the representation of one of these permutations.

The geometrical problem *to circumscribe a square to a given quadrigon* is a classical one. A terrific solution can be deduced from [1, Problem 9, Page 8]. Altshiller-Court's formulation

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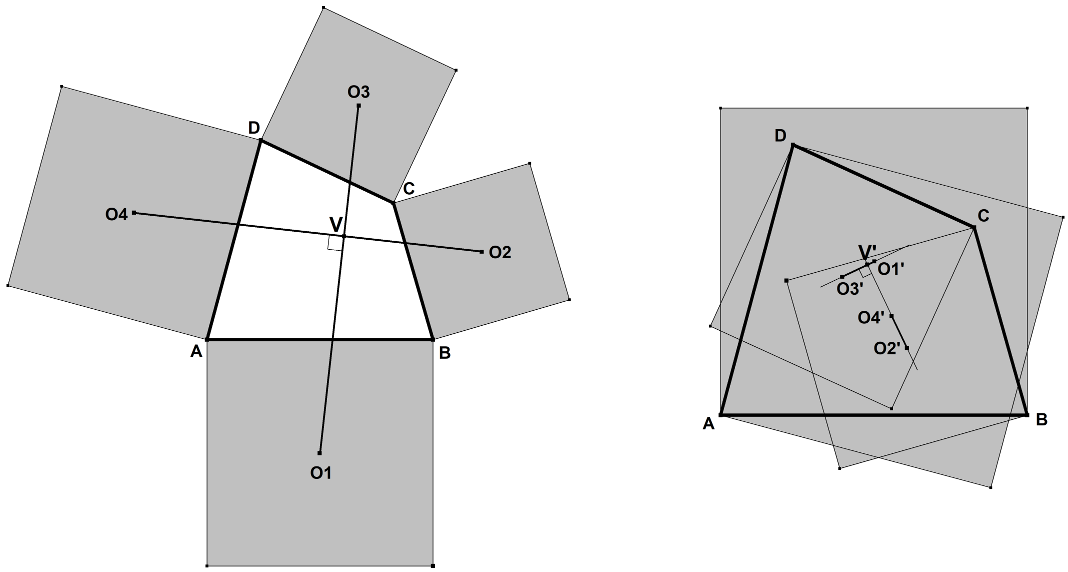

Figure 1: Van Aubel's theorem. External and Internal constructions.

is the following: *construct a square so that each side, or the side produced, shall pass through a given point.* It has to be noted that he dealt with four random points without order: a quadrangle. He found, in general, six solutions. Our formulation of the problem, in which the concept of a circumscribed square of a quadrigon is defined, is the following: *construct a square so that each side, or the side produced, shall pass through a vertex of a given quadrigon.* From the formal definitions of quadrigon and quadrangle, it is quite evident that a quadrangle contains three quadrigons [9] (item QA-3QG1). Therefore we expect, in general, two solutions.

1.2 The Van Aubel configuration

Van Aubel's theorem [7] states:

Theorem 1. *Given a convex quadrigon, on each side construct a square external to the quadrigon. Join the centers of the squares constructed over the opposite sides. Then the segments obtained are of equal length and orthogonal to each other.*

This theorem is quite general: it still holds true when the squares are erected internally on the sides of the given quadrigon and no matter if this quadrigon is convex, re-entrant or even crossed [2, Page 52] (in the latter case, external/internal ceases to have any meaning - so we must be careful with the square constructions [6]). The segments joining the centers of the squares erected either externally or internally on the opposite sides of the quadrigon has been referred to as Van Aubel segments in [6]. The outer Van Aubel segments, O_1O_3 and O_2O_4, and the inner Van Aubel segments, $O'_1O'_3$ and $O'_2O'_4$, are represented in Figure 1. The point of intersection of the outer Van Aubel segments, V, has been referred to as the first or outer Van Aubel point [6], [9], while the one of the inner Van Aubel segments, V', has been referred to as the second or inner Van Aubel point. We will use this nomenclature throughout the paper.

