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A Note on Some Generalizations of Monge's Theorem

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Abstract. We generalize Monge's theorem for $n+1$ pairwise homothetic sets (in particular convex bodies) in E^n in place of three disks in E^2. We also present a version for homotheties for pairs of vertices of a non degenerate simplex in E^n. It includes a reverse of Monge's theorem. Moreover, we give an analogon of Monge's theorem for the n-dimensional sphere and hyperboloid model of the hyperbolic space.

Key Words: Monge's theorem, Menelaus' theorem, Euclidean space, sphere, hyperbolic space

MSC 2020: 52A20 (primary), 52A21, 52A55

1 Introduction

For any two circles in a plane, an *external tangent* is a line that is tangent to both circles but does not pass between them. There are two such external tangent lines for any two circles. Each such pair of external tangents for circles of different size has a unique intersection point. The classic Monge's theorem states that for three such pairwise disjoint circles of different size the three intersection points of the external tangent lines given by the three pairs of circles always lie in a straight line. For instance, see the book [4] by Gardner. Walker [9] generalized Monge's Theorem for $n+1$ balls in the Euclidean n-space E^n and a hyperplane in place of the above straight line.

First we present a version of Monge's theorem for $n+1$ linearly independent points in place of the balls. This version includes also a reverse of Monge's theorem. Next we give a generalization of Monge's theorem for $n+1$ pairwise homothetic bounded sets (not obligatory disjoint) with homothety ratios over 1 in E^n in place of the $n+1$ balls in E^n. A good visualization is obtained by taking convex bodies in place of our bounded sets.

Moreover, we give spherical and hyperbolic n-dimensional analogons of Monge's theorem. Our proof applies the n-dimensional Menelaus' theorem generalization for the n-dimensional sphere S^n and hyperbolic space H^n by Ushijma [8]. Compare also the paper [1] by Akyopyan. Questions remain about possible analogs of Monge's theorem in Thurston geometries (see [7]), and in axiomatic geometry as for instance Guggenheimer [5] treats the Menelaus' theorem.

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2 Monge's theorem for a wide class of sets in E^n

Let us start with a proposition on a version of Monge's theorem for $n+1$ points in E^n instead of balls.

Recall that a set A of points in E^n is said to be *independent* if the affine span of any proper subset of A is a proper subset of the span of A.

Proposition 1. *Let the set of points $a_1, a_2, \dots, a_{n+1} \in E^n$ be independent. Consider the straight line L_{ij} containing $a_i a_j$ and a point $b_{ij} \in L_{ij}$ different from a_i and a_j for $i, j \in \{1, \dots, n+1\}$. For every $i < j$ denote by λ_{ij} the ratio of homothety with center b_{ij} which transforms a_j into a_i. We claim that the $n(n+1)/2$ points b_{ij} belong to one hyperplane if and only if $\lambda_{ij}^{-1}\lambda_{ik}\lambda_{jk}^{-1} = 1$ for every $i, j \in \{1, \dots, n+1\}$.*

Proof. Let us apply the variant of Theorem 2 of the paper [2] by Buba-Brzozowa in which we take into account lengths instead of oriented lengths. Her n-dimensional generalization of the classic Menelaus' theorem says that points b_{ij}, where $i, j = 1, \dots, n+1$ and $i < j$, belong to one hyperplane of E^n if and only if

$$\frac{|a_i b_{ij}|}{|b_{ij} a_j|} \cdot \frac{|a_j b_{jk}|}{|b_{jk} a_k|} \cdot \frac{|a_k b_{ik}|}{|b_{ik} a_i|} = 1.$$

Since $\frac{|a_i b_{ij}|}{|b_{ij} a_j|} = \lambda_{ij}$ for every $i, j \in \{1, \dots, n+1\}$, where $i < j$, we obtain that all our points b_{ij} belong to one hyperplane if and only if $\lambda_{ij}^{-1}\lambda_{ik}\lambda_{jk}^{-1} = 1$ for every $i, j \in \{1, \dots, n+1\}$, which is our thesis. □

Clearly, if we agree that the points $a_1, \dots, a_{n+1}$ in Proposition 1 are dependent (but still different), then the "if" part trivially holds true.

