Journal for Geometry and Graphics
Volume 25 (2021), No. 2, 271–293

Perspectives on Spatial Thinking

Günter Maresch[1], Sheryl A. Sorby[2]

[1] *University of Salzburg, Salzburg, Austria*
guenter.maresch@plus.ac.at

[2] *Michigan Technological University and University of Cincinnati, Houghton, Michigan and Cincinnati, Ohio, USA*
sheryl@mtu.edu

Abstract. At the end of the 19th century, the ability to think spatially was recognized as one of the crucial facets of human intelligence and intensive research has been carried out in this area by researchers from different branches of science since then. This paper presents the four relevant areas of science that deal with spatial thinking in depth, each approaching it from different perspectives, focusses on training and relations to science, mathematics, engineering, and technology (STEM) professions, and educational perspectives. The different approaches, key messages, and models are discussed in this paper, which, in the context of the historical genesis of spatial thinking research, opens the possibility of being able to see the topic of spatial thinking in a more comprehensive way. Fully understanding the historical context of spatial cognitive research will provide a solid foundation for further developments in this area. Finally, reference is made to the model of "the basic practices of spatial thinking", which was developed with the intention of extracting the essence from the findings of the four scientific areas that deal in depth with spatial thinking and to bring them together in one model.

Key Words: spatial thinking, spatial ability, spatial cognition, neurology

MSC 2020: 97C40

1. Introduction

The ability to think spatially is one of the most fundamental cognitive abilities of humans. It enables us to move in our environment, to aim at targets, to plan routes, to estimate distances and speeds, to recognize the position of spatial objects in relation to each other, to fill refrigerators and trunks skillfully, and to move bulky objects through stairwells and other rooms. Spatial thinking includes the ability to imagine spatial objects and to rotate (Figure 1), mirror, and move them purely by the power of our imagination, to imagine the position of several objects in relation to each other in space, and to imagine intersections of

ISSN 1433-8157/$ 2.50

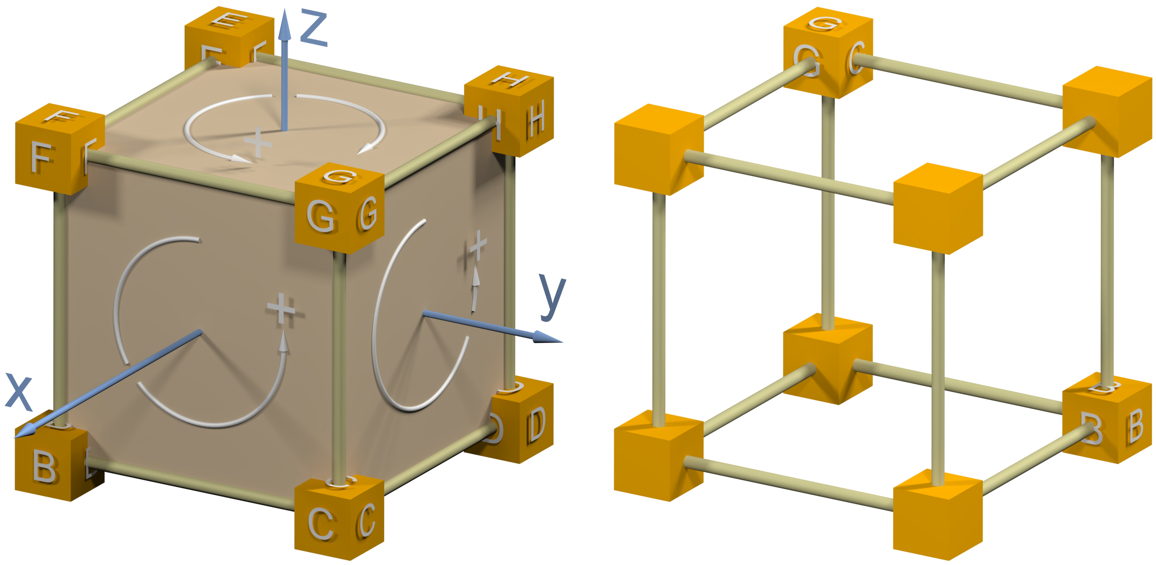

Figure 1: An exemplary task for the mental rotation of a cube: The cube given on the right is rotated along the given axes x, y, z until it reaches the position of the cube on the left. [51]

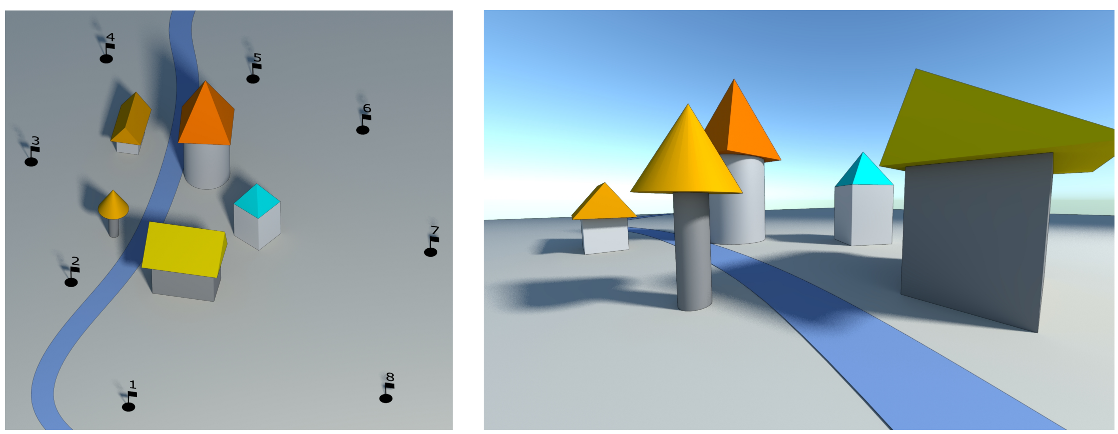

Figure 2: An exemplary task for the mental ability to take other spatial perspectives: The left image shows a photo of a small village. A photographer takes a photo of the village from positions 1–8. From which position was the photo on the right taken? [51]