In [6], it has been proved that the Van Aubel points, the common midpoints of the inner and outer Van Aubel segments and the midpoints of the quadrigon diagonals are concyclic. The segment joining the midpoints of the quadrigon diagonals and the segment joining the

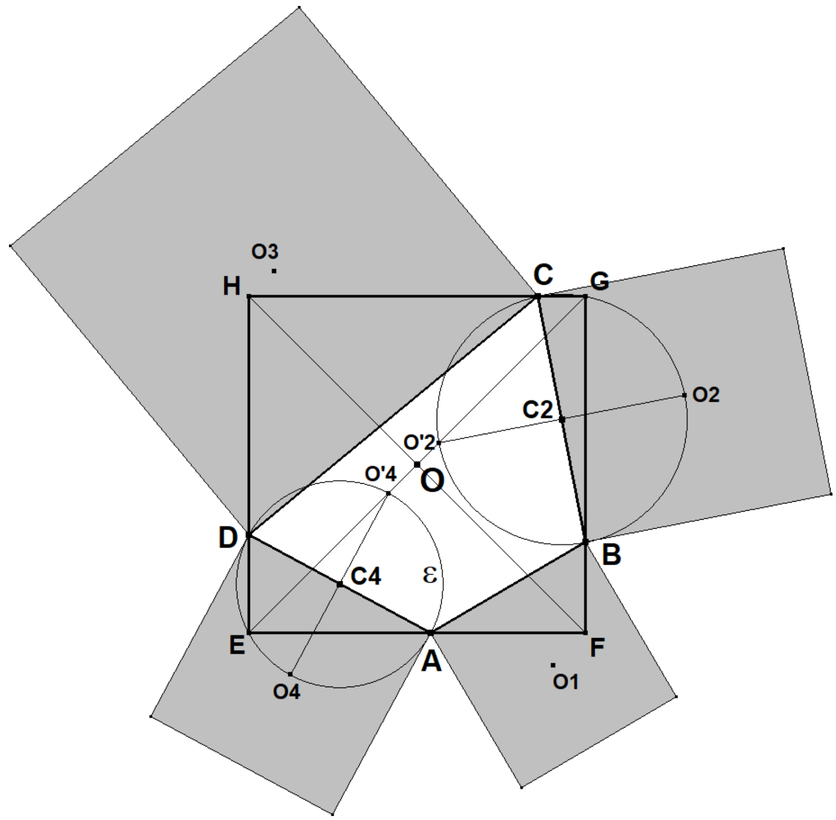

Figure 2: Proof sketch.

midpoints of the Van Aubel segments are two mutually orthogonal diameters of the circle. This circle will be referred to as the six-point circle.

2 Circumscribed squares and Van Aubel points

The following theorem represents the main result and original contribution of this work:

Theorem 2. *There are only two squares circumscribing a quadrigon and their centers coincide with the Van Aubel points.*

Proof. Let $ABCD$ be a given quadrigon. Let $EFGH$ be one of the circumscribed squares and O its center, as represented in Figure 2.

$EFGH$ is constructed such that its vertex E makes a right corner with AD, vertex F makes a right corner with AB, vertex G makes a right corner with BC and vertex H makes a right corner with CD. Thanks to Thales's converse theorem, E lies on circle ϵ with diameter AD. In this construction we choose E to lie externally to quadrigon $ABCD$ (on the same side as O_4 with respect to AD). Now, circle ϵ contains the center O_4 of the square constructed externally to the quadrigon over side AD and the center of the square constructed internally, O'_4 (again by Thales's converse theorem). Because E and O_4 are situated on the same half of the circle, we know that angles $\angle AEO'_4$ and $\angle AO_4O'_4$ subtend the same arc AO'_4, so they are both equal, thanks to the inscribed angle theorem [3]. They also measure the half of a right angle as $\angle AC_4O'_4 = \frac{\pi}{2}$ (trivially) and $\angle AEO'_4 = \angle AO_4O'_4 = \frac{1}{2}\angle AC_4O'_4$, thanks to proposition 20 from Euclid's Elements book III [4].

