Journal for Geometry and Graphics
Volume 25 (2021), No. 2, 205–225

Poncelet Porisms in Hyperbolic Pencils of Circles

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Abstract. The usual Poncelet porisms deal with polygons which are inscribed into one conic and circumscribed to another conic. A more general form of Poncelet porisms considers polygons whose sides are tangent to more than one conic of a pencil of conics. We shall study the case of poristic triangles inscribed into a circle c_1 with sides tangent to two further circles c_2, c_3 and all three circles shall be contained in a hyperbolic pencil of circles. In order to allow poristic triangle families, the radii and central distances of the circles are subject to certain algebraic relations. The main contribution of this article is to derive these relations for two special cases: In the first case, only proper circles are involved, while in the second case, we allow one circle to shrink to a point. We also pay attention to traces of triangle centers of the poristic families. Finally, we also provide closing conditions for three more types of circle pencils.

Key Words: Poncelet transverse, hyperbolic pencil of circles, closing condition, point orbit, 3-periodic billiard.

MSC 2020: 51M04 (primary), 51D30

1 Introduction

The incircle and the circumcircle of a triangle define a poristic family of triangles. To put it in another way: A triangle determines a poristic familiy of triangles sharing the incircle and the circumcircle (cf. [9]). On the other hand, two circles cannot be chosen independently in order to determine a poristic family. The inradius r, the circumradius R, and the central distance d have to satisfy the equation

ISSN 1433-8157/$ 2.50

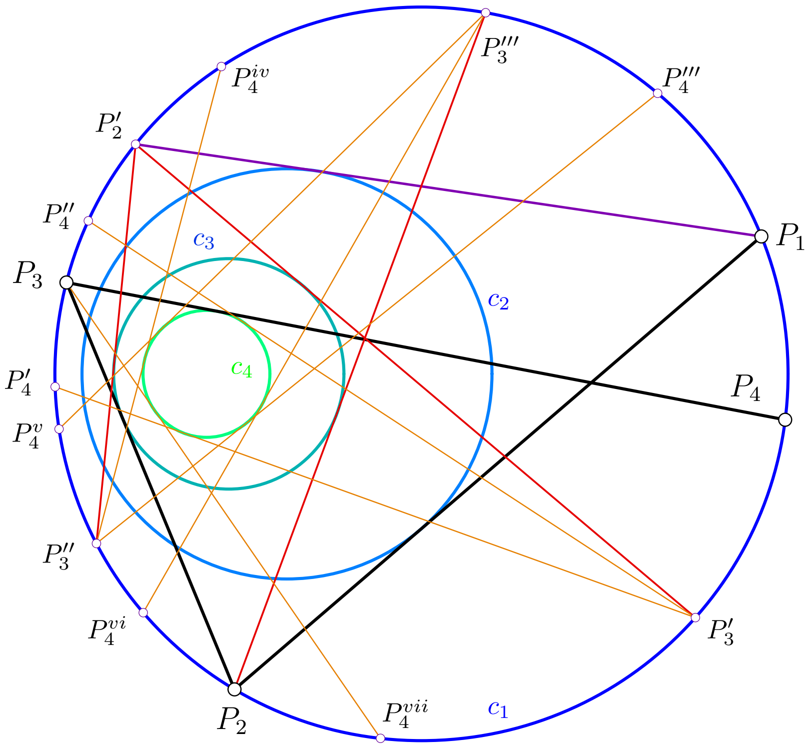

Figure 1: The tree of polygons interscribed to four circles of a hyperbolic pencil. Since there are two tangents from each point to each circle, the polygons spread out from P_1 and close after a given number of steps if the radii are subject to a certain relation.

$$\frac{1}{R-d} + \frac{1}{R+d} = \frac{1}{r} \tag{1}$$

which is usually ascribed to L. Euler (who published this result in 1765), but it was given in 1746 by W. Chapple (cf. [1]). Therefore, these kinds of porisms are frequently referred to as *Chapple's porisms*.

The Euler triangle formula (1) is just a special case of the many closing conditions for bicentric polygons. For some $n \in \mathbb{N} \setminus \{0, 1, 2, 3\}$, one may find these polynomial conditions on the radii R, r, and the central distance d in [5]. All the cases treated in [5] deal with *bicentric* n-gons, *i.e.*, n-sided polygons inscribed into one circle and circumscribed to another.

In the original version of Poncelet's porism (cf. [2]), the poristic families of bicentric n-gons appear to be a very special case described in one Lemma. Poncelet even showed that it is possible to find one-parameter families of interscribed n-gons to a (finite) sequence of conics in a pencil: Assume that c_i (with $i \in \{1, \ldots, N\}$) are conics of a pencil.[1] Let now P_1 be a point on c_1. A tangent from P_1 to c_2 may intersect c_1 in a point P_2, from which a tangent to c_3 is drawn and intersects c_1 in a point $P_3 \neq P_1, \ldots$. This results in a sequence of points $P_i \in c_1$ and lines $[P_i P_{i+1}]$ tangent to c_{i+1} and a *last* point P_{N+1}, see Fig. 1.

In the case of an n-gon, we shall not forget that the polygon is not unique whether it closes (*i.e.*, $P_1 = P_{N+1}$) or not.

Poncelet's most general result states:
If the polygon $P_1 \ldots P_{N+1}$ closes for one particular choice of P_1, then it closes for any choice of P_1.

[1]The type of pencil does not matter. From the projective point of view there are five different types, cf. [4].

In fact, the polygon closes anyhow: PONCELET also showed that the line $[P_1, P_{N+1}]$ envelopes a conic which belongs to the pencil.

PONCELET's general result contains the very special (and by no means trivial) configuration of a triangle with its incircle and circumcircle (CHAPPLE's porism) in two ways: On one hand, any triangle is interscribed between its incircle and its circumcircle, and on the other hand, the incircle and the circumcircle span a hyperbolic pencil of circles (which are of course conics).

In the following, we shall study triangles interscribed between three circles of a hyperbolic pencil. There are two cases to be distinguished:

(i) no circle is of radius zero,

(ii) exactly one circle is a zero circle.

These two cases have to be treated separately, at least in the algebraic approach. As is the case with CHAPPLE's porisms, the choice of circles is not free if we want the polygons to close without introducing a further circle.

Therefore, and motivated by the many experimental results given in [3], we determine conditions on the radii or central distances of the involved circles. It is sufficient to have a condition on the radii of the circles since the radii and the distances of the centers of the circles in the hyperbolic pencil determine each other mutually. Further, it means no restriction to construct triangles in a *normal form* of the hyperbolic pencil (with the zero circles placed at $(\pm 1, 0)$), since each hyperbolic pencil can be mapped to the standard pencil via a similarity transformation. In Sec. 2, we deal with the case of three circles none of which is allowed to be a zero circle. We first determine the closing condition, and then, we sketch how to derive the algebraic equations of the paths of the centroid and the orthocenter. We will not write down the algebraic equations of these traces due to their complexity.

It turns out that the algebraic approach delivers more than we expected in the beginning. Besides the path of a particular triangle center of the moving triangle (principal triangle), we find the path of the same center of the *opportunistic triangles* which occur with the principal triangle since we can draw two tangents from an exterior point to a circle. The latter fact causes the interscribed polygons to spread (as is illustrated in Fig. 1). Motivated by numerical experiments (cf. [3]), we shall have a closer look at the traces of incenters and excenters. In some special cases, these traces contain (parts of) circles.

Sec. 3 treats the case with one zero circle. At least from the constructive point of view, this seems to be a simpler case. However, from the algebraic stand point it is not. In Sec. 4, we shall add the closing conditions for triangle porisms in a hyperbolic pencil with four circles. The techniques used for that purpose do not differ from those in the beginning and so we do not lay down all the details. Further, the four circle case is not a case on its own right since the interscribed polygons do close in any case according to PONCELET's most general form of his theorem. Finally, we give the closing conditions for poristic families in some elliptic and parabolic pencils of circles. We shall not treat the very elementary case of pencils of concentric circles in detail.

2 Three proper circles

2.1 The closing condition

Following [4, p. 323], the equations of the circles of a hyperbolic pencil can be parametrized by one real parameter $t \in \mathbb{R}^\star := \mathbb{R} \setminus \{-1, 0, +1\}$ as

$$c(t): \ x^2 - 2tx + y^2 + 1 = 0. \tag{2}$$

The values $t = \pm 1$ change (2) into $(x \pm 1)^2 + y^2 = (x \pm 1 - \mathrm{i}y)(x \pm 1 - \mathrm{i}y) = 0$, and thus, they correspond to pairs of isotropic lines through the points $N_{1,2} = (\pm 1, 0)$ (the null circles in the pencil). The circle $c(0): \ x^2 + y^2 + 1 = 0$ carries no real point. The centers C and radii r of the circles in the pencil are

$$C(t) = (t, 0) \quad \text{and} \quad r(t) = \sqrt{t^2 - 1}. \tag{3}$$

The following symbolic computations are simplified by trying to write down everything in terms of polynomials or rational functions and by avoiding square roots whenever possible. This not only in the sense of *rational trigonometry* (cf. [10]), it could also be a new approach in the area of porisms resulting not necessarily in smooth triangle families, but rational or discrete ones.

Therefore, we reparametrize the family of circles (2) by

$$t \to \frac{1 + u^2}{2u} \quad \text{with } u \in \mathbb{R}^\star, \tag{4}$$

and thus, the centers and radii become

$$C(u) = \left(\frac{1 + u^2}{2u}, 0 \right) \quad \text{and} \quad r(u) = \frac{u^2 - 1}{2u}. \tag{5}$$

We assume that the vertices P_1, P_2, P_3 of the triangles Δ of the poristic family lie on the circle c_1 (defined by setting $t = t_1$ in (2)). Since the circumcircle c_1 of Δ admits the rational parametrization

$$c_1(\tau) = \left(r_1 \frac{1 - \tau^2}{1 + \tau^2} + m_1, r_1 \frac{2\tau}{1 + \tau^2} \right) \quad \text{with } \tau \in \mathbb{R}, \tag{6}$$

we can assume that $P_1 = c_1(T)$, $P_2 = c_1(U)$, and $P_3 = c_1(V)$ with pairwise different real parameters T, U, and V. In (6), the radius r_1 and the coordinate m_1 of the circumcenter are to be replaced by their rational equivalents (5) depending on the parameter $u_1 \in \mathbb{R}^\star$.