objects. These are all examples of the ability to imagine spatial objects cognitively and to manipulate them (rotate, mirror, cut, intersect, ...) mentally. In addition, the ability to think spatially includes the ability to mentally take other perspectives (Figure 2). Humans have the ability to imagine a building we know (e.g., the school we attended), recall the spatial layout and relations there, and, if necessary, describe certain paths (e.g., from a classroom to the cafeteria) to another person.

Spatial thinking has been a skill for humans, which is essential for survival, since time immemorial. Hunters, chasing wounded prey needed to keep a mental map of their path in order to return to their home by the shortest path at the end of the day. Gatherers needed to be able to recall where the best locations were for berries, mushrooms, or other necessities of life. More than ever, the ability to think spatially is needed in our modern technology-based world. An analysis of the major social changes since the turn of the millennium shows that our world is continuously becoming more digital, data-based, and visual [28]. While the media

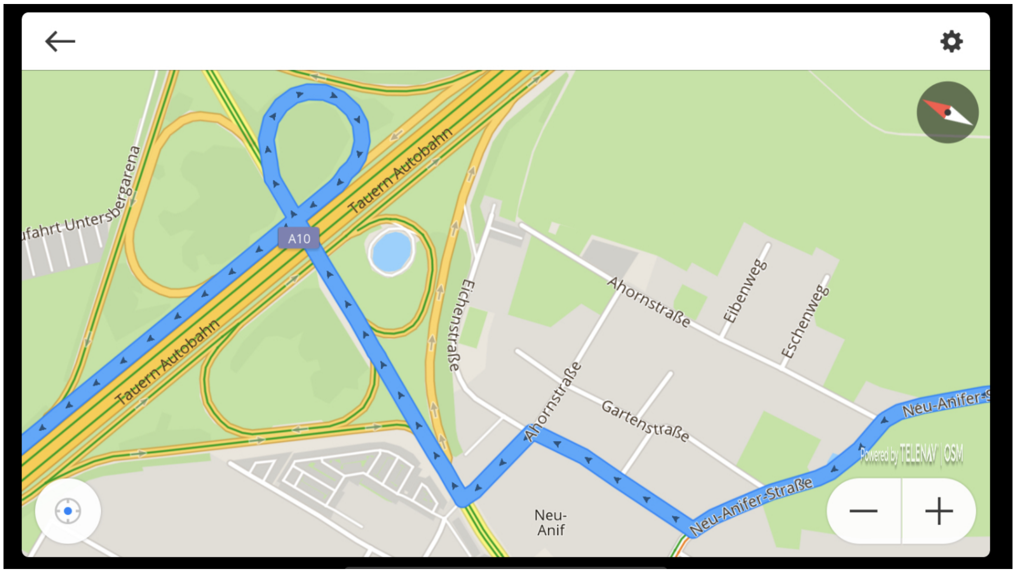

Figure 3: Route displayed on a navigation device

frequently reports digital progress and achievements, and discussions around phenomena such as big data and privacy in the context of data richness of modern times predominate, the fact that our world is becoming more visual seems to be little noticed. The term “little noticed” in this context is not to be equated with “little effect” or “little significance.” Quite the contrary. In almost all areas of our private and professional everyday life, we are increasingly confronted with visual information and challenges. We must recognize them, interpret them correctly, and make appropriate decisions based on this visual information. Navigation or GPS devices are a good example of this. These devices in our vehicles potentially help us enormously in finding the best way from A to B (Figure 3).

However, a closer examination clearly shows that there are some visual challenges associated with the use of these modern-day tools. With only a brief glance at the device, the user has to be able to recognize whether the device is correctly displaying the actual position of the car in the environment, which objects of the real environment are represented by corresponding individual graphic elements on the screen, and finally, which of the route options in the environment corresponds to the driving route displayed on the screen. And all of this mental processing and decision-making must be accomplished in a relatively short timeframe. This continuous translation of the moving symbols on the screen of the navigation device in real time is a cognitively challenging process that requires well-developed spatial thinking skills. Reports of accidents in which a driver misinterprets the route displayed on the navigation device (Figure 4) and mistakes a staircase for the next side street confirm this assumption.

