# A Quadratic Mapping Related to Frégier's Theorem and Some Generalisations 

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#### Abstract

We start with a special Steiner's generation of a conic section based on a quadratic mapping, which is connected to a well-known theorem of Frégier. The paper generalises this construction of a conic to higher dimensions and to algebraic mappings of higher order.


Key Words: conic section, Steiner's generation, Frégier point, quadratic mapping MSC 2020: 51 Mxx (primary), 51N15, 51N20, 51N35

## 1 Introduction: Frégier's Involution

The Theorem of Frégier (see for example [11]) states that,
given a conic $c$ in the Euclidean plane, all right angled triangles inscribed into $c$ such that the vertex $G$ at their right angle is fixed, have hypotenuses passing through one point, the Frégier point $F$ of $G$ with respect to $c$.

With this theorem we connect an involutoric quadratic mapping $\varphi$, which is called "Frégier involution", c.f. [10].

We put $F$ to the origin of a Cartesian frame and $G$ as unit point of the $x$-axis and use homogeneous coordinates, as

$$
F=(1,0,0) \mathbb{R}, \quad G=(1,1,0) \mathbb{R}, \quad P=(1, p, q) \mathbb{R}, \quad P^{\prime}:=P^{\varphi}=\left(1, p^{\prime}, q^{\prime}\right) \mathbb{R}
$$

We must intersect line $f=F P$ with a line $g^{\prime} \perp g=G P$ through $G$ (see Figure 1):

$$
f \ldots y=\frac{q}{p} x, g^{\prime} \ldots y=-\frac{p-1}{q}(x-1) \quad \Longrightarrow \quad p^{\prime}=\frac{p(p-1)}{p^{2}+q^{2}-p}, \quad q^{\prime}=\frac{q(p-1)}{p^{2}+q^{2}-p} .
$$



Figure 1: The Frégier mapping $\varphi$ transforms a line $l$ to a conic $l^{\varphi}$.


Figure 2: By replacing $\gamma$ by rotations $\gamma(\alpha)$, $\left(-\frac{\pi}{2}<\alpha<\frac{\pi}{2}\right)$, the set of $\varphi(\alpha)$ images of a line $l$ is a pencil of conics.

Omitting ideal points to be mapped by $\varphi$ we get the coordinate representation of $\varphi$ as

$$
\begin{equation*}
\varphi:(1, p, q) \mathbb{R} \mapsto\left(p^{2}+q^{2}-p, p(p-1), q(p-1)\right) \mathbb{R} \tag{1}
\end{equation*}
$$

It is obvious, how to extend (1) to ideal points, too, when using the coordinates $\left(x_{0}, x_{1}, x_{2}\right) \mathbb{R}$ for a general point $P_{1}$ in a projectively extended Euclidean plane:

$$
\varphi:\left(x_{0}, x_{1}, x_{2}\right) \mathbb{R} \mapsto\left(x_{1}^{2}+x_{2}^{2}-x_{0} x_{1}, x_{1}\left(x_{1}-x_{0}\right), x_{2}\left(x_{1}-x_{0}\right)\right) \mathbb{R}
$$

Remark 1. As a special Euclidean case we consider the ideal line $u$ of the plane of action as directrix line $l$. Here we get $f \| g_{1}, f \perp g_{2}$, and Steiner's construction becomes the interpretation as the "Theorem of Thales". For no other position of line $l$, there occurs a circle as $\varphi$-image of a line.

Furthermore, since the Frégier mapping $\varphi$ is involutoric, the Thales circle $u^{\prime}$ over segment $[F, G]$ maps to the ideal line $u^{\prime \prime}=u$. We conclude that the number of real intersections of a directrix line $l$ with $u^{\prime}$ is responsible for the type of the image conic $l^{\prime}$. In case of $l$ intersecting line $F G$ in an inner point of segment $[F, G]$, this conic $l^{\prime}$ can only be a hyperbola.