The equations of the three side lines of the triangle are

$$[P_1, P_2]\colon u_1(1 - TU)x + u_1(T + U)y + TU - u_1^2 = 0, \tag{7}$$

where the equations of $[P_2, P_3]$ and $[P_3, P_1]$ are obtained from (7) by replacing first $T \to U$, $U \to V$ and then $U \to V$, $V \to T$. Note that T, U, V have to be pairwise different in order to define pairwise different points P_1, P_2, and P_3 on the circle c_1. Therefore, the linear forms

$$T - U, \quad U - V, \quad V - T$$

are not zero and can be canceled whenever they occur as factors.

In order to obtain a poristic family of triangles, the sides $[P_1, P_2]$ and $[P_2, P_3]$ are tangent to c_2, while $[P_3, P_1]$ touches the circle c_3. The fact that $[P_3, P_1]$ is tangent to c_3 causes a loss of symmetry in the geometry as well as in the computations, but the polygon has to close.

However, we could also demand that the third side has to touch a fourth circle. Then, the computational complexity would increase dramatically. The closing condition for this case is given in Sec. 4.

It is a rather elementary task to determine the tangency condition for $[P_1, P_2]$ and c_2. For that purpose, we compute the resultant of the respective equations with respect to y and determine the discriminant of the resulting quadratic equation. It makes no difference if we compute the discriminant of the resultant with respect to x. Thus, the tangency between $[P_1, P_2]$, $[P_2, P_3]$ and c_2 is ruled by

$$\begin{aligned}&\#([P_1,P_2]\cap c_2)=1 \iff \mathcal{C}(T,U): 4u_2(u_1-u_2)(u_1u_2-1)(T^2U^2+u_1^2)+\\ &+2u_1(u_1u_2^2+u_1-2u_2)(2u_1u_2-u_2^2-1)TU-u_1^2(u_2^2-1)^2(T^2+U^2)=0,\\ &\#([P_2,P_3]\cap c_2)=1 \iff \mathcal{C}(U,V): 4u_2(u_1-u_2)(u_1u_2-1)(U^2V^2+u_1^2)+\\ &+2u_1(u_1u_2^2+u_1-2u_2)(2u_1u_2-u_2^2-1)UV-u_1^2(u_2^2-1)^2(U^2+V^2)=0,\end{aligned} \tag{8}$$

while the contact between $[P_3, P_1]$ and c_3 is described by

$$\begin{aligned}&\#([P_3,P_1]\cap c_3)=1 \iff \mathcal{C}(T,V): 4u_3(u_1-u_3)(u_1u_3-1)(T^2V^2+u_1^2)+\\ &+2u_1(u_1u_3^2+u_1-2u_3)(2u_1u_3-u_3^2-1)TV-u_1^2(u_3^2-1)^2(T^2+V^2)=0.\end{aligned} \tag{9}$$

Note that all three conditions (8) and (9) depend on u_1, since all vertices of the triangle lie on the circle c_1. The contact condition (9) depends further on u_3, because $[P_1, P_2]$ touches c_3; while both of the two equations (8) also depend on u_2, for they describe the contact with c_2.

In order to derive conditions on the radii r_i and the central distances m_i (coordinates of the centers) such that the three circles c_i allow for a Poncelet porism, we determine conditions on the parameters u_i. The latter can be transformed into conditions on the parameters t_i in the hyperbolic pencil of circles.

For that purpose, we eliminate two of the point parameters T, U, V, for example U and V from the contact conditions (8) and (9). (The choice of the variables to be eliminated does not change the result.)

From the first resultant

$$R_1 := \operatorname{res}(\mathcal{C}(U,V), \mathcal{C}(T,V), V),$$

we can cut out the factor u_1^4, since u_1 is not allowed to be zero.

The final resultant

$$R := \operatorname{res}(R_1, \mathcal{C}(T,U), U) = 2^8 \prod_{i=1}^{8} f_i.$$

factors into 8 polynomials, some of which depend on T. We shall see that only a few factors yield a condition on u_i such that the thus defined three circles allow a poristic family of triangles. The factors $f_1 = (u_3T^2 + u_1)^2$ and $f_2 = (T^2 + u_1u_3)^2$ are dispensable, since they would only allow a closing of one particular triangle for a specific T depending on the circles

c_1, c_3. The biquadratic factors

$$\begin{aligned} f_3 = &\Big(4u_2^2u_3(u_1^2+1) + ((u_2^2-1)^2 - 4u_2u_3(u_2^2+1))u_1\Big)^2 T^4 \\ &+2u_1\Big(4u_1u_2(u_2^2-1)^2(u_1u_2-1)(u_2-u_1)(u_3^2+1) + (8u_2^2(u_2^4+1)(u_1^4+1) \\ &-8u_2(u_2^2+1)^3u_1(u_1^2+1) + (u_2^8+28u_2^6+38u_2^4+28u_2^2+1)u_1^2)u_3\Big)T^2 \\ &+u_1^2\Big(u_1(u_2^2-1)^2u_3 + 4u_2(u_1u_2-1)(u_1-u_2)\Big)^2 \end{aligned}$$

and

$$\begin{aligned} f_4 = &\Big(4u_2^2(u_1^2+1) + (u_2^4u_3 - 4u_2^3 - 2u_2^2u_3 - 4u_2 + u_3)u_1\Big)^2 T^4 \\ &+2u_1\Big(8u_2^2u_3(u_2^4+1)(u_1^4+1) - 4u_2(u_2(u_2^2-1)^2(u_3^2+1) + 2(u_2^2+1)^3u_3)u_1(u_1^2+1) \\ &+(4u_2(u_2^2+1)(u_2^2-1)^2(u_3^2+1) + (u_2^8+28u_2^6+38u_2^4+28u_2^2+1)u_3)u_1^2\Big)T^2 \\ &+u_1^2\Big(4u_1^2u_2^2u_3 + ((u_2^2-1)^2 - 4u_2u_3(u_2^2+1))u_1 + 4u_2^2u_3\Big)^2 \end{aligned}$$

can be considered as polynomials in T and vanish identically (for all T) if, and only if, all their coefficients vanish simultaneously. For the factor f_3 this is the case if, and only if, $u_1 = u_2 = -1, 0, 1$. This would imply that at least one of the circles c_1 or c_2 becomes either a zero circle or $x^2+y^2+1=0$ which carries no real points. All other trivial solutions like $u_i = 0$ and $u_i = u_j$ (with $i,\ j \in \{1,2,3\}$ and $i \neq j$) are ruled out in each step of the computation. The same holds true for f_4.

The factor $f_5 = (u_1 - u_3)^2$ vanishes if, and only if, $u_1 = u_3$ implying $c_1 = c_3$ which is not allowed. Further, we have to discuss the factor $f_6 = (u_1u_3 - 1)^2$. From $u_1u_3 = 1$ we infer that these values are each others reciprocals. Since the rational expression (4) for t_i remains unchanged if we replace u with u^{-1}, $u_1 = 1/u_3$ implies that $t_1 = t_3$ and $r_1 = r_3$, *i.e.*, the circles c_1 and c_3 are identic. So far we have discussed six factors of R.

The last two factors depend on u_i exclusively:

$$\begin{aligned} f_7 &= (u_1(2u_1u_2 - u_2^2 - 1)^2u_3 - (u_1u_2^2 + u_1 - 2u_2)^2)^2, \\ f_8 &= (u_1(2u_1u_2 - u_2^2 - 1)^2 - (u_1u_2^2 + u_1 - 2u_2)^2u_3)^2. \end{aligned} \tag{10}$$

Both factors (although of multiplicity 2) depend linearly on u_3. Setting them equal to zero yields a condition on u_i such that the circles c_1, c_2, c_3 allow a one-parameter family of interscribed triangles. The two conditions (10) can then be solved for u_3 which gives

$$u_3 = \frac{(u_1u_2^2 + u_1 - 2u_2)^2}{u_1(2u_1u_2 - u_2^2 - 1)^2} \quad \text{and} \quad u_3 = \frac{u_1(2u_1u_2 - u_2^2 - 1)^2}{(u_1u_2^2 + u_1 - 2u_2)^2}, \tag{11}$$

which are, obviously, each others reciprocals.

In order to find a condition on the pencil parameters t_i, we eliminate u_i from f_7 and f_8 in (10). This is done by using the inverse of (4) although this mapping is not birational. Cutting out the constant factor 2^{32}, f_7 and f_8 become the same factor with multiplicity 8:

$$(2t_1t_2 - t_2^2 - 1)^2t_3 - 4t_1^3 + 8t_1^2t_2 - t_1(t_2^4 + 6t_2^2 - 3) + 4t_2(t_2^2 - 1). \tag{12}$$

This is due to the fact that f_7 and f_8 can be transformed into each other by the algebraic substitution $u_2 \to u_2$, $u_1 \to u_1^{-1}$, and $u_3 \to u_3^{-1}$.