In many other areas of our current private or professional lives, visual skills are increasingly a necessity, such as the appropriate placement of spatial objects in preparation for 3D printing, the control and navigation of drones, the navigation of mobile robots in canals, human surgery via minimally invasive procedures using laparoscopic devices, and in dental technology, where high-precision processes starting with 3D scans and CAD-supported planning are ultimately used to plan and realize implants, crowns, and braces. Soon we will encounter numerous other visual developments, such as Virtual Reality (VR) or Augmented Reality (AR), in our private and professional everyday lives that will require even better and more sophisticated spatial thinking skills than have been necessary before. For example, shopping in stores,

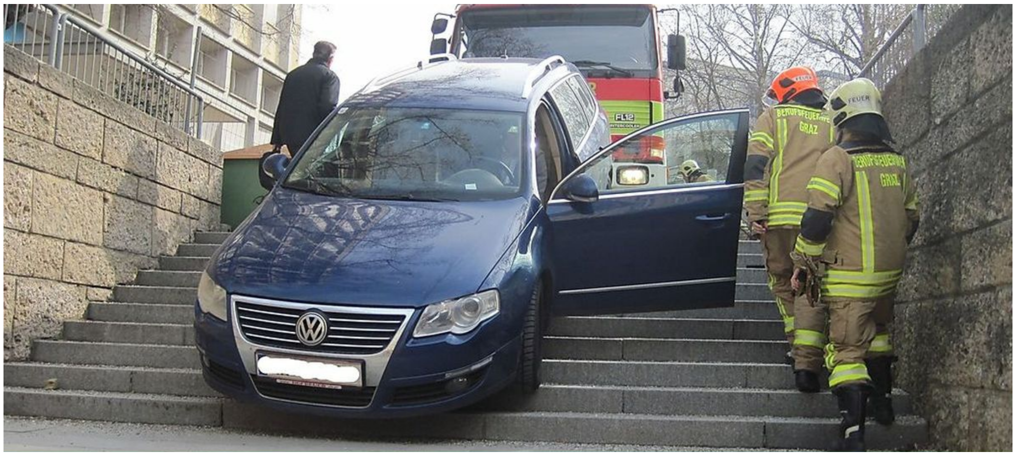

Figure 4: Photo of a car mishap in which a misinterpretation of the display of a navigation device is likely the cause of the accident (Photo: Professional fire department Graz; used by permission)

planning and viewing home (re)construction, buying houses, assembling furniture, setting up and installing technical devices, and much more will be supported to a greater extent by virtual environments afforded through VR and AR technologies.

2. Spatial Thinking

The term "spatial thinking" is used as an umbrella term covering various concepts such as spatial perception, spatial ability, visual perception, or spatial intelligence. An analysis of the descriptions and definitions of these terms shows that they are often described and used differently by people from different scientific fields [16, 24]. The sub-terms are not understood to be synonyms for the general term "spatial thinking," but refer to specific spatial sub-skills that are described and used differently in various contexts in the literature. The term "spatial thinking" is used as a unifying, common umbrella term for the human ability to direct optical stimuli received by the eye into the brain, to be able to interpret these stimuli, to be able to recognize spatial objects, to be able to mentally imagine spatial objects (with or without prior optical stimuli), to be able to manipulate these objects mentally, to be able to imagine taking other perspectives in space, to be able to perceive and interpret motion sequences, and to be able to execute spatial motor movements (Figure 5).

Basically, *visual-anatomical senses* are needed to be able to receive optical stimuli (Figure 5). The facets of *visual perception* help to filter and interpret the optical stimuli, supported by further sensory impressions such as hearing or kinesthetics, and thus to analyze them fundamentally. With the help of the fascinating possibility of purely mental operating with spatial objects, these objects can now be manipulated (purely mentally), or we can take other spatial perspectives purely mentally, and can thus use our *spatial ability.* If necessary, we also plan (fine)*motor movements,* execute them, see the results of the execution, and thus get into a loop-like sequence of spatial thinking (right arrows in the Figure 5). In contrast, in creative processes (such as designing a house) we do not need any optical stimuli to think spatial (see left arrow in the Figure 5). Here, we use our spatial ability to mentally design new things or to develop solutions to spatial-geometric problems.

The umbrella term "spatial thinking" includes the abilities in the following list. Figure 6 illustrates the first three items from this list.

