Journal for Geometry and Graphics
Volume 25 (2021), No. 2, 295–300

Pentagon from Side Using the Internal Golden Section

Giovanni Asti, Laura Asti

20067 Paullo, Milan, Italy
astigiovanni@virgilio.it

Abstract. A new construction of the pentagon starting from the side is presented. Differently from other versions that are already described in the literature it uses the internal, and not external, golden section. This is a point of strength from the educational point of view, as it shows another possible pathway to create a regular pentagon, helping the students to train their minds to consider many different possibilities in the geometrical constructions. Its Geometrography complexity and inexactness are within the ranges covered by the most famous constructions from the circumscribed circle, and comparable to those of the most commonly taught one from the side.

Key Words: pentagon, construction, side, golden section

MSC 2020: 51M04 (primary), 51M05

1 Introduction

Regular polygons have been studied since ancient Greek times (if not before) and despite nowadays Computer Aided Design (CAD) software makes their construction immediate for an arbitrary number of sides, drawing them only with compass and straightedge still conserves its appeal and theoretical interest.

Among the different polygons, the regular pentagon has always earned special attention, due in a minor part to its being the first true "poly"-gon (after triangle and square) and in a major part to its many peculiar properties (first of all its relation with the "divine" golden section).

Many constructions for it can therefore be found in the literature, starting from the works by Euclid [8] and Ptolemy [15] and arriving at our days. As a general classification, such constructions can be grouped on the basis of the starting point (approximate versions, e.g., the famous one by Dürer [7], will not be considered here):

- constructions from the circumscribed circle (e.g., [1–3, 6, 9, 14, 16, 17, 19, 20] or others reported on Web sites, e.g., [22, 23]);

ISSN 1433-8157/$ 2.50

- constructions from a segment having the same length of the side (e.g., [10]) or a length related to that of the side (e.g., [4]);
- constructions from a side (e.g., [5, 6, 16, 18, 21] or others reported on Web sites, e.g., [24, 25, 27]).

Focusing the attention on the latter group, these constructions are usually based on finding the *external* golden section point of the given segment. Then the majority of them determine the top vertex of the "golden triangle" and from it the two "external" vertices of the pentagon, while a minor number of constructions determine the "external" vertices first and the top vertex as a final step.

On the contrary, a construction is presented here that is based on determining the *internal* golden point in the given segment and from that constructing the "golden gnomons", thus locating the "external" pentagon vertices. The top vertex of the pentagon is finally identified from the latter.

2 Educational purpose

As already said, the advent of CAD systems made geometrical constructions immediate, so that it may seem that studying and teaching them "with compass, straightedge and square" is obsolete and no longer of importance. On the contrary, it still conserves its value. First of all, to form aware users of the software tools, able to check their output and if needed to overcome possible limitations (e.g., but not only, during software development). In addition, the study of the constructions thought by the different authors, the effort in finding new ones or new ways for some steps, and their analysis in terms of underlying concepts, proof, complexity, etc. are fundamental to keep geometrical knowledge and ability alive in the new generations, without reducing geometrical drawing and design to a mere "click a button" activity. More in general, analysing a process (in this case the construction) and searching for the most straightforward way to reach the objective is a recommended approach in many design activities (e.g., consider the "K.I.S.S." principle [26]). This is also a very useful exercise in reasoning – a "brain gym" – and as in other disciplines (e.g., fluid dynamics, with the development of Computational Fluid Dynamics), the availability of powerful resources of calculus should not make forget the primary role of the theoretical analysis in the continuation and improvement of knowledge. Specifically for the proposed construction, in the authors' opinion it is more straightforward than others available in the literature, and interesting as it evidences the repeated occurrence of the golden section in the construction.

3 Construction of a pentagon from side using the internal golden section.

The proposed construction requires only a straightedge and a compass (if the latter is modern, noncollapsing, the construction requires less steps). It is graphically summarized in Fig. 1, and its steps are as follows:

1. Draw a line segment AB, that will be the first side of the pentagon.
2. Find the midpoint of AB: e.g. by opening the compass with aperture AB and pointing first in A and then in B draw the two circles $C_1 = A(AB)$ – meaning with this notation a circle with center in A and radius AB – and $C_2 = B(AB)$; their intersections are C and D; the intersection of the line segment CD with the pentagon side AB is E, the

midpoint of AB.