Setting the latter polynomial equal to zero, we find an analog to the Euler formula (1) relating the circumradius R, the inradius r, and the central distance of the circum- and the incircle of a triangle. It allows to express t_3 in terms of a rational function depending on t_1, t_2 as

$$t_3 = \frac{4t_1^3 - 8t_1^2 t_2 + (t_2^4 + 6t_2^2 - 3)t_1 - 4t_2(t_2^2 - 1)}{(2t_1 t_2 - t_2^2 - 1)^2}.$$

Finally, we can derive a condition on the radii r_i in order to allow a poristic family of the above described type. We use (5) in order to eliminate u_i from (10) and arrive at

$$\begin{aligned}&\Big((4r_1^2r_2^2 - r_2^4 + 4r_1^2)^2 r_3^2 + 2r_1(4r_1^2r_2^2 + r_2^4 + 4r_1^2)(4r_1^2 - 4r_2^2 - r_2^4)r_3 + \\ &\quad + r_1^2(r_2^8 - 8r_1^2r_2^4 - 8r_2^6 + 16r_1^4 - 32r_1^2r_2^2)\Big) \cdot \\ &\cdot\Big((4r_1^2r_2^2 - r_2^4 + 4r_1^2)^2 r_3^2 - 2r_1(4r_1^2r_2^2 + r_2^4 + 4r_1^2)(4r_1^2 - 4r_2^2 - r_2^4)r_3 + \\ &\quad + r_1^2(r_2^8 - 8r_1^2r_2^4 - 8r_2^6 + 16r_1^4 - 32r_1^2r_2^2)\Big) = 0.\end{aligned} \tag{13}$$

We shall summarize our results in

Theorem 1. *Let c_1, c_2, c_3 be three circles of a hyperbolic pencil given by the equations* (2). *These circles allow a poristic one-parameter family of interscribed triangles $P_1P_2P_3$ such that c_1 is the common circumcircle, $[P_1, P_2]$, $[P_2, P_3]$ are tangent to c_2 while $[P_3, P_1]$ is tangent to c_3 if their center coordinates t_i satisfy* (12).

The fact that the condition (13) on the radii splits into two factors mirrors the fact that the involved circles are not necessarily nested, *i.e.*, they may lie on different sides of the *straight circle* $x = 0$ (corresponding to $t = \infty$) in the hyperbolic pencil.

2.1.1 Computing point paths

Now, we have derived the condition on the radii of the circles defining the poristic triangle family. In order to compute the equations of the traces of at least some simple (rational) triangle centers, we assume that u_3 is related to u_1 and u_2 via one of the relations in (11). From (8) and (9) (which are now dependent because of a suitable choice of u_i) we cannot easily extract expressions for U and V as functions depending on T. Therefore, it is not possible to parametrize the families of triangles traversing the various Poncelet families.

For some centers (like the centroid or the orthocenter), we can go the following way: We compute the centroid

$$X_2 = \frac{1}{3}(P_1 + P_2 + P_3)$$

and the orthocenter X_4 of the triangles $P_1P_2P_3$ (labeling of triangle centers according to [6], [11]) using the initial representations of the points P_i as points on c_1 depending on T, U, and V. This yields paramatrizations of X_2 and X_4 depending in the parameters T, U, V. Since the center X_4 is a linear combination of the fixed point X_3 and the (moving) point X_2, the trace of X_4 is similar to that of X_2 with X_3 as the center of similarity and scaling factor 3. Analogous results hold true for all other triangle centers on the Euler line $\mathcal{L}_{2,3} = [X_2, X_3]$.

We first eliminate U from

$$X_2[1] - x = 0 \quad \text{and} \quad X_2[2] - y = 0$$

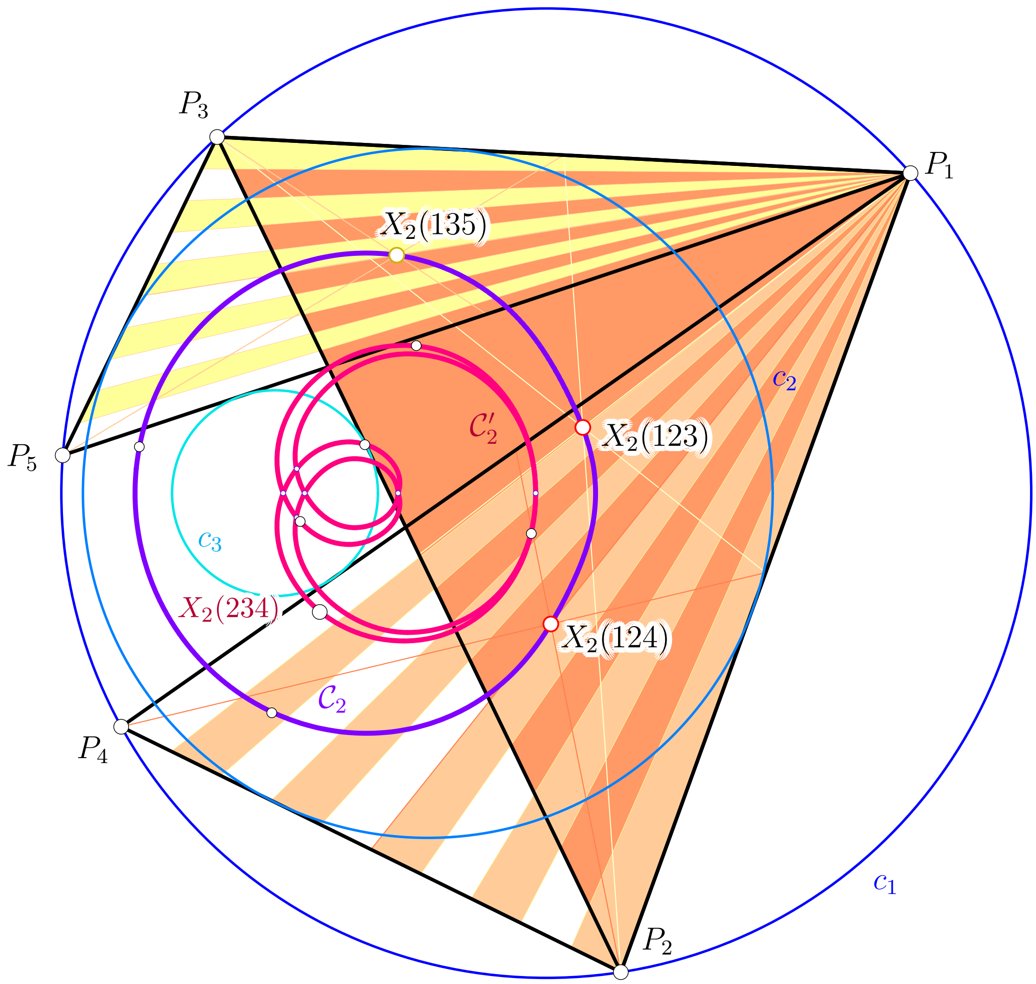

Figure 2: A triangle $P_1P_2P_3$ (dark orange) with sides tangent to c_2 and c_3 together with two opportunistic triangles $P_1P_2P_4$ (light orange) and $P_1P_3P_4$ (yellow). The trace of the centroid consists of two curves (red, violet).

using the first equation of (8). (Here and in the following, $X_2[i]$ means the i^{th} component of the coordinate vector X_2.) Subsequently, we use (9) (where we have inserted one of the values for u_3 chosen from (11)) in order to eliminate V. In the third step, T is eliminated from both polynomials related to either coordinate function of X_2.

In the case of the centroid, we find a polynomial $\mathcal{P}_2$ of degree 128 (in the variables x and y) which factors into 8 different polynomials

$$\mathcal{P}_2 = \prod_{i=1}^{8} p_i^{\mu_i}.$$

The degrees d_i and the multiplicities μ_i of p_i are

$$d = (16, 16, 20, 12, 12, 20, 16, 16) \quad \text{and} \quad \mu = (1, 1, 1, 2, 1, 1, 1, 1).$$

The sextic factor (*i.e.*, the fourth factor) with multiplicity two turns out to be the equation of a part of the trace $\mathcal{C}_2$ of X_2 as shown in Fig. 2. This can also be checked by inserting the parametrization $X_2(T, U, V)$ of the centroid and subsequent simplification using the conditions (8), (9), and u_3 from (11).

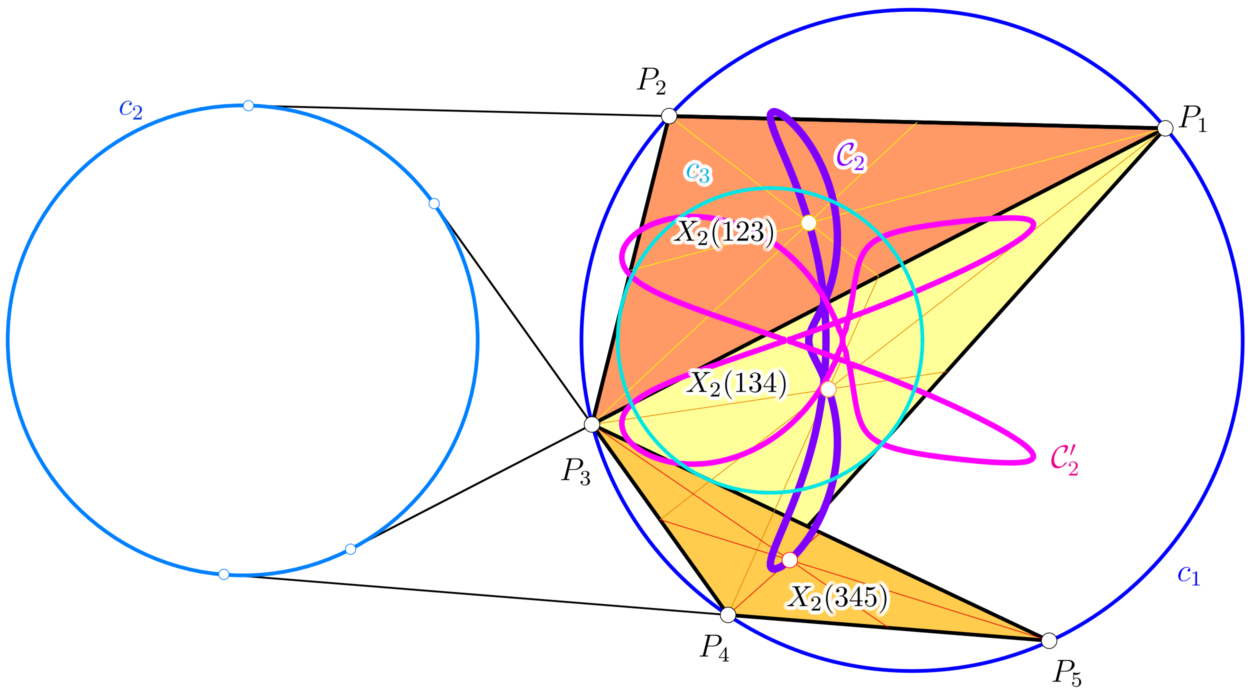

Figure 3: One circle, say c_2, may lie on the other side of the straight circle in the pencil (second axis of symmetry): The centroid traces a sextic $\mathcal{C}_2$ and the degree 12 curve $\mathcal{C}_2'$ is the locus of all centroids of opportunistic triangles.