The singularity object of $\varphi$ consists of the points $F$ and the twice to count point $G$, and line $F G(y=0)$ with multiplicity two and the line normal to $F G$ through $G(x=1)$, (c.f. [10]). As the construction is based on right angles which define an involutoric projectivity $\gamma$ in the pencil of lines through $G$, the mapping $\varphi$ is involutoric, too. In [10] it is therefore called "Frégier involution". The isotropic lines through $G$ are the only fixed lines of $\gamma$, therefore the points $P=(1,1, \pm i) \mathbb{R}$ are the only fixed points of $\varphi$.

We focus on the fact that $\varphi$ maps a line $l,(G, F \notin l)$, to a regular conic through $G, F$ (intersecting $F G$ orthogonally in $G$ ), see Figure 1. Obviously the pencils $F(f)$ and $G\left(g^{\prime}\right)$ are related in a projectivity $\sigma$, which is the product of a perspectivity $\pi: g \mapsto f$ (with perspectivity axis $l$ ) and the rotation $\gamma: g \mapsto g^{\prime}$. Therefore, $l^{\varphi}$ is the result of Steiner's generation of $a$ conic via projective pencils of lines.

In the following sections we shall deal with generalisations of this Steiner-construction based on projective pencils at different places of action.

A first, still Euclidean generalisation of the construction shown in Figure 1 might replace the right-angle rotation $\gamma$ by an arbitrary rotation, see Figure 2. As a rather projective geometric generalisation, which directly follows from that, we replace line $l$ by a conic and


Figure 3: $l$ intersects $G F$ in an inner point of $[G, F], m$ is an ellipse.


Figure 4: For $l \| G F$ the pencil of conics generated by $\gamma(\alpha)$ has collinear midpoints.
$\gamma$ by an arbitrary projectivity. This way it is possible to generalise the quadratic Frégier involution also to higher degree transformations.

A second generalisation deals with 3D-versions of the original construction, Figure 1. Here we can expect quadratic mappings again, which can be further generalised to higher degree algebraic mappings.

Finally we consider the original construction in non-Euclidean and Minkowskian planes. While, because of the projective geometric nature of the Steiner generation, non-Euclidean versions do not deliver essentially new results, Minkowskian versions depend on the gauge-set and the orthogonality concept.

As the topic belongs to Elementary Geometry in a wider sense, we might proceed in the following way: What do we observe, if modifying the original recipe? Can we extract properties from these observations? This means to find sufficiently good arguments for an observation such that one can consider it finally as a theorem-like statement. Due to classical results in Projective Geometry (see e.g. [4] or [2]) and of conics (see e.g. [5]) it is possible to deduce such statements via coordinate-free synthetic reasoning, a proving method, which is typical for (Elementary) Geometry.

## 2 Euclidean Versions of Frégier's Mapping and Steiner's construction

We start with replacing the $90^{\circ}$-rotation $\gamma: g \rightarrow g^{\prime}$ by a rotation $\gamma(\alpha)$ by an arbitrary but fixed angle $\alpha$. As such a rotation still is a projectivity within the pencil $\{g\}$, the final result of $l$ due to Steiner's construction is again a conic through $F, G$. By varying $\alpha$ we receive a pencil of conics, see Figure 2. This pencil contains $l \cup G F$ as real singular $2^{\text {nd }}$ degree curve. Besides the fixed points $F, G$, the fixed points on $l$ are conjugate imaginary.

The Frégier mappings $\varphi(\alpha),-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, modified by the rotations $\gamma(\alpha)$ mentioned above, deliver a pencil of conics as the images of line $l$, and the mappings $\varphi(\alpha)$ map also each conic of this pencil again to this fixed pencil. The poles of a line with respect to a pencil of conics fulfil a conic, (see [5]), therefore, the conics of our fixed pencil have centres fulfilling a conic $m$ (see Figures 2 and 3). According to Remark 1, if $l$ intersects $G F$ in an inner point of $[G, F]$, the fixed pencil consists only of hyperbolas. As there is no parabola, $m$ has no ideal points and therefore is an ellipse. If $l$ is parallel to $G F$, then $m$ degenerates into a line, (see Figure 4).