Theorem 1. *Assume that for sets $C_1, \dots, C_{n+1} \subset E^n$ and for every $i, j \in \{1, \dots, n+1\}$ with $i < j$ there are unique homotheties h_{ij} of ratios over 1 such that $h_{ij}(C_j) = C_i$. Then the $n(n+1)/2$ centers of these homotheties are in one hyperplane.*

Proof. Consider the $\frac{1}{2}n(n+1)$ homotheties h_{ij} such that $h_{ij}(C_j) = C_i$, where $i, j \in \{1, \dots, n+1\}$ and $i < j$ (for three convex bodies in E^2 see Figure 1). Of course, for every three homotheties h_{pq}, h_{pr}, h_{qr}, where $p < q < r$ we have $h_{pr}(h_{qr}^{-1}(h_{pq}^{-1}(C_p))) = C_p$. By the uniqueness of the homotheties h_{ij}, there are points $o_1, \dots, o_{n+1}$ such that $h_{ij}(o_j) = o_i$ for every $i, j \in \{1, \dots, n+1\}$ with $i < j$. From the "if" part of Proposition 1 we get our thesis. □

If $C_1, \dots, C_{n+1}$ from Theorem are centrally-symmetric convex bodies, we may say that Monge's theorem holds for $n+1$ balls of different sizes of a normed n-dimensional space in place of $n+1$ balls of Euclidean space. A particular case is for the two-dimensional L_p spaces in the paper [3] by Ermiş and Gelişgen. The author thanks them for sharing their preprint [3] which was mobilizing for formulating the above theorem.

The assumption that homotheties h_{ij} are unique holds if the sets are bounded and nonempty. It also holds for some unbounded sets. For instance when in E^2 we take C_i as the intersection of half-planes $x \geq 0$, $y \geq 1/i$, $x + y \geq 4 - i$, where $i = 1, 2, 3$. Then all b_{ij} are different and lie on the line $x = 0$. If we exchange $y \geq 1/i$ into $y \geq 0$ here, all b_{ij} coincide and still are on $x = 0$. For example the assumption does not hold for any family of $n+1$ translates of a half-space in E^n; the thesis may be not true for some homotheties with ratios over 1 between them. We let the reader to show that if the assumption does not hold then there are some positive homotheties for which Theorem is still true.