3. Determine the internal golden section of AB (i.e., find the segment, smaller than AB, whose length divided by the length of AB gives the golden ratio) with one of the usual constructions, e.g.,
   a) Draw a line segment perpendicular to AB and having length $AB/2$, e.g., by drawing three circles $C_3 = B(AE)$, $C_4 = E(AE)$, $C_5 = F(AE)$ where F is the intersection between C_4 and CD. The point of intersection $G = C_3 \cap C_5$ is on the vertical of B and at a distance AB from B.
   b) Draw the line segment AG and the circle $C_6 = G(GB \cong AB/2)$; their intersection is point H. AH is congruent with the internal golden section of segment AB.
4. Draw the circle $C_7 = A(AH)$, whose intersection with CD is I.
5. Extend the line segment passing through A and I until it intersects C_2 in J, that is another vertex of the pentagon. The symmetric vertex K can be likewise found by extending the line segment BI until it intersects C_1.
6. Draw the two circles $C_8 = J(AB)$ and $C_9 = K(AB)$; $L = C_8 \cap C_9$ is the top vertex of the "golden triangle" and fifth vertex of the pentagon.

The proof of the proposed construction is immediate, given that the diagonal of the pentagon is equal to the side plus its golden section. Triangles AIB and ABJ are both isosceles and they share angle BAJ. Therefore they are similar triangles. AH is congruent to the golden section φ of AB, so AB is congruent to the golden section of AJ and $AJ = AB(1+\varphi)$.

As additional notes, it can be observed that I is one of the intersection points of the pentagram inscribed in the pentagon itself, and the same holds for point M that is the intersection between C_2 and C_7 within the pentagon. Thus triangle IAM is one of the five triangles of the pentagram and IM is a side of the inner pentagon (so – as it is well known – AM is the golden section of AB, IM is the golden section of AM, and so on).

4 A possible modification of the construction.

Another possible construction of the pentagon is to follow the previously described steps up to step 3, then C_2 intersects C_7 in M. The intersection of $C_{10} = M(AH)$ and C_1 is vertex K. The top vertex L can also be found drawing $C_{11} = K(AH)$ and $C_{12} = J(AH)$ that intersect C_1 and C_2 respectively in other two points of the pentagram. Two final circles centered in the latter points and having radius AH intersect in vertex L.

5 Complexity and inexactness of the proposed construction.

In this section the proposed construction is analysed from the point of view of Geometrography, calculating its *complexity* and *inexactness* — named according to Merikoski and Tossavainen [13], and coinciding with the "simplicity" and "exactitude" in the original work by Lemoine [11].

The complexity and inexactness are defined on the basis of the number of the elementary operations for geometrical constructions, as follows [13]:

- Basic operation L1: place the ruler through a given point.
- Basic operation L2: draw a line.
- Basic operation C1: place one leg of the compass on a given point.