Surprisingly, a second factor of $\mathcal{P}_2$ is annihilated by the parametrization of the centroid. It is a factor of degree 12 which describes a curve $\mathcal{C}_2'$ of genus 1 having 6-fold points at the absolute points of Euclidean geometry. It is the trace of centroids of *opportunistic triangles*, *i.e.*, triangles which are also results of the construction (computation) and whose sides also fulfill the contact conditions.

As can be seen in Fig. 2, the triangle $P_1P_2P_3$ can be viewed as a principal solution and traverses one family. The triangles $P_2P_1P_4$ and $P_3P_1P_5$ are opportunistic: They come along with the principal solution and satisfy closing and tangency conditions. The existence of opportunistic triangles is caused by the fact that there exist two tangents from P_1 to c_2 and from each intersection of these tangents with the circumcircle c_1 there exist two further tangents to c_3. The curve $\mathcal{C}_2'$ houses the traces of centroids of opportunistic triangles.

A triple of circles from a hyperbolic pencil may not necessarily be a triple of mutually nested circles. As shown in Fig. 3, the appearance of the sextic trace $\mathcal{C}_2$ of the centroid may change its shape. Nevertheless, the algebraic properties remain unchanged even if one circle lies not in the interior of c_1, *i.e.*, it lies on the *the other side*.

Finally, we note that the trace $\mathcal{C}_4$ of the orthocenter X_4 is the image of $\mathcal{C}_2$ under the central similarity with the midpoint of c_1 (common circumcenter X_3 of the poristic triangles) as the center, since X_2, X_3, and X_4 are collinear for all triangles. The factor of similarity equals $\overline{X_4X_2} \cdot \overline{X_3X_2}^{-1} = -2$.

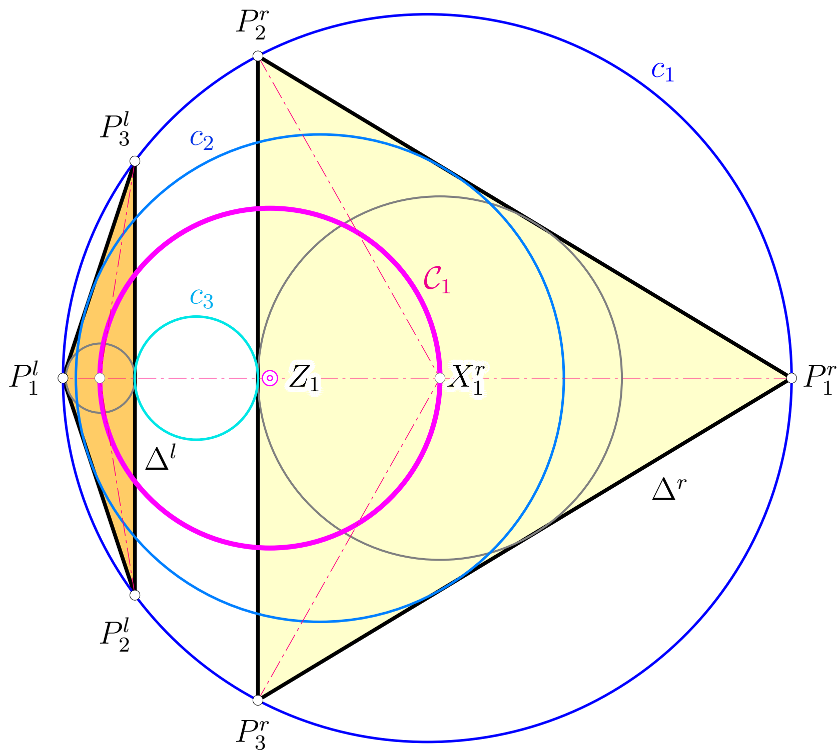

Figure 4: The circular trace of the incenter X_1 for nested circles c_1, c_2, c_3.

2.1.2 Experiments

The incenter of a triangle is the first in C. Kimberling's exhaustive list, see [6], [11]. This is probably caused by its very simple representation

$$X_1 = 1 : 1 : 1$$

in terms of trilinear coordinates. However, it is doubtful if X_1 deserves this prominent position. (In terms of barycentric coordinates, the centroid X_2 would be in the first place.) The computation as well as the construction of the incenter bear on non-rational operations, such as the normalization of vectors, or equivalently, the construction of angle bisectors. Moreover, the incenter as the center of a tritangent circle of a triangle is only one of four such points which is even true in a projective setting (see [8]) and in rational trigonometry or universal geometry (cf. [10]).

Numerical experiments have shown that the incenter of the triangle $\Delta = P_1P_2P_3$ traces at least an oval curve $\mathcal{C}_1$ (cf. [3]). Moreover, this trace was so close to circles in almost all cases that it was near to suggest that $\mathcal{C}_1$ is a circle. As we shall see, in some special cases, we are able to show that $\mathcal{C}_1$ is really a circle. Fig. 4 does not only illustrate the results of numerical experiments which showed that the incenter X_1 of $P_1P_2P_3$ moves on a curve that looks like a circle $\mathcal{C}_1$. It is not at all obvious that $\mathcal{C}_1$ is a circle and at least for the case where two sides of $P_1P_2P_3$ are tangent to the same circle, say c_2, in the pencil we can give the equation of this circle and state:

Theorem 2. *Let c_1, c_2, c_3 be three nested circles from a hyperbolic pencil of circles which allow a one-parameter family of poristic triangles $\Delta = P_1P_2P_3$ such that c_1 is the common circumcircle and the sides $[P_1, P_2]$ and $[P_3, P_1]$ are tangent to c_2 while $[P_2, P_3]$ is tangent to*

c_3. *Then, the trace* $\mathcal{C}_1$ *of the incenter* X_1 *of the triangles* Δ *is a circle which is not contained in the hyperbolic pencil.*

Proof. The assumption that $[P_1, P_2]$ and $[P_3, P_1]$ are tangent to c_2 guarantees that there exist two poses of the triangle $P_1P_2P_3$ which are symmetric with respect to the axis a of the circle pencil (cf. Fig. 4):

(i) $\Delta^l = P_1^l P_2^l P_3^l$ with P_1^l being the *left point* of $a \cap c_1$ and

(ii) $\Delta^r = P_1^r P_2^r P_3^r$ with P_1^r being the *right point* of $a \cap c_1$.

Without loss of generality, we may at first assume that $u_i > 1$ (for $i \in \{1, 2, 3\}$) hold. Secondly, the assumptions $u_1 > u_2$ and $u_1 > u_3$ shall guarantee that the circle c_1 is the largest one, and therefore, the points P_2 and P_3 are always real. Then, we have $P_1^l = (u_1^{-1}, 0)$ and $P_1^r = (u_1, 0)$. Further, $P_{2,3}^l = (u_3^{-1}, \pm y_l)$ and $P_{2,3}^r = (u_3, \mp y_r)$ with u_3 being one of (11) and

$$y_l = \frac{2(u_1^2-1)(u_2^2-1)\sqrt{u_2(u_1u_2-1)(u_1-u_2)}}{(u_1u_2^2+u_1-2u_2)^2} \quad \text{and} \quad y_r = y_l u_3.$$

In order to find the incenter of Δ^l and Δ^r, it is sufficient to intersect the interior angle bisector at P_2^l and P_2^r (or P_3^l and P_3^r) with $a: \ y = 0$. This yields the surprisingly simple coordinate representations of the *left* and *right* incenter $X_1^l = (\xi_l, 0)$ and $X_1^r = (\xi_r, 0)$ with

$$\xi_l = \frac{2u_1^2 - 2u_1u_2 + u_2^2 - 1}{u_1u_2^2 + u_1 - 2u_2} \quad \text{and} \quad \xi_r = \frac{u_1^2 - u_1^2u_2^2 + 2u_1u_2 - 2}{u_1(2u_1u_2 - u_2^2 - 1)}.$$

(Note that only the substitution $u_1 \to u_1^{-1}$ yields $\xi_l \to \xi_r$.) Since ξ_l and ξ_r are not each other's reciprocals, the points X_1^l and X_1^r cannot be joined by a circle from the underlying hyperbolic pencil. Now, we compute the Thales circle $\mathcal{C}_1$ on the segment $X_1^l X_1^r$ and find the circle

$$\begin{aligned} \mathcal{C}_1\colon\, & u_1(2u_1u_2 - u_2^2 - 1)(u_1u_2^2 + u_1 - 2u_2)(x^2+y^2) \\ &+(u_1^3u_2^4 - 4u_1^4u_2 + 6u_1^3u_2^2 - 8u_1^2u_2^3 + u_1u_2^4 + u_1^3 + 6u_1u_2^2 + u_1 - 4u_2)x \\ &-(u_1^2u_2^2 - u_1^2 - 2u_1u_2 + 2)(2u_1^2 - 2u_1u_2 + u_2^2 - 1) = 0 \end{aligned} \tag{14}$$

with the radius

$$\rho = \frac{1}{2}\frac{(u_1u_2^4 + 4u_1^2u_2 - 6u_1u_2^2 - 3u_1 + 4u_2)(u_1^2-1)}{u_1(2u_1u_2 - u_2^2 - 1)(u_1u_2^2 + u_1 - 2u_2)}.$$

In order to verify that the equation of $\mathcal{C}_1$ is the equation of the trace of X_1, we can compute a parametrization and show that it annihilates the circle equation which definitely needs a CAS. □

The circle $\mathcal{C}_1$ with the equation (14) is only a part of the complete picture shown in Fig. 5. The excenters ot the triangle Δ move on a more complicated curve that contains one circle and two further closed loops. In comparison with Chapple's porism (where the three excenters of a triangle Δ move on a single circle, cf. [9, Thm. 3.2]), the poristic trace is broken up into three components since the tangency of Δ's sides to the unique incircle is replaced by tangencies to different circles. In Thm. 3 we shall give the equations of the two circles in the case of non-nested circles c_1, c_2, c_3.