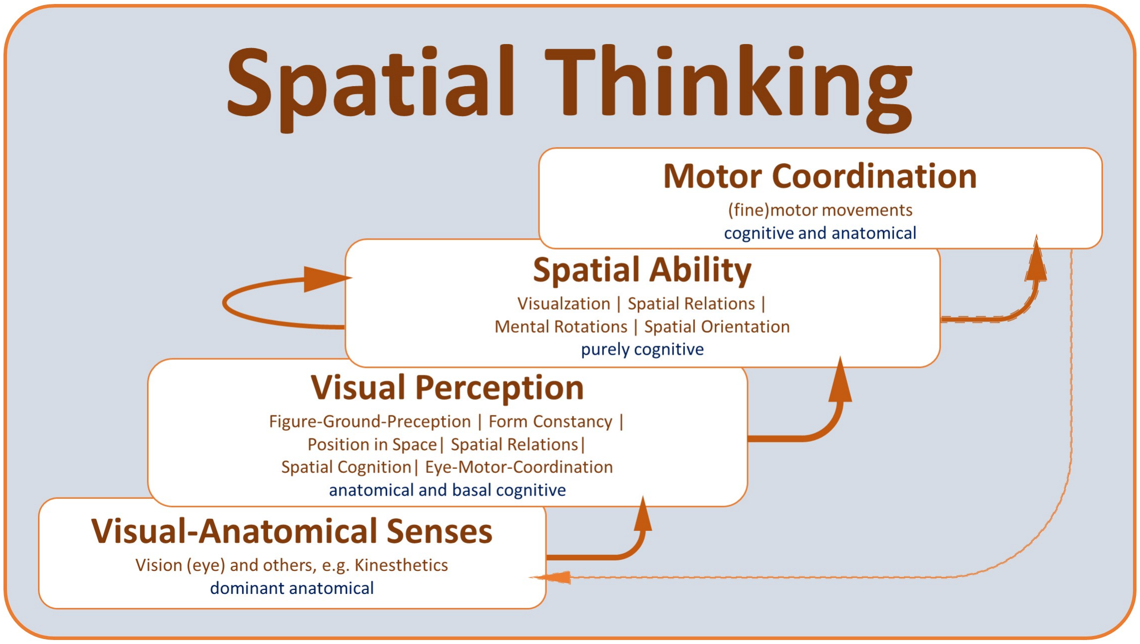

Figure 5: The successive stages of the process of spatial thinking

- Receiving the optical stimulus (via the eye) and forwarding it via various stations to the brain
- Interpreting the optical stimulus and cognitively comparing the optical stimulus to experiences at the various stations from the eye to the brain and thus being able to "recognize" or learn new spatial objects
- Planning and executing spatial motor reactions to optical stimuli
- Imagining spatial objects in the mind (without prior optical stimulus)
- Mentally manipulating spatial objects through rotating, moving, mirroring, or scaling
- Taking other perspectives in space purely mentally.

3. Spatial Thinking Skills as a Prerequisite for STEM Professions

The ability to think spatially not only supports what we do in our private lives, but also facilitates or promotes a professional career in the fields of mathematics, computer science, natural sciences, engineering, and technology (STEM). In 2009, Wai and his colleagues [48] convincingly demonstrated, through the analysis of longitudinal data obtained in the 1960s, that the well-developed ability to think spatially is a key qualification for successful careers in STEM fields. The spatial thinking and other skills of 400,000 high school students in grades 9 through 12 in the U.S. were measured using various tests. Eleven years after graduation, a further survey of these 400,000 graduates was conducted to determine their professions. Survey data showed that 45% of those with a PhD in STEM fields were in the top 4% on the spatial ability test during their high school years. Nearly 90% of those who earned a doctorate in STEM fields were among the top 23% in spatial ability testing eleven years previously (Figure 7). The results from this study impressively demonstrate that the ability to think spatially is one of the key qualifications for being able to succeed professionally in the fields of mathematics, information and communication technologies, natural sciences, engineering, and technology.

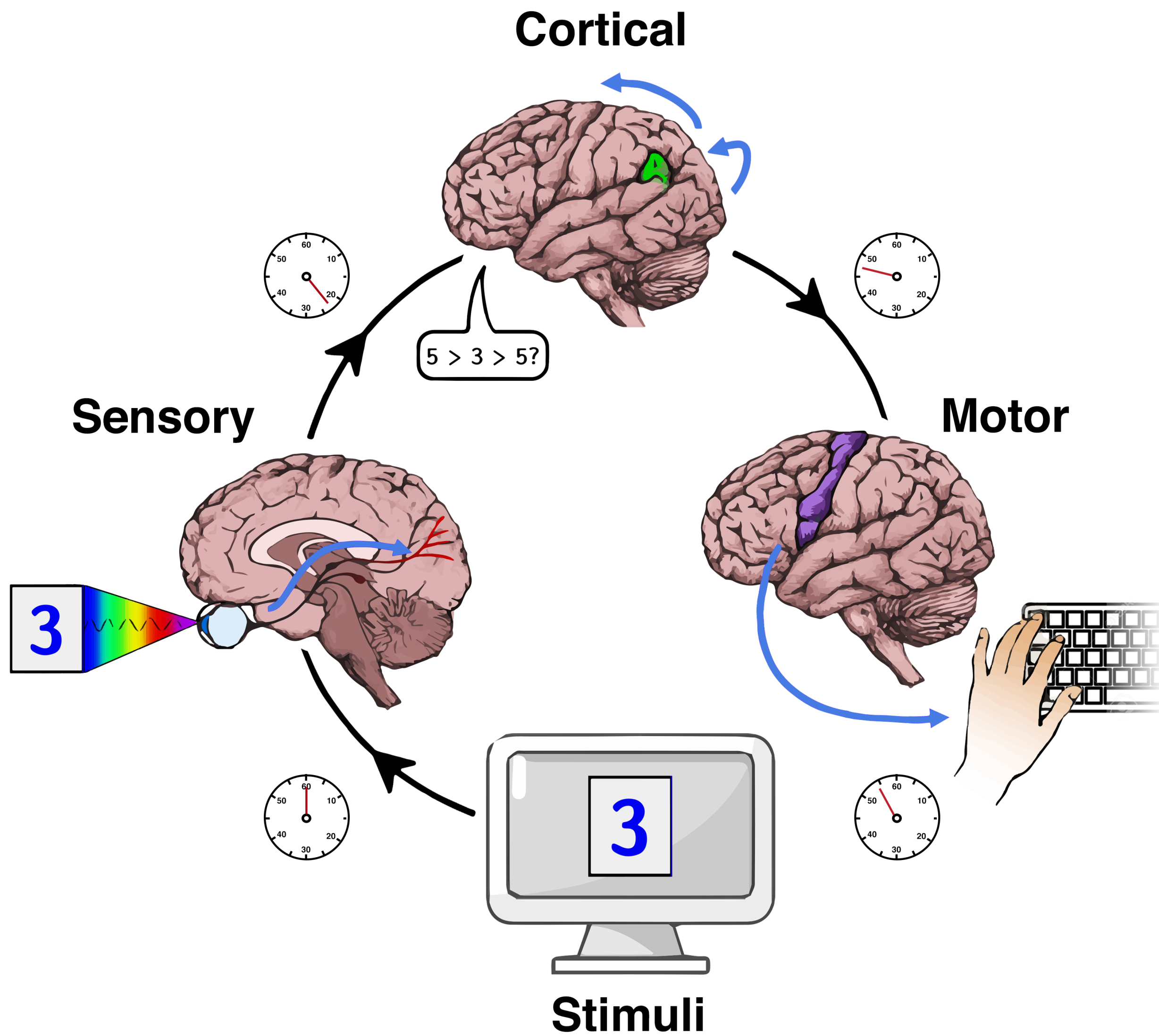

Figure 6: Step-by-step presentation of an activity that requires spatial reasoning. Here: typing the digit "3". [`https://commons.wikimedia.org/wiki/File:Reaction_time_stages.png` Emily Willoughby, CC BY-SA 4.0 `https://creativecommons.org/licenses/by-sa/4.0`]