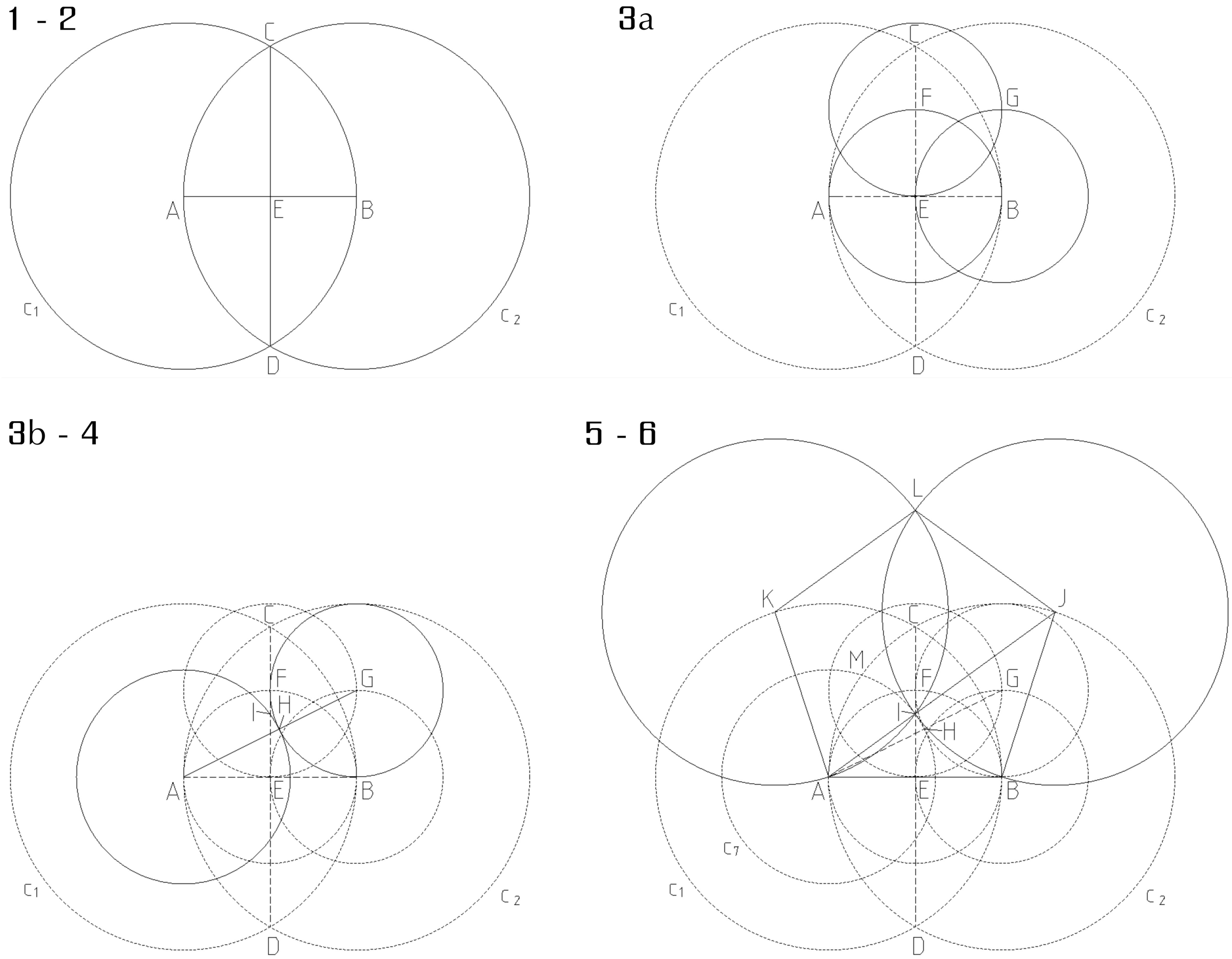

Figure 1: Sketch of the steps of the proposed construction.

- Basic operation C2: place one leg of the compass on an indeterminate point of a given line.
- Basic operation C3: draw a circle.
- Complexity of the construction: sum of the total numbers of occurrences of all the basic operations in the construction.
- Inexactness of the construction: sum of the numbers of occurrences of the L1, C1 and C2 basic operations (those in which placing is involved) in the construction.

Therefore, if the numbers of occurrences of the single basic operations are indicated with l_1, l_2, c_1, c_2, c_3, the complexity is $l_1 + l_2 + c_1 + c_2 + c_3$, the inexactness is $l_1 + c_1 + c_2$, and a construction can also be identified by the Lemoine's 5-tuple $(l_1, l_2, c_1, c_2, c_3)$.

Table 1 details the basic operations needed for the proposed construction, with reference to the steps enumerated in Sec. 3. No operation of type C2 is needed.

The resulting complexity is 34 with an inexactness of 21. The Lemoine's 5-tuple is (8,4,13,0,9). So the proposed construction has parameters of merit comparable with the most famous constructions: as for those from the circle, their complexity is in range 15–45 and inexactness in the range 10–26 [13]; other constructions may reach values over 100 [12]; concerning the constructions from the side, the one that is likely the most commonly taught ([6]) has complexity of 33 (so one point lower), but inexactness of 24 (so three points higher).

Step	*Operations*	*Complexity*	*Inexactness*
1	–	–	–
2	2 L1, L2, 3 C1, 2 C3	8	5
3	2 L1, L2, 5 C1, 4 C3	12	7
4	2 C1, C3	3	2
5	4 L1, 2 L2	6	4
6	3 C1, 2 C3	5	3
Total	8 L1, 4 L2, 13 C1, 9 C3	34	21

Table 1: Complexity and inexactness of the proposed construction.

References

[1] B. Atkin: *Constructing an inscribed regular pentagon.* Math. Gaz. **72**(460), 119, 1988. doi: 10.2307/3618923.