Since $\angle AEG$ is also equal to the half of a right angle (EG is the diagonal of square $EFGH$ with A on side EF) angles $\angle AEG$ and $\angle AEO'_4$ are the same and therefore E, G and O'_4 are collinear. In the same way, the center O'_2 of the internal constructed square over BC lies on the square diagonal EG. O'_2 and O'_4 are the endpoints of one of the inner Van Aubel segments which, for what just said, superimposes over the square diagonal EG. The same reasoning applies for the other Van Aubel segment $O'_1O'_3$ (not represented) which superimposes over the square diagonal FH. These two Van Aubel segments meet at the second (inner) Van Aubel point (not represented in the figure). It follows that the center of circumscribed square $EFGH$, O, coincides with the second Van Aubel point. Analogously, it can be proven that when O_4 is chosen (internally) on the other side of AD, the center of the other circumscribed square coincides with the first Van Aubel point.

With O_4 chosen on the external half circle ϵ, exactly one solution is found: the solution in which the diagonals of the squares superimpose with the inner Van Aubel segments, in this case. On the other hand, if O_4 is chosen on the internal half circle, exactly one other solution is found. For any other quadrigon forming right corners at E, F, G, H, its diagonal lines are not aligned with the given Van Aubel segments, so $\angle AEG$ will differ from $\frac{\pi}{4}$. We deduce that there are only two circumscribed squares about a quadrigon, whose centers are the Van Aubel points. $\square$

2.1 Discussion

Theorem 2.1 holds true in the general case when the inner and outer Van Aubel segments are defined. There is a particular case that should be mentioned: when the given quadrigon $ABCD$ is iso-ortho-diagonal (its diagonals are of equal length and orthogonal to each other), the inner Van Aubel segments degenerate into points and the inner Van Aubel point is not defined. In this case, any quadrigon forming right corners at E, F, G, H, will be a square [1, Problem 9, Page 8], [5]. It can be proved that all the possible circumscribed squares of a given iso-ortho-diagonal quadrigon are in a one-to-one correspondence with the six-point circle points of the quadrigon (the centers of the squares). We do not present here a proof of these latest statements that can be left for a future work or as a nice entertainment exercise for the readers. Anyway, this particular case has been extensively treated in [5].

In contrast to the particular configuration chosen for the proof, when the form of the quadrigon changes (being convex, concave or even crossed) the location of the vertexes for the quadrigon on the sides of the circumscribed square are not easily predicted, but the theorem still holds true. Therefore we prefer not making any statement regarding the relation between the position of the vertexes of the quadrigon on the square sides and which Van Aubel point will be its center. Though, we point out that the Van Aubel points hierarchy established with our nomenclature is merely historical and quite meaningless: consider a crossed quadrigon, for example, in such a case the external/internal constructions are not definable [6].

2.2 How to circumscribe a square to a given quadrigon

A solution to this problem can be directly deduced from Altshiller-Court's College Geometry [1, Problem 9, Page 8]. Here we propose two alternative solutions.