4. Spatial Thinking Skills as a Prerequisite for Successful Learning at School

The ability to think spatially is not only fundamentally important for coping with many activities in our private and professional everyday lives, but is also an asset that clearly helps to facilitate or enable learning in different areas. In the 1970s, Frostig found clear correlations between spatial skills and the ability to learn to read and write. Cheng and Mix [7] have found that at basic mathematical activities such as transforming simple terms, for example, $5 + \_\_ = 7$ to $\_\_ = 7 - 5$, those students who have good spatial thinking skills have a significant advantage. Other studies with similar results show overall that well-developed spatial thinking skills enable and help people in activities far beyond the STEM domain [15, 44, 48].

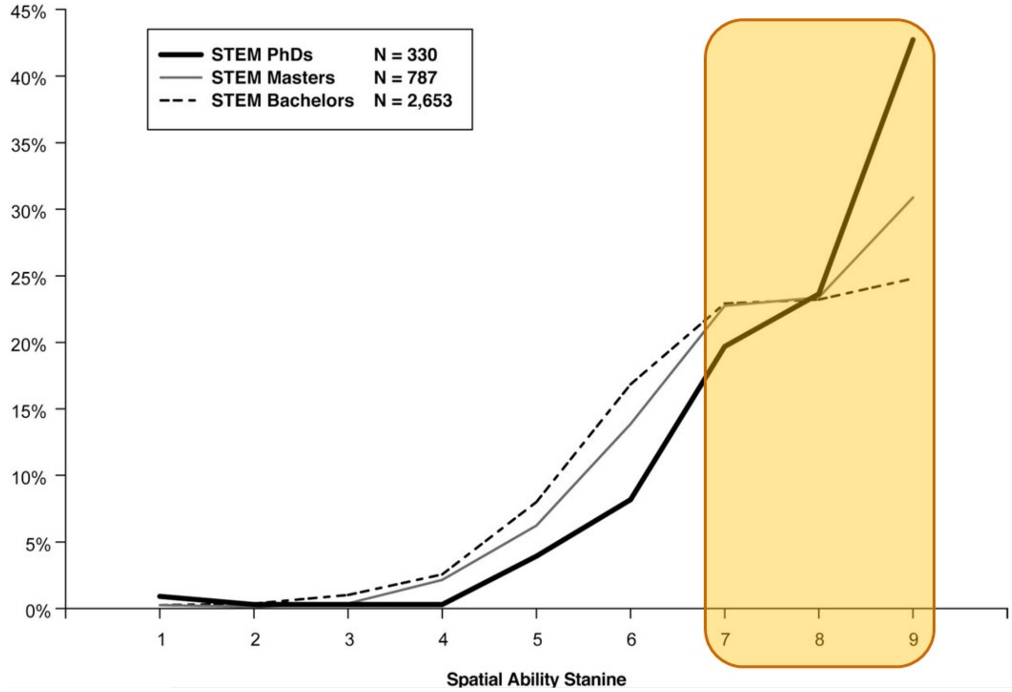

Figure 7: How well did doctoral graduates, master's graduates, or bachelor's graduates working in STEM fields perform on spatial ability tests during their high school years [48].

5. Trainability of Spatial Thinking Skills

The importance of the ability to think spatially has been articulated in previous sections of this paper. In a logical next step, therefore, the requirement for the extensive development of these skills in the context of pre-college education is advocated. Before we delve into this, however, the aspect of whether spatial thinking can be trained or whether it represents an innate ability that is resistant to change must be considered. In general, two opposing viewpoints can be found in the literature of the 20th century. On the one hand, some believe that the ability to think spatially is genetically determined ability and cannot be enhanced through training [35]. On the other hand, it is often stated that the ability to think spatially can be trained and improved to a high degree and that the *active* training of spatial thinking skills is crucial for how well this ability is developed in an individual [12, 13]. The current state of research assumes that both theories are likely true and do not represent mutually exclusive ideas. Montello, Grossner and Janelle state: "One common misunderstanding in this regard is the notion that just because some trait is genetically determined, it is necessarily immutable, and that because some trait is modifiable, it must be caused by experiences after birth (or conception) rather than by genetics. Neither of these complementary claims are true." [28]. The ability to think spatially is to a certain extent genetically inherent in individuals, but it can also be enhanced to a high degree through training (e.g., [13]). Similarly, some individuals seem to have a genetic predisposition to learning music, but others without this genetic advantage can still learn to play the piano, although perhaps not as well as those with the genetic predisposition. Montello and his colleagues further state: "Spatial learning in educational and everyday settings is important because it holds the promise of improving spatial thinking, which in turn holds the promise of contributing to a host of desirable outcomes, including generating economic development, making more user-friendly and functional technology, fostering equitable access to employment, and generally helping people realize their potential" [28, p. 10]. In addition to the general usefulness of spatial thinking skills for STEM and other fields, explicit and implicit active training of spatial thinking skills in pre-college education appears to be promising. It is postulated that a varied, structured training program for spatial thinking skills in school will significantly promote and support all students in being more successful in several areas of their private and professional lives.