Until now we have assumed $u_i > 0$ ($i \in \{1, 2, 3\}$). Now, we shall discuss the effect of other choices of u_i. For $u_i \neq 0$ we observe that u_i^{-1} leads to the same circle c_i since (4) does not change under the substitution $u_i \to u_1^{-i}$. This holds also true for negative u_i. If now one of the values u_i is negative, say $u_2 < 0$, then c_2 is no longer in the interior of c_1. Such a

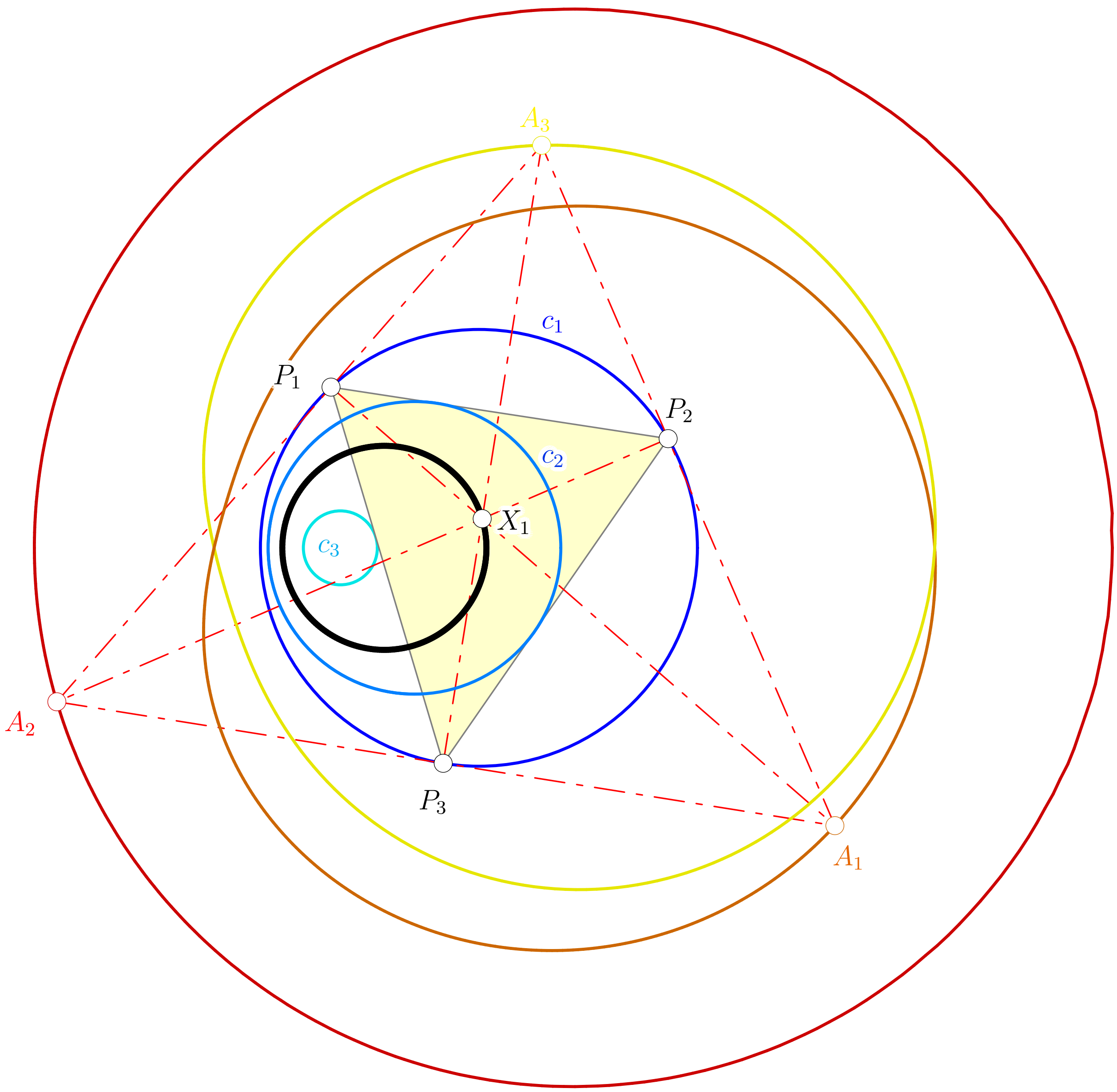

Figure 5: The trace of the centers of the triangles' tritangent circles in the case of a nested circle triple.

case is illustrated in Fig. 6. The curves shown in Fig. 6 are determined numerically and the coloring of the different parts of the curves correspond to different triangle shapes. Whenever a triangle collapses, its incenter happens to lie on the common circumcircle c_1. As long as the center of the interior tritangent circle remains in the interior of c_1, the path of the center is drawn black. The red, orange, and yellow parts are the traces of centers of tritangent circles of Δ if these centers are excenters. The transition from an incenter to an excenter happens precisely at the cusps of the black curve. The cusps are located at the contact points of the common tangents of c_1 and c_2. Fig. 6 also indicates that the orbit of the centers of the four tritangent circles of Δ move on two circles and two a additional closed curves and all four branches belong to the same algebraic curve. Situations like these are a good reason to make no difference between the incenter and the excenter of a triangle and to simply speak about the four tritangent circles of a triad of lines as indicated in [7, 8].

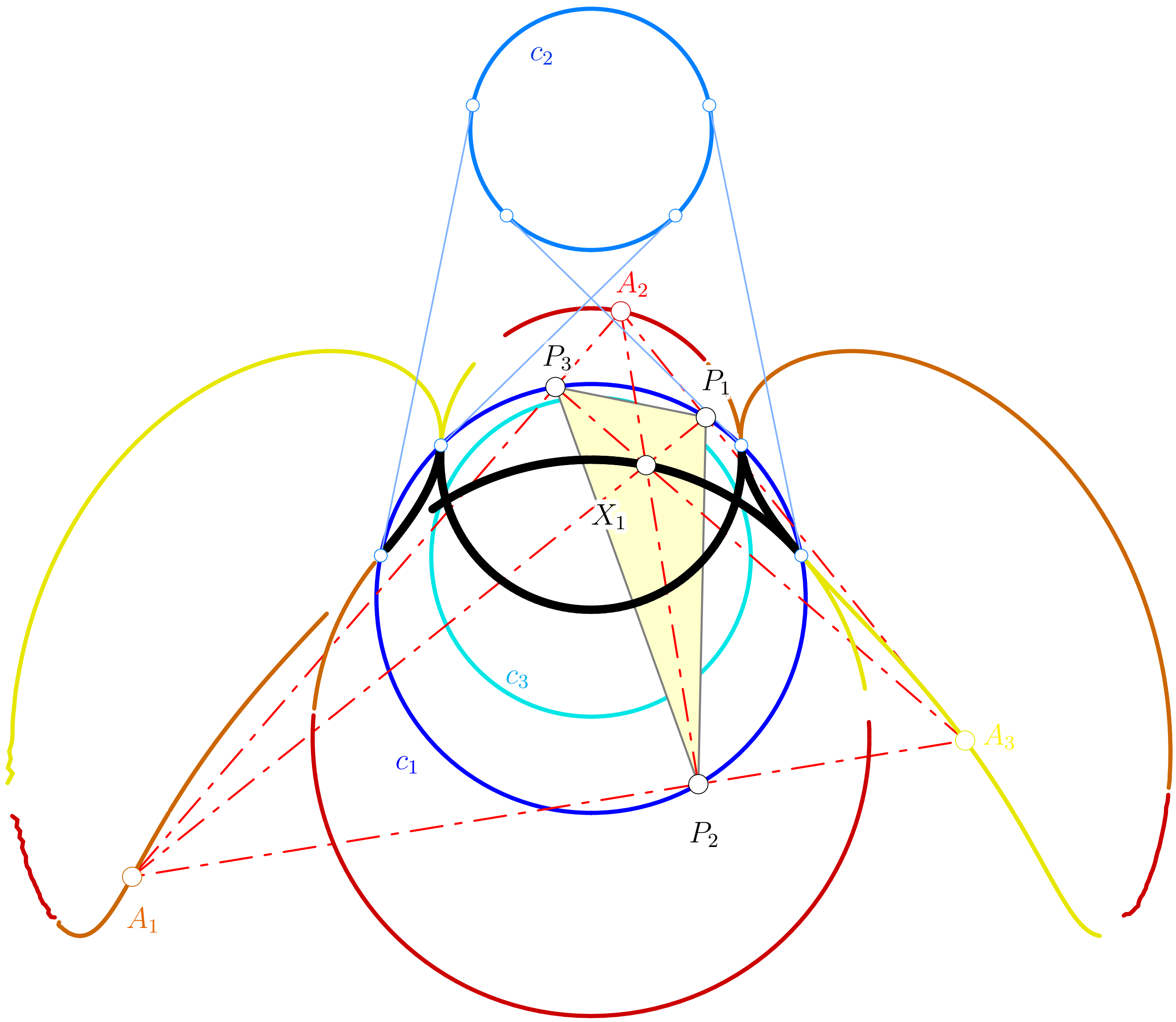

Figure 6: A circle configuration with c_2 *outside*: The centers of some of the tritangent circles run on a circle as long as these centers are incenters, even if the initial circles are not nested as long as there exist symmetry poses of the triangles.