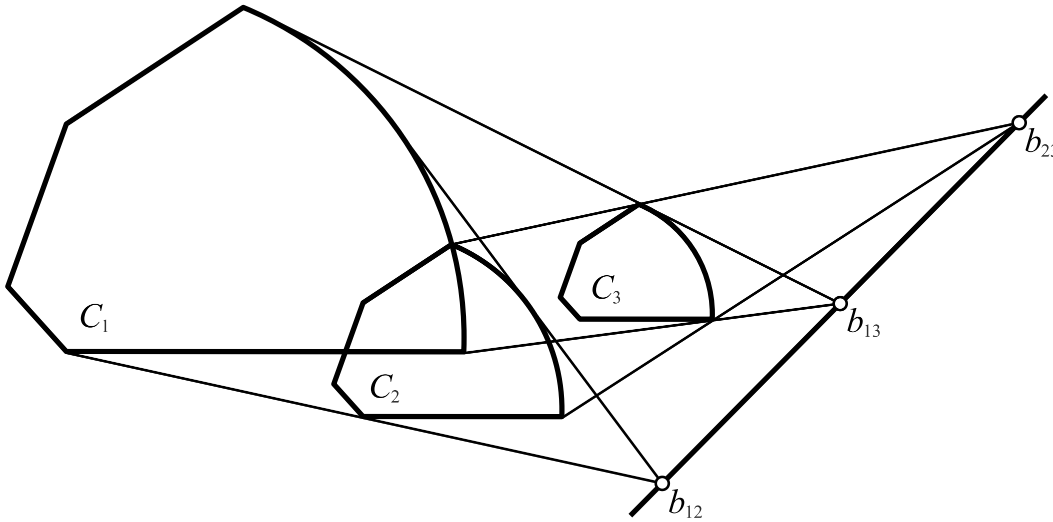

Figure 1: Illustration of Theorem in case of three convex bodies in E^2

3 Analogons of Monge's theorem in spherical and hyperbolic spaces

Below by X^n we denote both the sphere S^n and the hyperboloid model H^n of the hyperbolic n-dimensional space. By a *hyperplane* and a *line* of X^n we mean a subset of X^n isometric to X^{n-1} and X^1, respectively. Usually, for S^n they are called an $(n-1)$-*dimensional subsphere* and a *great circle,* respectively. By the *distance* $|xy|$ of points $x, y \in X^n$ (which are not opposite if $X^n = S^n$) we mean the length of the geodesic joining x and y. By the *arc* xy between x and y we mean the set of points p such that $|xp| + |py| = |xy|$. We say that any such p *is between* x *and* y.

For different points $c, p, r \in X^n$ such that p is between c and r, and $|cr| = \lambda \cdot |cp|$, we say that r is the *image* of p under the X^n-*homothety* with center c and ratio λ. Clearly $\lambda \geq 0$. Observe that $\lambda < \frac{\pi}{2}$ for S^n.

We call a set $A \subset X^n$ (embedded in E^{n+1}) to be *independent* if the set $A \cup \{o\}$, where o is the origin of E^{n+1}, is independent in E^{n+1}.

The Menelau's theorem on S^2 is recalled in Proposition 66 of the book [6] by Rashed and Papadopoulos. Recently their generalizations for S^n and H^n are given in Theorem 4 of Ushijima [8]. From this result, similarly to the proof of our Proposition 1 for E^n, we get the following Proposition 2 (illustrated in Figure 2 for S^2) on the analogous variant of Monge's theorem for $n+1$ points on S^n and H^n. Here by λ_{ij} we mean $\frac{\sin |a_i b_{ij}|}{\sin |b_{ij} a_j|}$ for S^n and $\frac{\sinh |a_i b_{ij}|}{\sinh |b_{ij} a_j|}$ for H^n.

Proposition 2. *Let $a_1, a_2, \dots, a_{n+1}$ be independent points of X^n. Take into account the line L_{ij} containing $a_i a_j$. Denote by b_{ij} a point different from a_i and a_j in L_{ij} such that $a_j \in a_i b_{ij}$ for $i, j \in \{1, \dots, n+1\}$. For every $i < j$ denote by λ_{ij} the ratio of the X^n-homothety with center b_{ij} which transforms a_j into a_i. Then the $\frac{1}{2}n(n+1)$ points b_{ij} are in one hyperplane of H^n if and only if $\lambda_{ij}^{-1} \lambda_{ik} \lambda_{jk}^{-1} = 1$ for all $i, j \in \{1, \dots, n+1\}$.*

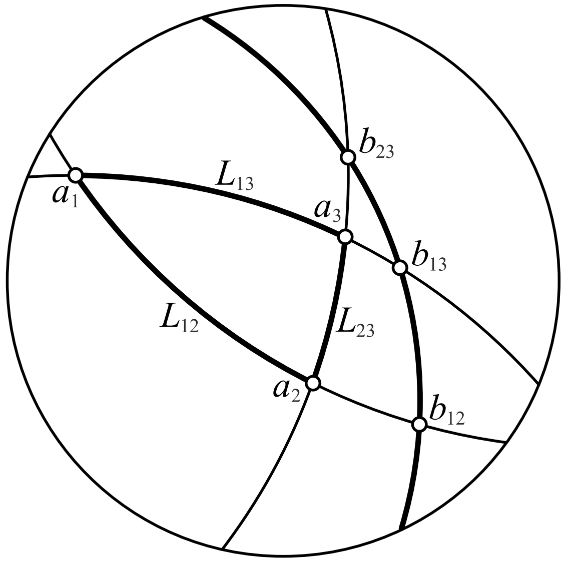

Figure 2: Illustration of Proposition 2 for S^2

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