6. The Four Perspectives on Spatial Thinking

The insight that spatial thinking (1) is an important life and educational skill, (2) is one of the key competencies required for full participation in our modern society, and (3) can be improved by explicit and purposeful training, leads directly to the question of how this human ability can best be developed and enhanced in a meaningful and structured way. To this end, the first step requires an in-depth examination of the published literature which deals with spatial thinking. In this section, the essence of this literature is summarized in a succinct way and the reader is thus offered the opportunity to view the topic area of spatial thinking from a comprehensive perspective and to see it in a larger context. It is noteworthy that exactly four scientific disciplinary areas have been identified, which have dealt with spatial thinking from different traditions and perspectives and for different purposes. These four major domains and the relevant perspectives on spatial thinking are described in the following paragraphs.

6.1. Developmental Psychology

Developmental psychology is – in regard to spatial thinking – mainly concerned with the question of how spatial thinking develops over a person's lifetime. In most cases, the individual phases, which can last months or years, are considered from the time of birth until the age of 12 to 14 or beyond. In the second half of the 20th century, relevant developmental psychological spatial thinking models were developed by Stückrath [39], the researcher couple van Hiele and van Hiele-Geldorf [46], Piaget and Inhelder [33], Thurstone [42] and Bruner [3, 4].

In his model, Stückrath [39] describes the stages of development of *spatial orientation* such that in early childhood, at the age of 0–1 years, a child generally develops the so-called *"Leibraum"* (body space), followed by the *"Ichraum"* (I space) at the age of 2–6 years. At the age of 6–14 years, the *"Handlungsraum"* (action space) and the *"Laufraum"* (walking space) develop, each subdivided into three sublevels.

In 1978, the von Hieles [46] described five levels that are passed through in the development of geometric and spatial thinking: (0) Visualization (recognize figures by appearance alone), (1) Analysis (analyzing geometric figures and relations), (2) Abstraction (initial deduction and reasoning), (3) Deduction (basic geometric reasoning and geometric deduction), and (4) Rigor (stringent, abstract geometry).

Piaget [31] divides the general stages of a child's development of spatial thinking skills into four stages and describes these stages within the framework of his *theory of cognitive development.* The four stages are: (1) *sensorimotor stage.* We pass through this stage at the age of about $0–1\frac{1}{2}$ years in development. This is followed by the (2) *preoperational stage,* which is evident at approximately $1\frac{1}{2}–7$ years of age. This phase is followed by stage (3), the *concrete operational stage.* This phase is generally experienced by children between the ages of 7–11/12 years. Finally, children develop further to phase (4), the *formal operational stage,* which is observable from the age of 11 or 12. It is only from this phase that children

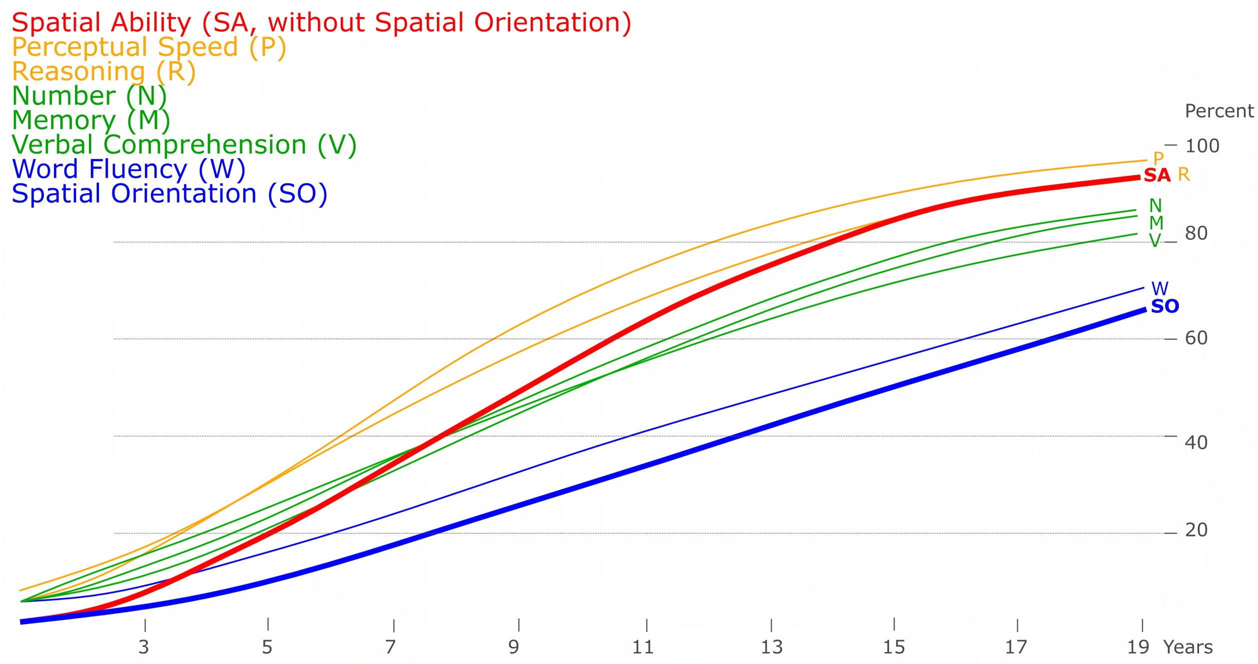

Figure 8: Progress curve of eight intelligence domains. Seven primary intelligence factors according to Thurstone and additionally spatial orientation (SO). The blue intelligence domains develop rather slowly, the green ones at an average rate. The orange intelligence areas develop fastest according to Thurstone's observations.