We are able to give the equations of the circular paths of the centers of Δ's tritangent circles if the three circles from the hyperbolic pencil are not nested:

Theorem 3. *Let c_1, c_2, c_3 be three circles of a hyperbolic pencil of circles. Assume that c_2 lies not in the interior of c_1 and the triple of circles allows a poristic family of triangles $\Delta = P_1P_2P_3$ such that $[P_1, P_2]$ and $[P_2, P_3]$ are tangent to c_2 while $[P_2, P_3]$ Then, the trace $\mathcal{C}_1$*

of the centers of tritangent circles of the contains the two circles

$$\begin{aligned}\mathcal{K}_1\colon\, & u_1u_3\left(\sqrt{u_1(u_1^2-1)}+u_1\sqrt{u_1-u_3}\right)\left(x^2+y^2\right)\\ & -\sqrt{u_1u_3}\left(\left((u_1u_3+1)\sqrt{u_3}-(u_1-u_3)\sqrt{u_1}\right)\sqrt{u_1^2-1}-\left(u_1^2-2\sqrt{u_1u_3}-1\right)\sqrt{u_1-u_3}\right)x\\ & \quad +u_3\left(1-(u_1^2-1)\sqrt{u_1u_3}\right)\sqrt{u_1-u_3}+\sqrt{u_3}\left(u_3+u_3\sqrt{u_1u_3}-u_1\right)\sqrt{u_1^2-1}=0,\end{aligned} \tag{15}$$

$$\begin{aligned}\mathcal{K}_2\colon\, & u_1u_3(u_1^2-1)\left(\sqrt{u_3(u_1^2-1)}+\sqrt{u_1-u_3}\right)\left(x^2+y^2\right)\\ & \qquad -\sqrt{u_1u_3}(1-u_1^2)\left((2u_1\sqrt{u_1u_3}+u_3(u_1^2-1))\sqrt{u_1-u_3}\right)\\ & \qquad\quad +\left(\sqrt{u_1}(1+u_1u_3)+\sqrt{u_3}(u_1-u_3))\sqrt{u_1^2-1}\right)x\\ & \quad +\sqrt{u_1u_3}(u_1^2-1)\left(u_1\sqrt{u_1(u_1^2-1)}+\sqrt{u_1-u_3}\left(u_1\sqrt{u_1u_3}+u_1^2-1\right)\right)=0\end{aligned} \tag{16}$$

centered at the points

$$C_{1,2}=\frac{\sqrt{u_3}}{2u_1u_3}\left(u_3(u_1^2+1)\pm(\sqrt{u_1u_3}+1)\sqrt{(u_1-u_3)(u_1^2-1)},0\right).$$

Proof. Due to symmetry reasons, the circular parts $\mathcal{K}_1$ and $\mathcal{K}_2$ of the curve $\mathcal{C}$ are centered on the axis of the hyperbolic circle pencil. Both are Thaloids of segments on the axis bounded by the interior and exterior angle bisectors of two triangles $\Delta_l=P_1^lP_2^lP_3^l$ (*the left one*) and $\Delta_r=P_1^rP_2^rP_3^r$ (*the right one*) in symmetry pose, cf. Fig. 7. Thus, we may assume that the vertices of the triangles are

$$\begin{aligned} P_1^l&=(u_1^{-1},0), & P_{2,3}^l&=\left(u_3,\pm\sqrt{\frac{(u_1u_3-1)(u_1-u_3)}{u_1}}\right),\\ P_1^r&=(u_1,0), & P_{2,3}^r&=\left(u_3^{-1},\mp\sqrt{\frac{(u_1u_3-1)(u_1-u_3)}{u_1u_3^2}}\right).\end{aligned} \tag{17}$$

Note that u_2 does not show up in the above representations of triangle vertices. However, this is not necessary as long as u_i fulfill (11).

Now, we can compute the centers of the tritangent circles of the left and right triangle $\Delta_l=P_1^lP_2^lP_3^l$ and $\Delta_r=P_1^rP_2^rP_3^r$ which simplifies to the computation of the intersection of a pair of bisectors with the symmetry axis $y=0$ of the circle pencil. The Thaloids on the respective intersection points are the circles given in (15) and (16) and its is elementary to verify that the above given points $C_{1,2}$ are their centers. □

We can also confirm that the contact points of the four common tangents lie in pairs on the circles $\mathcal{K}_1$ and $\mathcal{K}_2$.

If the triangles interscribed to the circles of the hyperbolic pencil do not share symmetries with the circles, the trace $\mathcal{C}_1$ of the incenter also looses its symmetries. This would be the case if one of the two lines which are tangent to c_2 would touch a further circle, say $c_4\neq c_2,c_3$.

Moreover, as we can observe in Fig. 8, the trace of the incenter X_1 becomes a cusped curve. This is also true for the traces of the incenters of the opportunistic triangles. The

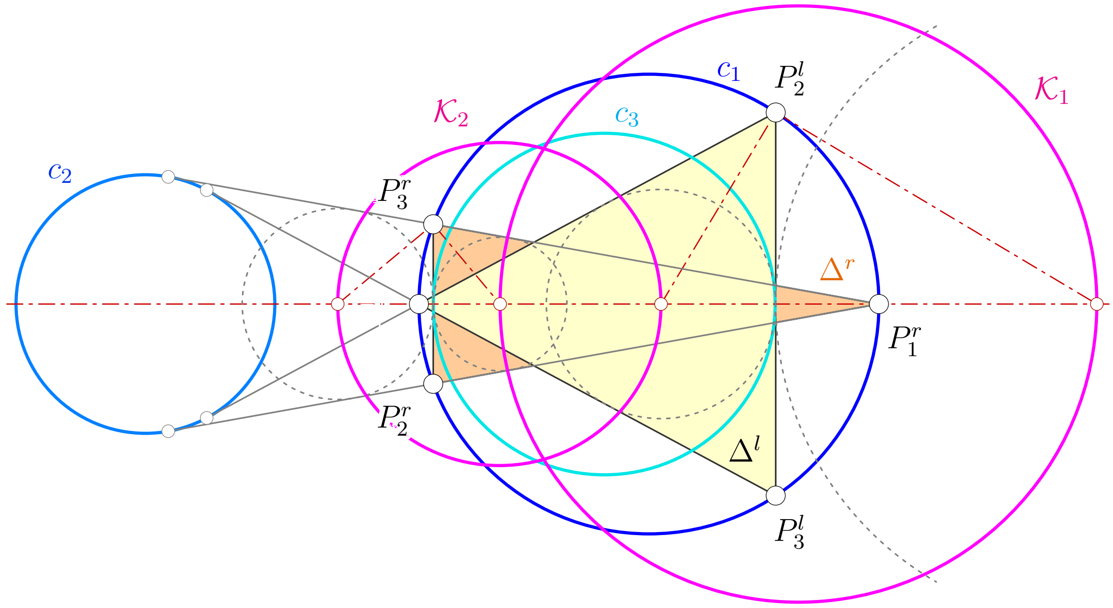

Figure 7: The construction of the circular parts of the paths of X_1.

cusps (singularities) of $\mathcal{C}_1$ correspond to degenerate triangles: Such triangles lie in common tangents of the involved circles and will not become entirely real if all three circles are nested. If one circle, say c_2, lies outside c_1 (and c_3), then there exist 8 real common tangents of which four lead directly to the cusps of $\mathcal{C}_1$. The cusps are located on c_1. The traces of incenters of the opportunistic triangles share some cusps which correspond to degenerate triangles that belong to different (combinatorial) types of opportunistic triangles.

Fig. 8 shows three more cusped curves which are the traces of incenters of opportunistic triangles. The cusped curves are the traces of *true* incenters. Whenever an incenter changes to an excenter (this happens at the cusps), its path is no longer in the interior of c_1. The fourth circle c_4 which is the envelope of the third triangle side is not displayed as well as the exterior branches of $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$, and $\mathcal{C}'''$.

3 One zero circle

3.1 The closure condition

Again, we assume that the we deal with circles in a hyperbolic pencil of circles with equations (2). Like in the previous case, c_1 shall be the circumcircle of the triangles in the poristic family. The line $[P_1, P_2]$ shall pass through the zero circle $c_0 = [1, 0]$ (the *right one*). Further, the line $[P_2, P_3]$ shall be tangent to the circle c_2 and the terminal segment $[P_3, P_1]$ shall touch the circle c_3.

We start with the point P_1 which can be parametrized by

$$P_1 = \left(\frac{T^2 + u_1^2}{u_1(1 + T^2)}, \frac{(u_1^2 - 1)T}{u_1(1 + T^2)}\right) \quad \text{with } T \in \mathbb{R} \tag{18}$$

according to (6).