Figure 5: For $G, F$ symmetric to $l$ Steiner's construction $\gamma(\alpha)$ generates a pencil of concentric equilateral hyperbolas.


Figure 6: Steiner's construction in the case of $F$ being an ideal point. The midpoints of the resulting conics fulfil a parabola $m$.

Remark 2. Again we consider the ideal line $u$ of the plane of action as directrix line $l$ and apply the Frégier mappings $\varphi(\alpha)$ to it. Here Steiner's construction becomes the interpretation as the "Inscribed Angle Theorem". The pencil of conics then becomes a pencil of circles. For no other position of line $l$, due to rotations $\gamma(\alpha)$, there occurs a circle in the resulting pencil of conics.

Remark 3. If we choose line $l$ as the symmetry line of segment $[G, F]$, the pencils of lines $\{f\}$, $\left\{g_{1}\right\},\left\{g_{2}(\alpha)\right\}$ are congruent, and Steiner's construction based on $\varphi(\alpha)$ delivers a pencil of equilateral hyperbolas, see Figure 5. In this case $m$ degenerates to a point, i.e., the hyperbolas are concentric.

Remark 4. Let $F$ be an ideal point. By a little synthetic reasoning we get that the conics, which are the images of line $l$ under $\varphi(\alpha)$, then have midpoints on a parabola $m$ through $F$ and $l \cap G F$, see Figure 6.

As long as we use Euclidean rotations at $G$, putting $G$ into an ideal point does not make sense.
Remark 5. As a rather trivial generalisation we mention the dual $\varphi^{*}$ of the Frégier involution $\varphi$ : Let a pair of lines $f=F^{*}, g=G^{*}$ and an involutoric projectivity $\gamma: g \rightarrow g$ be given. A line $p=P^{*}$ intersects $f$ in $F$ and $g$ in $G$. Its $\varphi^{*}$-image $p^{\prime}$ connects $F$ with $G^{\gamma} \in g$. Then $\varphi^{*}$ maps a point $l^{*}=L$, considered as a pencil of lines, to the set of tangents of a conic $L^{\prime}$. Figure 7, left, shows the generic case of the givens in a projectively closed Euclidean plane. Thereby involution $\gamma$ is the reflection in $g$ with fixed point $G_{0}$. Figure 7, right, shows a case, where $g$ is the ideal line of the plane and $\gamma$ the "absolute involution" ruling Euclidean orthogonality.

This allows to interpret a well-known classical kinematic construction of a parabola as the dual of a Frégier mapping: The $\varphi^{*}$-image of a point $L$ is the envelope of a leg of a right angle, if the vertex slides along a line $f$, while the other leg passes through $L$.

## 3 Generalisations of Frégier's Mapping and its Steiner Construction

Remark 4 in the former section indicates to replace $\gamma(\alpha)$ by adding an arbitrary projectivity $\delta:\{g\} \mapsto\{h\}$ from the pencil $G:\{g\}$ to a pencil $H:\{h\}$. A generic point $P($ with $P F:=f)$


Figure 7: Examples of dual Frégier mappings $\varphi^{*}(f, g, \gamma)$ applied to a point $L$. Left: The involution $\gamma: g \rightarrow g$ is the reflection at $G_{0} \in g$. Right: $g$ is the ideal line and $\gamma: g \rightarrow g$ the "absolute involution."


Figure 8: Projective generalisation to Frégier's involution.
is then mapped to $P^{\varphi}=f \cap h$. When mapping a directrix line $l$ we will receive a conic $l^{\varphi}$ because Steiner's construction of a conic via projective pencils of lines still remains valid. Therefore, $\varphi: P \mapsto P^{\varphi}$ is still a quadratic mapping. It is a projective generalisation to Frégier's involution. For the construction of $\delta$ one could e.g. use a "Steiner conic" $s$ (c.f. [2]) through $G$ and $H$, see Figure 8. (In Figure 8 a circle acts as Steiner conic $s$ ). It turns out that a line $l$ is mapped to a conic $l^{\varphi}$ though $F, H$ and the intersections $l \cap s$.