are generally able to grasp and comprehend abstract structures. This is also the stage where (simple) mathematical proofs can be comprehended for the first time. In the book "The child's conception of space" (1956) [32] – the original edition was written in French in 1948 with the title "La représentation de l'espace chez llenfant" – Piaget together with Inhelder describes in detail the development of spatial thinking skills from birth to young adulthood. Developmental stages such as stereognostic perception, projective space and Euclidean space are postulated in detail.

Thurstone's work [42] turns to the question of how what he sees as the seven primary mental abilities develop through childhood. According to Thurstone, these primary mental abilities are number, verbal comprehension, space (divided into spatial ability and spatial orientation), memory, reasoning, word fluency, and perceptual speed.

In Figure 8, the factor space (spatial ability; SA) is specifically highlighted; the factor is shown with a thick curve. As shown by the SA curve, we see that the greatest potential for improvements in spatial thinking is present at the age of about 3–15 years. From the age of about 14 years, about 80% of the spatial thinking skills are typically developed. Current research results [50] demonstrate that this curve can be confirmed for almost all areas of spatial thinking. However, it should be noted that spatial thinking skills can continue to be developed into adulthood. Sorby has developed and implemented a course for developing spatial thinking aimed at first-year engineering students. Through this course, young adults (approximate age 18–19 years) have made significant improvements in their spatial thinking ability. In a meta analysis, [45] demonstrated that spatial thinking skills are highly malleable into adulthood.

The factor of spatial orientation develops much more slowly than the spatial ability factor, and the least rapidly of all intelligence factors Thurstone identified. The development curve for spatial orientation (the lowest curve in Figure 8; SO) runs slightly below the plotted curve W of word fluency. It is noted that spatial orientation is developed to nearly 100% by the age of 25.

The U.S. developmental psychologist Bruner [3, 4] developed the so-called "EIS principle" as one of his numerous principles and theories. The name of this principle stands as an acronym for the three representation modes and development stages *enactive, iconic,* and *symbolic.* According to Bruner, learning succeeds when the first developmental stage is experienced enactively. In this stage, children should pursue numerous activities or be encouraged with a wide variety of activities based on haptic experiences — active grasping, folding, gluing, running, hopping and other activities. According to [18], this haptic activity is a necessary preliminary stage in the development of later deep understanding. In a further developmental stage, children are able to make iconic representations of the haptically grasped objects. At the third developmental stage children and adolescents can deal with learning content on a symbolic level. Taking a simple cube as an example, children should start by haptically grasping a wooden cube, for example, and experiencing it actively in as many different situations as possible (e.g., throwing, stacking, rolling). At the iconic stage of development, young children should be able to make sketches and simple construction drawings of cubes in a variety of representational forms. Finally, from the symbolic stage of development, students can make calculations (such as surface or volume) on a cube symbolically thus treating it on an abstract level.

Most of the developmental psychologists mentioned in this chapter have explicitly created models for the development of spatial thinking skills. Bruner's EIS principle does not explicitly address spatial thinking, but it can be applied specifically for the development of spatial thinking skills, since it contains very large structural parallels to the four previously mentioned models of Stückrath, van Hiele/van Hiele-Geldorf, Piaget/Inhelder, and Thurstone.

6.2. Visual Perception

Visual perception is in many ways a preliminary stage of spatial ability. Spatial ability addresses – as the name already aptly describes – the ability of a person to mentally imagine spatial objects and to mentally transform them, to recognize relations between several of these mental objects and to mentally take different spatial perspectives. Spatial ability is the cognitive ability of a person to imagine spatial objects, to be able to mentally rotate, mirror, shift or intersect them, and to be able to mentally take different perspectives in space. This applies to the ability to imagine a scene in a picture from a different perspective or to be able to mentally move to other places and therefore to be able to describe routes, for example. Spatial ability is a cognitive ability of our human brain (see also Figure 5). Optical stimuli or movements in space are not required; it is a purely cognitive function. Some refer to spatial ability as seeing something in the "mind's eye." It should be noted that the same areas are activated in the brain when looking at a physical object and when only thinking of these objects. However, the areas are activated in two entirely different ways. During the process of seeing, the brain areas are activated via the optical nerve, which transmits the optical stimuli starting from the retina of the eyes towards the brain, and during thinking spatially the same brain areas are activated only via cognitive imagination of the corresponding physical objects.

Visual perception is the preliminary stage of anatomical reception and basal-cognitive processing of optical stimuli. The optical stimuli received via the eyes are filtered by the brain or its upstream areas and relevant information is extracted and interpreted. These anatomical and basal cognitive steps finally enable the “recognition” of spatial objects by matching them with memories or learning new spatial objects and storing them. Visual perception is thus an important preliminary step to then be able to reason cognitively about spatial objects and plan actions, if necessary.

There is a clear and distinct difference between the terms “spatial ability” and “visual perception.” Visual perception is the physiological ability to receive optical stimuli via the eyes and their retina and to transmit them via the corresponding basal upstream brain areas to the primary visual cortex and finally to recognize spatial objects or, if necessary, to learn them. This is contrasted with spatial ability, which is a mental cognitive operation (rotating, mirroring, intersecting, . . .) with objects in the mind’s eye or the ability to take other spatial perspectives entirely mentally.