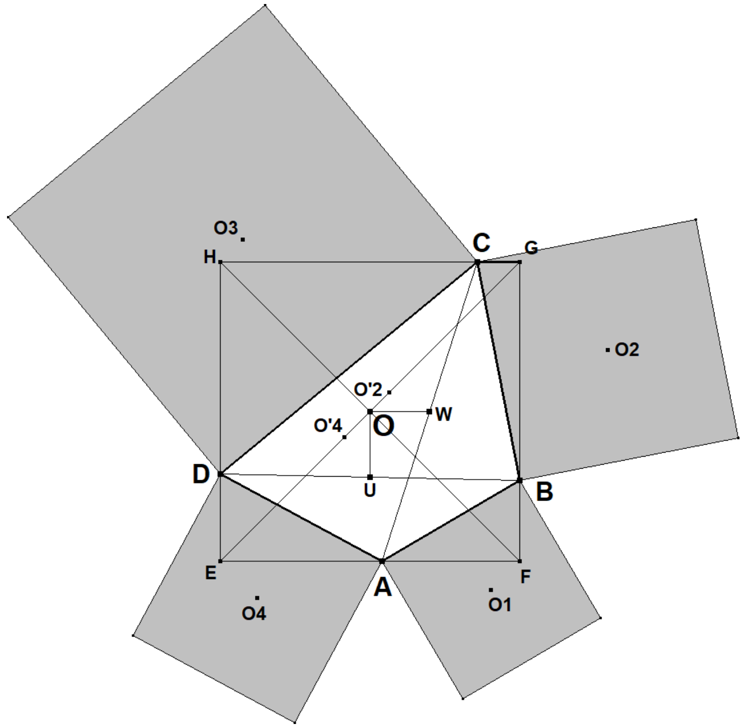

Figure 3: A lemma for a second alternative construction.

Construction-1 [8]

The presented proof can be exploited straightaway for a construction of the circumscribed squares: draw the circles with the sides of the quadrigon as diameters. Draw the midpoints of the internal half circles, O_2' and O_4' represented in Figure 2. Draw the lines through the midpoints of the internal half circles constructed on opposite sides, $O_2'O_4'$, obtaining the diagonal lines of the circumscribed square with center the second Van Aubel point. The other intersections of these diagonal lines with the circles, E and G, are the vertexes of this circumscribed square. If we repeat the construction using the midpoints of the external half circles, we obtain the circumscribed square with center the first Van Aubel point.

Construction-2

Let U and W be the midpoints of the quadrigon diagonals BD and AC, respectively (see Figure 3). Since O is the midpoint of EG and U is the midpoint of BD (by definition) and since EG and BD are spanned between the two parallel sides of the circumscribed squares EH and GF, it must be that OU is parallel to these two sides, by Thales's intercept theorem. Analogously, OW is parallel to the other two sides EF and GH (WO and UO meet at $O = V'$ forming a right angle). With this consideration, we deduce the following alternative construction.

Once the first Van Aubel point, V, is drawn, draw lines through A and C parallel to WV and, through B and D parallel to UV, like in Figure 4. The circumscribed square $QRST$ with center V and diagonals superimposed on the Van Aubel segments O_1O_3 and O_2O_4 is formed. Analogously, the circumscribed square with center V' and diagonals superimposed on the inner Van Aubel segments $O_1'O_3'$ and $O_2'O_4'$ is formed.