From Figure 8 we derive the following generalisations:

1) Let points $F, G$ and $H$ and an arbitrary conic $s$ with $H \in s, G \notin s$, be given and adapt the mapping $\varphi$ and construction in Figure 8 in the following way: A generic point $P$ defines lines $f:=F P, g:=G P$ and the intersections $\{1,2\}=s \cap g$. Therewith, we get a pair of image points of $P$, which are defined as $P_{1}^{\varphi}:=f \cap H 1, P_{2}^{\varphi}:=f \cap H 2$, see Figure 9 , and the mapping $\varphi: P \rightarrow\left\{P_{1}^{\varphi}, P_{2}^{\varphi}\right\}$ is of third degree.

In Figure 9, left, $G$ is chosen as the centre of a circle $s$, even so the construction is purely of projective nature. If $G$ is an inner point of the conic $s$, a suitable collineation transforms the givens to this special Euclidean situation. If $G$ is outside $s$, a suitable collineation transforms these givens to an ideal point $G$ and a circle $s$, (Figure 8, right). The $\varphi$-image of a line $l$ is a rational cubic $l^{\varphi}$ with a double point (or cusp or isolated node) at $H$, it passes through $F$ and the intersections of $l$ and $G F$ with $s$. (If one of these intersection points is collinear with $F H$, then $l^{\varphi}$ degenerates into $F H$ and a conic).
2) Similar to 1) we now put $G \in s, H \notin s$ and adapt the mapping $\varphi$ and the construction


Figure 9: Cubic analogues of Frégier's involution.


Figure 10: Another cubic analogue of Frégier's involution.


Figure 11: A fourth-degree analogue of Frégier's involution
in Figure 8 in the following way: A generic point $P$ defines lines $f:=F P, g:=G P$ and $g$ intersects $s \backslash\{G\}$ in a single point 1. Therewith, we get the image point $P^{\varphi}$ of $P$ as $P^{\varphi}:=f \cap H 1$, see Figure 10. As can be seen from the image $l^{\varphi}$ of a generic line $l$ the mapping $\varphi: P \rightarrow P^{\varphi}$ is of third degree and not involutoric.
3) Now we start with $G \notin s, H \notin s$. As in Case 1) the analogue construction generates a mapping $\varphi: P \rightarrow\left\{P_{1}^{\varphi}, P_{2}^{\varphi}\right\}$ with $P_{1}^{\varphi}:=f \cap H 1, P_{2}^{\varphi}:=f \cap H 2$, Figure 11. This figure also shows that the image $l^{\varphi}$ of a generic line $l$ is a rational curve of $4^{\text {th }}$ degree. It has 3 double points (in algebraic sense) at $F, H$ and the third at line $l$. It passes through the intersections of $l \cap s$ and $G F \cap s$.

Remark 6. The constructions of the curves $l^{\varphi}$ in Figures 8-11 allow a kinematic interpretation. There occur slider-crank mechanisms, which make tangent constructions of the $l^{\varphi}$ possible. This rather page-consuming topic will be omitted here.

## 4 nD-versions of Frégier's Involution and Steiner's Construction

In this section we extend the quadratic mapping (1) to higher dimensions. The generalisations shown in Figures 8-11 are then "prototypes" for obvious further generalisations.

1) Let $F, G$ and $P$ be arbitrary points in a Euclidean $n$-space, $f:=F P, g:=G P$. We define a mapping $\varphi: P \mapsto P^{\prime}$ by $P^{\prime}:=P^{\varphi}=f \cap \Gamma$ with hyperplane $\Gamma \perp g$. If $F$ is the origin


Figure 12: The quadratic 3D-Frégier-involution $\varphi$ applied to a line $l$.