[2] M. Bataille: *Another Compass-Only Construction of the Golden Section and of the Regular Pentagon.* Forum Geom. **8**, 167–169, 2008.

[3] W. Blythe, ed.: *Geometrical Drawing. With notes and examples.* Cambridge University Press, 1900.

[4] F. Boon: *Construction of a regular pentagon.* Math. Gaz. **20**(238), 153, 1936. doi: 10.2307/3606303.

[5] E. Budden: *Construction of a regular pentagon.* Math. Gaz. **3**(59), 381–382, 1906. doi: 10.2307/3603726.

[6] C.-H. Chang: *Apply Discovery Teaching Model to Instruct Engineering Drawing Course: Sketch a Regular Pentagon.* Procedia - Social and Behavioral Sciences (International Educational Technology Conference - IETC2012) **64**, 457–466, 2012.

[7] A. Dürer: *Underweysung der Messung, mit dem Zirckel und Richtscheyt (Treatise on Mensuration with Compass and Ruler).* https://www.cut-the-knot.org/pythagoras/DurerPentagon.shtml.

[8] Euclid: *Elements IV.11*, ca. 300 BC.

[9] K. Hofstetter: *A Simple Compass-Only Construction of the Regular Pentagon.* Forum Geom. **8**, 147–148, 2008.

[10] K. Hofstetter: *A Simple Ruler and Rusty Compass Construction of the Regular Pentagon.* Forum Geom. **8**, 61–62, 2008.

[11] E. Lemoine: *La Géométrographie, ou l'Art des Constructions Géométriques.* Gauthier-Villars, Paris, 1893.

[12] J. Mackay: *The geometrography of Euclid's problems.* Proc. Edinburgh Math. Soc. **12**, 2–16, 1893. doi: 10.1017/S0013091500001565.

[13] J. Merikoski and T. Tossavainen: *Two Approaches to Geometrography.* J. Geom. Graphics **13**(1), 15–28, 2010.

[14] D. Nelson: *A regular pentagon construction.* Math. Gaz. **61**(417), 215–216, 1977. doi: 10.2307/3617225.

[15] Ptolemy: *Almagest I.10*, ca. 2nd century AD.

[16] D. Raynaud: *Abū al-Wafā' Latinus? A study of method.* Historia Math. **39**(1), 34–83, 2012. doi: 10.1016/j.hm.2011.09.001.

[17] H. Richmond: *A Construction for a Regular Polygon of Seventeen Sides.* Quarterly J. Pure Appl. Math. **26**, 206–207, 1893.

[18] G. Sansò: *The Euclidean right triangle: remarks on the regular pentagon and the golden ratio.* Lettera Matematica **4**(3-4), 189–194, 2016. doi: 10.1007/s40329-016-0145-1.

[19] A. Sparavigna and M. Baldi: *Japanese Temple Geometry: A Digital Sangaku About a Regular Pentagon and the Golden Ratio.* SSRN Electronic Journal 2016. doi: 10.2139/ssrn.2836615.

[20] A. Sparavigna and M. Baldi: *A Study of the Regular Pentagon with a Classic Geometric Approach.* hal-01295771v2, 2016. https://hal.archives-ouvertes.fr/hal-01295771/document.

[21] A. Van der Heyden: *Construction of a regular pentagon.* Math. Gaz. **3**(57), 323–324, 1906. doi: doi.org/10.2307/3604995.

Internet Sources

[22] S. Brodie: *Inscribing a regular pentagon in a circle - and proving it* http://www.cut-the-knot.org/pythagoras/pentagon.shtml.

[23] R. Hu: *Nine constructions for the regular pentagon* http://www.geocities.ws/robinhuiscool/Pentagon.html.

[24] A. La Manna: *Costruzione del pentagono con il metodo della sezione aurea. La Scienza del Disegno.* www.lascienzadeldisegno.it/disegno_geometrico_11.html.

[25] Seyed: Post in the Mathematics StackExchange forum. https://math.stackexchange.com/questions/2582818/constructing-a-regular-pentagon-of-a-desired-length.

[26] Wikipedia: *KISS principle.* https://en.wikipedia.org/wiki/KISS_principle.

[27] Wikipedia: *Pentagon.* https://en.wikipedia.org/wiki/Pentagon.

Received January 9, 2022; final form January 26, 2022.