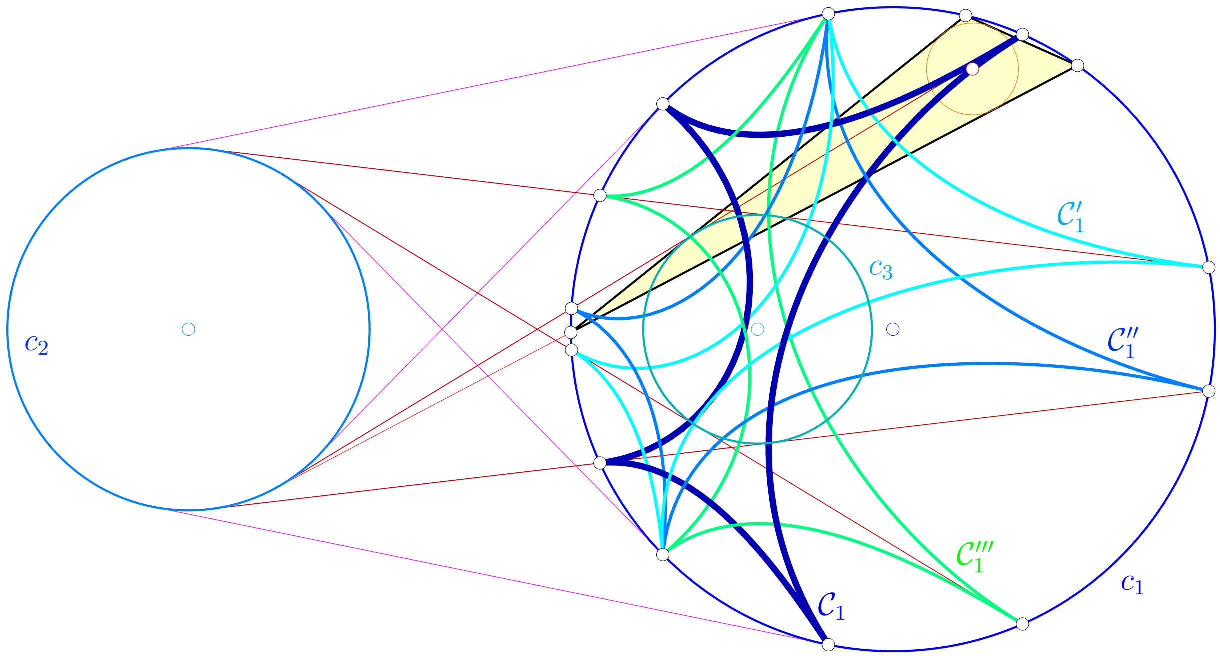

Figure 8: The curve $\mathcal{C}_1$ of incenters of the principal triangle and the curves $\mathcal{C}_1'$, $\mathcal{C}_1''$, $\mathcal{C}_1'''$ of the opportunistic triangles: Cusps are incenters of degenerate triangles.

The line $[P_1, c_0] = [P_1, P_2]$ intersects the circumcircle c_1 at P_2 which therefore obtains the parametrization

$$P_2 = \left(\frac{u_1(1+T^2)}{T^2+u_1^2}, \frac{T(1-u_1^2)}{T^2+u_1^2}\right) \quad \text{with } T \in \mathbb{R}. \tag{19}$$

For the point $P_3 \in c_1$ there exists a parameter $U \neq T \in \mathbb{R}$ such that

$$P_3 = \left(\frac{U^2+u_1^2}{u_1(1+U^2)}, \frac{(u_1^2-1)U}{u_1(U^2+1)}\right) \quad \text{with } U \in \mathbb{R}. \tag{20}$$

Now, U is to be determined such that the lines $[P_2, P_3]$ and $[P_3, P_1]$ touch c_2 and c_3. For that purpose, we first determine the equations of the latter lines and find

$$\begin{aligned}
&[P_2, P_3]\colon (u_1U+T)x + (TU-u_1)y - u_1T - U = 0,\\
&[P_3, P_1]\colon u_1(TU-1)x - u_1(T+U)y + u_1^2 - TU = 0.
\end{aligned}$$

Secondly, we derive the contact conditions of these lines with the circles c_2 and c_3, *i.e.*, we compute the resultants of the linear equations and the equations of the respective circles, and subsequently, we determine the discriminants of the resulting quadratic equations. This yields

$$\begin{aligned}
\mathcal{C}_{23}\colon\ &(T^2U^2+u_1^2)(u_2^2-1)^2 + 4u_2(u_1-u_2)(1-u_1u_2)(T^2+U^2)\\
&+2(u_1u_2^2+u_1-2u_2)(2u_1u_2-u_2^2-1)TU = 0,\\
\mathcal{C}_{31}\colon\ &4(T^2U^2+u_1^2)u_3(u_1u_3-1)(u_3-u_1) + u_1^2(u_3^2-1)^2(T^2+U^2)\\
&-2u_1(u_1u_3^2+u_1-2u_3)(2u_1u_3-u_3^2-1)TU = 0.
\end{aligned} \tag{21}$$

The two equations (21) have to be fulfilled by infinitely many pairs of (T, U), and therefore, they have to be *linearly* dependent. Thus, the resultant of $\mathcal{R} = \operatorname{res}(C_{23}, C_{31}, U)$ with respect to one circle parameter, say U, has to be fulfilled by all T in $\mathbb{R}$. (It is also possible to eliminate

T and discuss the resulting polynomial in U. This leads to the same closing condition.) We compute

$$\mathcal{R} = \operatorname{res}(C_{23}, C_{31}, U)$$

and observe that $\mathcal{R} = \varphi_1^2 \cdot \varphi_2$, *i.e.*, it factors into two polynomials with

$$\varphi_1 = 4u_2u_3(u_1^2+1) - u_1((u_3+1)^2(u_2^2+1) + 2u_2(u_3-1)^2) \tag{22}$$

and the nearly symmetric polynomial

$$\varphi_2 = \sum_{i=0}^{4} c_{2i} u_1^{4-i} T^{2i} \quad \text{with } \underline{c_{2i} = c_{8-2i}} \quad \text{for } i = 0,1,2,3,4.$$

Herein, the coefficients c_{2i} are

$$\begin{aligned}
c_8 &= (4(u_1^2+1)u_2u_3 + ((u_3-1)^2(u_2^2+1) - 2(u_3+1)^2u_2)u_1)^2,\\
c_6 &= 4(2u_3(u_2^2+1)(u_1^2+1) - ((u_2^2+1)(u_3+1)^2 - 2u_2(u_3-1)^2)u_1)\\
&\quad \cdot (2u_2(u_3^2+1)(u_1^2+1) - ((u_2^2+1)(u_3-1)^2 + 2u_2(u_3+1)^2)u_1),\\
c_4 &= 2(8(u_1^4+1)(u_2^2u_3^2(u_2^2+u_3^2+2) + u_2^2+u_3^2) - 8u_1(u_1^2+1)(u_2^4+1)(u_3+1)^2\\
&\quad + u_2(u_2^2+1)(u_3^2-u_3+1)(u_3-1)^2 + 2u_2^2(u_3^2+1)(u_3+1)^2).
\end{aligned}$$

It turns out that the polynomial φ_2 is independent of T if, and only if, all coefficient vanish simultaneously. This is only the case if $u_i = \pm 1$ which implies $t_i = \pm 1$ (for all $i \in \{1,2,3\}$) which is excluded by assumption (otherwise c_i are only zero circles).

Therefore, the only relevant part of $\mathcal{R}$ is the factor φ_1 from (22). With (4), we can rewrite (22) in terms of t_i which yields the surprisingly simple relation

$$t_2t_3 - 2t_1 + t_2 + t_3 - 1 = 0. \tag{23}$$

Assuming that $u_3 = c = \text{const.}$ and $c \neq 0, \pm 1$, then $\varphi_1(u_1, u_2, c) = 0$ from (22) describes a cubic curve in the $[u_1, u_2]$-plane. Independent of $u_3 = c$, the cubic curve has a singularity at $(-1,-1)$, and thus, it admits a rational parametrization

$$\left(\frac{(c+1)^2(\tau+1)\tau}{(c+1)^2\tau + (c-1)^2}, \frac{4c\tau}{(\tau+1)((c+1)^2\tau + (c-1)^2)} \right), \quad \text{with } \tau \in \mathbb{R}.$$

From (22), we can derive a condition on the radii of the circles c_1, c_2, and c_3 to allow for a porism. For that purpose, we eliminate u_i using (5) and find

$$\begin{gathered}
r_2^8r_3^8 - 16r_3^4(r_1^2r_3^2 + 2r_1^2 + 3r_3^2 + 4)r_2^6\\
+2^5(3r_1^4r_3^4 - r_1^2r_3^6 + 8r_1^4r_3^2 - 4r_1^2r_3^4 - 2r_3^6 + 8r_1^4 - 8r_1^2r_3^2)r_2^4\\
-2^8r_1^2(r_1^2 - r_3^2)(r_1^2r_3^2 + 2r_1^2 - r_3^2)r_2^2 + 2^8r_1^4(r_1^2 - r_3^2)^2 = 0.
\end{gathered} \tag{24}$$

Collecting our results, we can formulate in analogy to Thm. 1 the following:

Theorem 4. *Let c_1, c_2, c_3 be three circles of a hyperbolic pencil given by the equations* (2) *with the (right) zero circle $c_0 = (1,0)$. These circles allow a poristic one-parameter family of interscribed triangles $P_1P_2P_3$ such that c_1 is the common circumcircle, $[P_2, P_3]$ is tangent to c_2, $[P_3, P_1]$ is tangent to c_3, and $[P_1, P_2]$ passes through c_0 if their center coordinates t_i satisfy* (23)*, which implies that their radii satisfy* (24).

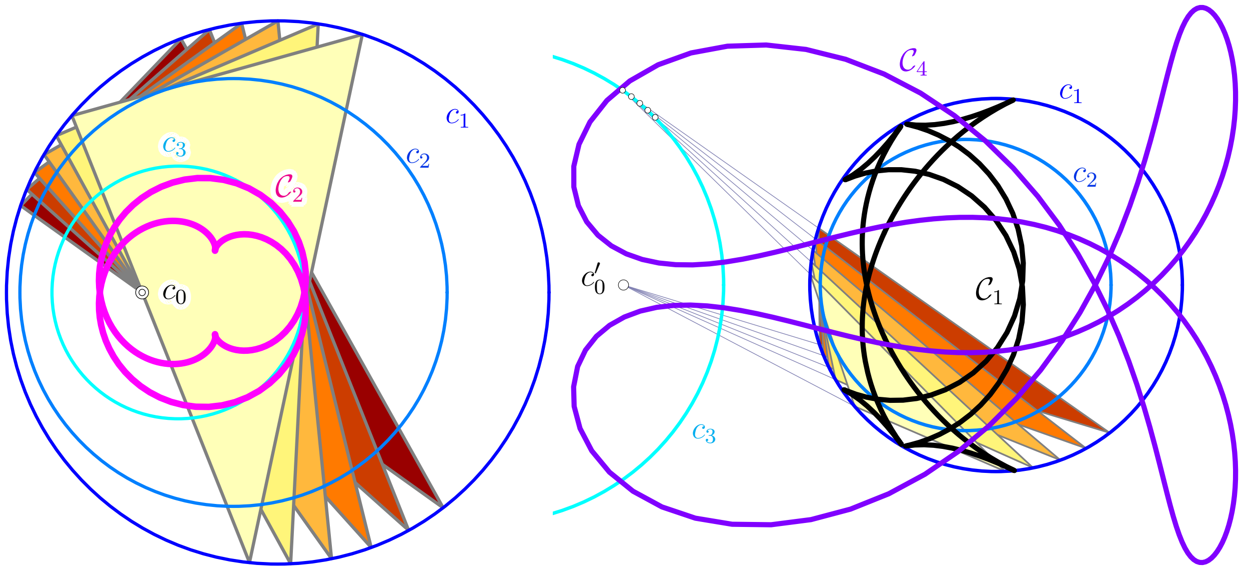

Figure 9: Circles c_1, c_2, c_3 from a hyperbolic pencil including a zero circle: poristic trace $\mathcal{C}_2$ of the centroid (left), poristic trace $\mathcal{C}_1$ and $\mathcal{C}_4$ of the incenter and the orthocenter (right).