Figure 13: The 3D-Frégier-involution $\varphi$ applied to a line $l$, visualised as top-view projection onto plane $F \vee l$.
of a Cartesian frame and if we write vectors from $F$ to a point $X$ as $\vec{X}$ and from a point $Y$ to $X$ as $\overrightarrow{Y X}$, then, with $F=(0,0,0), G=(1,0,0)$ and

$$
\Gamma \ldots(\vec{X}-\vec{G}) \cdot(\vec{P}-\vec{G})=0, \quad f \ldots \vec{X}=t \vec{P}, t \in \mathbb{R}
$$

follows

$$
\begin{equation*}
\overrightarrow{P^{\prime}}=\frac{\vec{G}(\vec{P}-\vec{G})}{\vec{P}(\vec{P}-\vec{G})} \cdot \vec{P} \tag{2}
\end{equation*}
$$

as the $n$-dimensional equivalent to (1), and $\varphi$ turns out to be a quadratic mapping for any dimension $n$. A line $l \subset \mathbb{R}^{3}$ with $X(r) \in l, \vec{X}(t)=\vec{P}+r \vec{Q}$ transforms into $\vec{X}^{\prime}(r)$ as

$$
\vec{X}^{\prime}(r)=\left(\begin{array}{l}
x_{1}^{\prime}(r)  \tag{3}\\
x_{2}^{\prime}(r) \\
x_{3}^{\prime}(r)
\end{array}\right)=\frac{p_{1}+r q_{1}-1}{\sum_{i=1}^{3}\left(p_{i}+r q_{i}\right)^{2}-\left(p_{1}+r q_{1}\right)}\left(\begin{array}{c}
p_{1}+r q_{1} \\
p_{2}+r q_{2} \\
p_{3}+r q_{3}
\end{array}\right),
$$

and we show such an image of a line in Figure 12 for $n=3$.
For the case $n=2$ the $\varphi$-image of the ideal line is the Thales-circle over segment $[F, G]$, for dimensions $n>2$ it follows therefore that the image of the ideal hyperplane becomes the Thales-hypersphere over $[F, G]$.

Remark 7. Because $f(P) \subset F \vee l$, the image curve $l^{\varphi}$ lies in the plane $F \vee l$. This allows a construction of $l^{\varphi}$ via the antipolarity of the distance-circle $d_{G}$ of $G$ in the plane $F \vee l$ with simple descriptive geometric methods (and again Steiner's construction of a conic). This antipolarity $\pi$ rules the orthogonality between lines and planes through $G$, c.f. [2]. In Figure 13 the top-view projection onto plane $F \vee l$ is shown.

Figure 13 gives rise to a new possibility for a generalisation of the $n D$-Frégier-involution $\varphi$ : Replace the singular polarity, ruling the orthogonality within the bundle of lines and (hyper-) planes through $G$ by a regular one $\pi: P \rightarrow \Gamma_{P}\left(\Gamma_{P}\right.$ the "polar hyperplane" to $\left.P\right)$. Then $P^{\prime}=P^{\varphi}:=F P \cap \Gamma_{P}$, and we again get a quadratic mapping $\varphi$.
2) Another generalisation might replace the hyperplane $\Gamma$ (used for (1)) by a hypercone of revolution $\Gamma(\alpha)$ with the (one-dimensional) axis $g=G P$ and halve apex angle $\alpha$, and we intersect it with $f=F P$ getting two image points $P^{\prime}, P^{\prime \prime}$ of $P$. (By the way, this gives a new


Figure 14: Frégier-mapping based on a hyperbolic line-congruence $\mathcal{F}$ (axes $f_{1}, f_{2}$ and the right-angle involution $\pi$ within the pencil of planes through line $g$ ).
interpretation of the Figures 3-6, when taking the rotations by $\alpha$ and $-\alpha$ as forming a "cone in the plane".) Obviously, as equations of $\Gamma(\alpha)$ and of $f$ and with $F$ as origin one has

$$
\Gamma(\alpha) \ldots \frac{(\vec{X}-\vec{G})(\vec{P}-\vec{G})}{|\vec{X}-\vec{G}||\vec{P}-\vec{G}|}=\cos \alpha, \quad f \ldots \vec{X}=t \vec{P}, t \in \mathbb{R}
$$