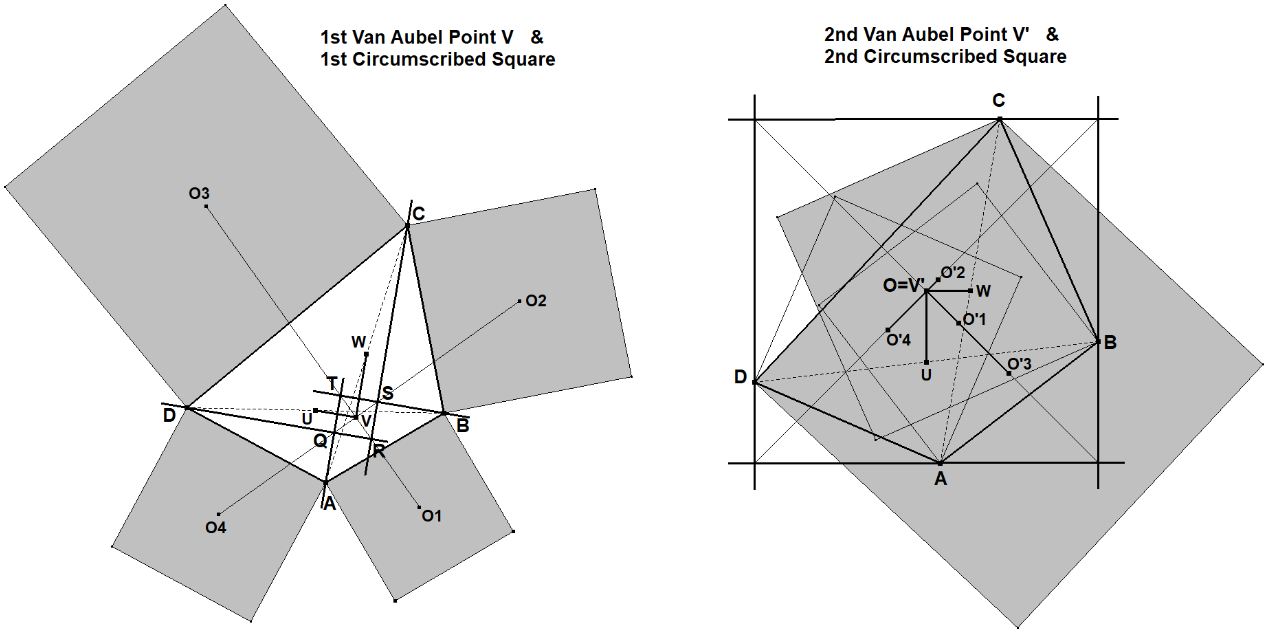

Figure 4: Construction-2 of the circumscribed squares.

3 Epilogue

Incidentally, as the centers of the circumscribed squares coincide with the Van Aubel's points, V and V', which make right angles with U and W (the midpoints of the diagonals of quadrigon $ABCD$), they lie on the six-point circle.

This circle also allows us to construct all possible rectangles circumscribed of a given quadrigon. For any point P on this circle, the directions UP and WP are defined. By the same arguments developed for construction-2 of the circumscribed squares, we deduce that P will be the midpoint of the diagonal EG of the corresponding circumscribed rectangle (see Figure 5). So it will be the center of the corresponding circumscribed rectangle. Also, with such a construction, it is not difficult to prove that all the possible rectangles circumscribing the given quadrigon are in a one-to-one correspondence with the six-point circle points (the centers of the rectangles). We can then circumscribe a rectangle to quadrigon $ABCD$ and center P on the six-point circle by drawing lines through A and C parallel to UP and lines through B and D parallel to WP. Again, from the proof discussion presented in section 2, there are only two points P on the six-point circle for which the rectangles are a square, being the Van Aubel points. Other way, we might say that the six-point circle is the locus of the centers of the circumscribed rectangles of the quadrigon.

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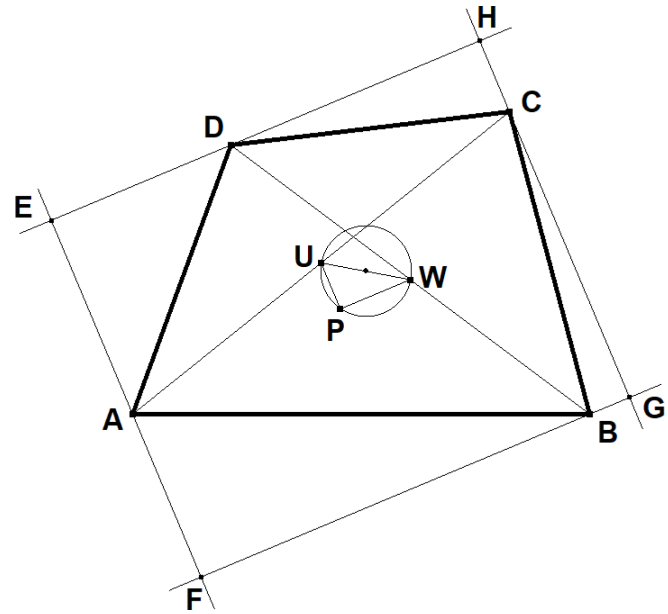

Figure 5: Circumscribed rectangle with center on the six-point circle

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