3.2 The other zero circle

In the previous subsection, we have chosen the zero circle $c_0 = (1, 0)$ (on the right side). If we replace c_0 with $c_0' = (-1, 0)$, *i.e.*, the *left* zero circle, then the equation equivalent to (22) relating the pencil parameters u_i of the three circles reads

$$4u_2u_3(u_1^2 + 1) + u_1((u_3 - 1)^2(u_2^2 + 1) - 2u_2(u_3 + 1)^2) = 0 \tag{25}$$

and is a planar cubic curve for fixed $u_3 = c = \text{const.}$ with $c \neq 0, \pm 1$ with an isolated double point at $(1, 1)$. Therefore, the totality of circle triples allowing a Poncelet porism in the above mentioned sense can be parametrized by

$$\left(\frac{-(c-1)^2(t+1)t}{(t(c-1)^2 + (c+1)^2}, \frac{4tc}{(t+1)(t(c-1)^2 + (c+1)^2)} \right).$$

Eliminating u_i with (4) from (25) we obtain the analog to (23) for the Poncelet variant with the *left* zero circle

$$t_2t_3 + 2t_1 - t_2 - t_3 - 1 = 0. \tag{26}$$

Similar to 4, we can summarize our results in:

Theorem 5. *Let c_1, c_2, c_3 be three circles of a hyperbolic pencil given by the equations* (2) *with the (left) zero circle $c_0' = (-1, 0)$. These circles allow a poristic one-parameter family of interscribed triangles $P_1P_2P_3$ such that c_1 is the common circumcircle, $[P_2, P_3]$ is tangent to c_2, $[P_3, P_1]$ is tangent to c_3, and $[P_1, P_2]$ passes through c_0 if their center coordinates t_i satisfy* (26)*, which implies that their radii satisfy* (24).

Surprisingly, the condition on the radii of the circles c_i mentioned in Thm. 5 equals the condition in Thm. 4, *i.e.*, the choice of the zero circle does not effect the condition on the radii.

Fig. 9 shows the two different versions of Poncelet porisms with three proper circles an a zero circle c_0. The left-hand side of Fig. 9 shows the variant with the *right* zero circle c_0. The trace $\mathcal{C}_2$ of the centroid is also displayed. The right-hand side of Fig. 9 displays a porism with the *left* zero circle c_0' and a circle c_3 encircling the point c_0'. The traces $\mathcal{C}_1$ (black) and $\mathcal{C}_4$ (violet) of the incenter and the orthocenter are also shown. The locus $\mathcal{C}_1$ of the incenter has six cusps (two are two-fold) which stem from degenerate triangles in the poristic family.

In any case, the loci of the centers X_1, X_2, X_4 (and most probably of many others) consist of two branches and can, therefore, never be rational curves.

Surprisingly, the limits of the orthocenters of the *flat* triangles are proper points, and thus, the curve $\mathcal{C}_4$ shown in Fig. 9 (right) has no real points at infinity.

3.3 Equations of point orbits

In the present case (with one zero circle), it is possible to parametrize the traces of the triangle vertices explicitly in terms of one real parameter T. By virtue of (18) and (19), this is obvious for the points P_1 and P_2. Assuming that u_i are chosen such that the equations (21) are dependent, then we can solve (for example) the first equation with respect to the parameter U and find

$$U = (u_1u_2^2 + u_1 - 2u_2)(2u_1u_2 - u_2^2 - 1)T \pm \frac{2(u_2^2-1)\sqrt{u_2(u_1u_2-1)(u_1-u_2)(T^2+1)(T^2+u_1^2)}}{T^2(u_2^2-1)^2-4u_2(u_1u_2-1)(u_1-u_2)}.$$

Note that $\mathcal{C}_{23}$ and $\mathcal{C}_{31}$ given in (21) are elliptic quartic curves with their only singularities (ordinary double points) at the points $0 : 1 : 0$ and $0 : 0 : 1$ in the projectively closed $[T, U]$-plane.

With the presence of algebraic parametrizations of the triangle vertices P_1, P_2, P_3 it is possible to parametrize the trace of any triangle center. The crucial point is the implicitization (which cannot be done automatically, even with Maple) and it is not so easy to prove that the degree of the curves $\mathcal{C}_2$ and $\mathcal{C}_4$ equals 12.

4 Further closing conditions

In this section, we give the closing condition for poristic triangles interscribed between four circles in a hyperbolic pencil. Further, we deliver two closing conditions for parabolic pencils of circles. This list is far from being complete.

It is not necessary to write down the computation of these conditions in detail, since the techniques used for that purpose do not differ very much from those used in Sec. 2 and Sec. 3.

4.1 Four circles of a hyperbolic pencil

As we have promised earlier, we also give the closing condition for four different circles of a hyperbolic pencil. The four circles c_i ($i \in \{1, 2, 3, 4\}$) with centers $(t_i, 0)$ and radii $r_i = \sqrt{t_i^2 - 1}$ of the hyperbolic standard pencil allow a one-parameter family of interscribed triangles if t_i are subject to

$$4t_1^4 - 4(\sigma + \pi)t_1^3 + (\omega^2 + 6\omega - 3)t_1^2 - 2(\omega - 1)(\sigma + \pi)t_1 + (\pi + \sigma)^2 - 4\omega = 0,$$

where we have used the abbreviations

$$\sigma = t_2 + t_3 + t_4, \quad \omega = t_2t_3 + t_3t_3 + t_tt_2, \quad \pi = t_2t_3t_4.$$

Again, we have assumed that c_1 is the circumcircle of $P_1P_2P_3$ and the each other circle is tangent to exactly one line of the triangle. A condition on the four radii can also be computed by eliminating t_i from the latter equation with (4). It turns out to be of degree 24.

4.2 Some simple examples from parabolic pencils

The following examples of closing conditions were just bycatch and yield comparably simple relations between circle parameters (in the pencil) or radii of the circles.

4.2.1 Four generic circles

The circles c_1 of a parabolic pencil can be given by their equations as

$$c_i\colon x^2 - 2t_i x + y^2 = 0$$

with $t_i \in \mathbb{R} \setminus \{0\}$ and $i \in \{1,2,3,4\}$ (see [4]). Again c_1 is assumed to be the common circumcircle of the triangles. In this case, the circles c_i are centered at $(t_i, 0)$ and have the radii t_i. Poristic families of triangles $P_1P_2P_3$ whose sides $[P_1, P_2]$, $[P_2, P_3]$, $[P_3, P_1]$ are tangent to the circles c_2, c_3, c_4 show up if the four radii satisfy

$$4\delta t_1^3 - \sigma^2 t_1^2 + 2\delta\sigma t_1 - \delta^2 = 0,$$

where $\delta = t_2t_3t_4$ and $\sigma = t_2t_3 + t_2t_4 + t_3t_4$.

4.2.2 Concentric circles

Let c_1 be the circumcircle with radius r_1 and the concentric circles c_i with radii r_i and $i \in \{2,3,4\}$. Then, it is obvious that the sides of the triangle $P_1P_2P_3$ inscribed into c_1 with sides $[P_1, P_2]$, $[P_2, P_3]$, $[P_3, P_1]$ tangent to the circles c_2, c_3, c_4 has the following side lengths: $\overline{P_1P_i} = 2\sqrt{r_1^2 - r_i^2}$ for all $i \in \{2,3,4\}$. Therefore, the triangles of the poristic family are of equal size and perform a pure rotation about X_3 (the center of c_1). Consequently, all centers (except X_3) of the triangle $P_1P_2P_3$ move on circles while the triangle traverses the poristic family.

The well-known formula $4RF = abc$ (relating the three side lengths a, b, c with the area F and the circumradius of a triangle) yields the relation

$$(r_1^3 - r_1r_2^2 - r_1r_3^2 - r_1r_4^2 - 2r_2r_3r_4)(r_1^3 - r_1r_2^2 - r_1r_3^2 - r_1r_4^2 + 2r_2r_3r_4) = 0$$

between the four radii in order to allow a poristic family.

4.2.3 Final remarks

There are still many metric special types of pencils of conics left to discus and to look for closing conditions for poristic triangles and n-gons with arbitrary numbers of vertices. The computational approach towards these conditions shown so far may cause troubles for sufficiently high n. It is also questionable whether our approach is an efficient one. For low n, we are at least able to give closing conditions, and in some simple or special cases, we can derive equations of poristic traces of triangle centers. We cannot expect that the traces and their computation are as simple as it is for the Chapple porism in [9]. At least from the number theoretic point of view, an entirely rational approach and a search for entirely rational solutions (families of poristic n-gons) may be interesting. However, this could be done in a future paper.

5 Acknowledgements

The authors are really grateful to the reviewers for serious and constructive comments.

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Received September 27, 2021; final form November 18, 2021.