3) More abstractly formulated, the original Frégier-involution is based on a linear set of lines, namely the pencil $\{F P\}$, and a singular polarity, namely the right-angle involution within the pencil $\{G P\}$. This interpretation opens up for a set of further generalisations for $n$-dimensional projective spaces $\mathbf{P}^{n}$ :

Definition 1. In $\mathbf{P}^{n}$ let a linear set of lines $\mathcal{F}=\{f\}$ and a regular or singular polarity $\pi$ be given. A Frégier-mapping $\varphi: \mathbf{P}^{n} \rightarrow \mathbf{P}^{n}$ maps a generic point $P$ (of the space spanned by $F$ ) to the intersection $P^{\prime}$ of the line $f_{P} \subset \mathcal{F}$ with the polar hyperplane $P^{\pi}$ of $P$

$$
\begin{equation*}
P^{\prime}=P^{\varphi}:=f_{P} \cap P^{\pi} \tag{4}
\end{equation*}
$$

For example, in the case of the projectively closed Euclidean 3 -space $\mathbf{P}_{e}^{3}$ we use a linear line-congruence $\mathcal{F}$, (i.e. a hyperbolic or parabolic or elliptic net, see [8]) instead of $\{F P\}$, and we replace the right-angle involution in $\{G P\}$ e.g. by a singular polarity $\pi$ with a line $g$ as exceptional set of points, ( $\pi$ maps each plane through $g$ to its orthogonal plane through $g$ and we visualise this in Figure 14.) The Frégier-image of a generic line $l$ is a curve of third degree $l^{\varphi} \subset \mathbf{P}_{e}^{3}$.

To end with this sort of generalisations we mention that it is possible to generate algebraic Frégier mappings of higher degree, when admitting $\mathcal{F}=\{f\}$ to be an algebraic manifold of lines in $\mathbf{P}^{n}$, meaning that through each proper point $P$ there are, in the algebraic sense, $k$ lines $f_{P}^{i}, i=1 \ldots k$. Furthermore, if one replaces the (real or complex, regular or singular) hyperquadric $\Gamma$ defining a polar system $\pi$ by an algebraic hypersurface $\Gamma^{(m)}$ of degree $m$, the former polar hyperplane $P^{\pi}$ has to be replaced by the "first polar hyperplane" $\Gamma_{P}^{(m-1)}$

$$
\begin{equation*}
P \mapsto\left\{P_{1}^{\prime}, \ldots, P_{k(m-1)}^{\prime}\right\}=P^{\varphi}=\left\{f_{P}^{i} \cap \Gamma_{P}^{(m-1)}, i=1 \ldots k\right\} \tag{5}
\end{equation*}
$$

## 5 Cayley-Klein and Minkowskian Versions of Frégier's Mapping

a) Classical Non-Euclidean Spaces We have covered this topic with Definition 1 already in projective planes, and it is therefore obvious that all sub-geometries of the projective


Figure 15: The Frégier-image of a line $l$ in a hyperbolic plane (Cayley-Klein model).


Figure 16: Frégier image of a line $l$ in a Minkowski plane $\mathcal{M}_{c}$ with unit-circle $c$ and Birkhoff left-orthogonality $\dashv$.
geometry will deliver a quadratic Frégier involution $\varphi$. We content ourselves with visualising the classical Frégier-involution $\varphi$ in a hyperbolic plane, Figure 15, as an example of a classical Cayley-Klein geometry.

Similar to the Euclidean plane (Figure 1), the type of the conic $l^{\varphi}$ depends on the position of $l$ with respect to the $\varphi$-image $\omega^{\prime}$ of the "conic at infinity" $\omega$, the limit conic of the hyperbolic plane. For proper points $F, G$ the curve $\omega^{\prime}$ is rational of fourth degree and lies completely within the hyperbolic disk. It has a node in $F$ and touches itself at $G$ with a tangent normal to $F G$, see Figure 15. The $\varphi$-image of line $l$, which intersects $F G$ at an inner point of segment $[F, G]$ will therefore always intersect $\omega$ in four points.

It is obvious, how to adapt the generalisations above to Cayley-Klein-spaces, so we can omit further considerations of this topic.
b) Minkowski Planes Finally, we introduce Frégier's mapping in (projectively enclosed) affine planes endowed with a Banach-Minkowski norm, see [9]. Well-known examples for norms in an affine plane endowed with a coordinate frame are the so-called $p$-norms $\|(x, y)\|_{p}:=\sqrt[p]{x^{p}+y^{p}}$, $p \in \mathbb{R} \cup \infty$. For $p=1$ one receives the "Manhattan norm", $p=2$ describes the Euclidean case, $p=\infty$ means the "maximum norm" $\|(x, y)\|_{\infty}:=\max (x, y)$. But any centrally symmetric, closed convex curve $c$ can act as unit-circle and defines a norm.

There are several possibilities to define orthogonality concepts in Minkowski plane $\mathcal{M}_{c}$ with unit-circle $c$, (see e.g. [1, 3, 6, 7]). Some are more adapted to a certain problem than others, some are symmetric, i.e., for two lines $p, q$ it holds $p \perp q \Rightarrow q \perp p$, some are not symmetric and one distinguishes "left-orthogonality" $p \dashv q$ and "right-orthogonality" $q \vdash p$. A geometric orthogonality relation, which will be used here, is "Birkhoff-orthogonality" with respect to a unit-circle $c \subset \mathcal{M}_{c}$.

Definition 2 (Birkhoff [1]). The radius $p$ of the unit circle $c$ at $P \in c$ (and all parallel lines) are left-orthogonal to the support (tangent) $q$ of $c$ at $P$ (and all parallel lines).

If $c$ is $C^{1}$-smooth and strictly convex, $\dashv$ is a $1-1$ relation of the parallel pencils of $\mathcal{M}_{c}$. If $c$ contains segments, there is an interval of directions with the same left-normal, and if $P$ is
a corner of $c$, then the radius $p$ is left-orthogonal to an interval of directions.
Definition 3. The Frégier mapping $\varphi: \mathcal{M}_{c} \rightarrow \mathcal{M}_{c}$ with respect to $F, G$ and $\dashv$ maps $P$ to $P^{\prime}:=f \cap g^{\dashv}, f=F P, g=G P, g \dashv g^{\dashv}$.

In Figure 16, for practical constructive reasons, the unit-circle $c$ is the union of two symmetric circular arcs and therefore has corners. If we map a line $l$, the image $l^{\varphi}$ will therefore contain a segment. Because of the special givens in Figure $16 l^{\varphi}$ consists of a line segment and an arc of a curve of only $4^{\text {th }}$ degree, even so $l$ is chosen arbitrarily.

Many of the former generalisations can surely be adapted for Frégier mappings in normed spaces of arbitrary dimensions. As a start for further investigations we content ourselves with this introduction to the topic.

## 6 Final Remarks and Conclusion

In the paper Frégier mapping and its properties are described and generalised. Frégier mapping is derived from the Frégier theorem [10], which proved to be very fruitful. The basic construction transforms a straight line to a conic which is the result of Steiner's generation of a conic via projective pencils of lines. It turns out that the projective geometry approach is very powerful compared to analytical approach when for instance the right-angle rotation is replaced by an arbitrary rotation. In this way it is possible to generalise the quadratic Frégier involution also to higher degree transformations.

Another generalisation deals with 3D-versions of the original construction, which is generalised to higher degree algebraic mappings. Investigation of further 3D-versions seems to be promising in future work of the authors.

Finally the original construction is described in non-Euclidean and Minkowskian planes.
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