Visual perception has been a topic of research for at least 1000 years. In the time of antiquity, Euclid, Ptolemy, and Aristotle made some remarks about perception and the first attempts to explain the nature of human perception. The first comprehensive work on perception was written by Alhazen (1021). In his book Kitāb al-Manazir (1021; English translation by Al Haytham [1]) he described in very detail the physical, optical, and mathematical facets of perception. In the centuries that followed, Leonardo da Vinci and other researchers made several contributions on perception and the process of seeing. In 1867 Helmholtz wrote the groundbreaking work “Handbuch der physiologischen Optik” (English translation: handbook of physical optics), which describes in great detail the anatomical and physical properties of the eye and deals fundamentally with the phenomenon of visual perception.

Frostig created a model of five sub-areas of visual perception. These are: (1) figure-ground perception, (2) perception of form constancy, (3) perception of position in space, (4) spatial relations, and (5) eye-motor coordination [10]. The subdivision of visual perception tasks into these five different domains is well suited to identify differentiated degrees of proficiency of students’ visual perception. Visual perception research is generally concerned with children from the age of about 3–12 years. Research has been conducted with even younger children in terms of their spatial thinking skills by developmental psychologists.

Kozhevnikov [2, 19] has conducted a series of experiments examining the link between spatial skill levels and creativity. In her work, she measured spatial skill levels using a standard instrument for mental rotations (mental rotation test (MRT), [47] and object visualization levels using an instrument designed for that type of task [19]. The object visualization instrument measured visual perceptual skills, e.g., how distinct shapes were within a surrounding area, changes in shading and so on. Her research has demonstrated that there is a distinct difference between technical creativity and artistic creativity. Further, people who have one type of creativity do not necessarily have the other type. Engineers typically have high spatial skill levels as measured by the MRT and high technical creativity. In fact, many of the engineers she tested had poor perceptual skills. Conversely, artists typically have high perceptual skills, and high artistic creativity, but often have low skills in spatial visualization as measured by a test such as the MRT.

6.3. Factor-Analytic Models

During the first half of the 20th century, the construct of human intelligence was examined in more detail by various researchers in the so-called “pre-factorial phase” of spatial thinking research [25]. Intelligence was no longer viewed in a one-dimensional and undifferentiated way, but different models of human intelligence were established that described multidimensional differentiable domains of intelligence [6, 38, 40]. For example, Thurstone's intelligence model [40] included seven different primary mental abilities: space (including spatial ability and spatial orientation), perceptual speed, numerical ability, memory, reasoning, word fluency, and verbal relations. Generally, all the intelligence models established at that time included the intelligence factor of "space" (spatial ability).

The factor of spatial ability (space) was subsequently investigated in more detail by numerous researchers in the second half of the 20th century in the so-called "factorial phase" of spatial thinking research [25]. Methodologically, a factor-analytical approach was often utilized meaning that test participants were given a variety of tasks related to plane or spatial geometry. Subsequently, in the analysis of the test results by means of factor-analytical mathematical calculations, researchers tried to determine which of the test items the participants solved with somewhat equal probability or which further tasks b, c, d, ... were also solved correctly if the participant solved given task "a" correctly. This factor-analytical approach, with suitable underlying mathematical calculation models, provides the opportunity to see which test items are "related" to one another in terms of content, i.e., which test items have similarities in terms of content and are therefore solved with similar considerations and also with fairly equal probability. With the help of these methods, specific areas – called "factors" – are found that are related in terms of content, thus establishing different, separable, areas of spatial ability. With the help of factor-analytic methods, numerous models of spatial ability have been established during the factorial phase of spatial intelligence research (1950–1994) [5, 8, 14, 20, 21, 21, 27, 34, 41]. These models are united by a common effort to find appropriate subdomains of spatial ability by factor-analytic means. In many cases, researchers formulated models with two or three factors. Some models had up to nine different factors [5, 22]. In his remarkable work "Räumliches Vorstellungsvermögen" [24] (spatial ability; 1994), Maier summarized and discussed the factor-analytic models of spatial ability in detail for the first time in the German language. In this book, he proposes five of the mentioned factors as an overall coherent essence of the various identified factors of spatial ability. The years after 1994 were characterized by using the various factor-analytical models developed previously and testing these models for internal consistency and selectivity of the factors and thus their viability. It became clear that the models of spatial ability that were developed in a factor-analytic way were not very robust, partly due to the lack of discriminatory power of the factors described. From about 2010 onwards, it became clear that participants who perform spatial ability tasks usually use significantly different individual strategies and therefore do not always use the strategies intended by the developers of the tests. Therefore, it became clear that using a factor-analytic analysis of spatial ability test results cannot serve to develop a viable model of spatial ability [37].

The period from 1994 onwards is also called the "post-factorial phase" of spatial ability research (Figure 9) [25], since it is recognized that individual solution strategies play an important role in the processing of tasks, and that other aspects, such as working memory, dynamics and motor skills, also significantly influence the probability of solving spatial ability tasks, and that these aspects must therefore also be taken into account in an overall view of spatial thinking. The factor-analytic perspective of research on spatial ability has made a significant contribution to the analysis of the space intelligence facet in its almost 50 years (about 1950–1994), has provided numerous approaches and suggestions, and has thus stimulated many further developments in the area of spatial ability research.

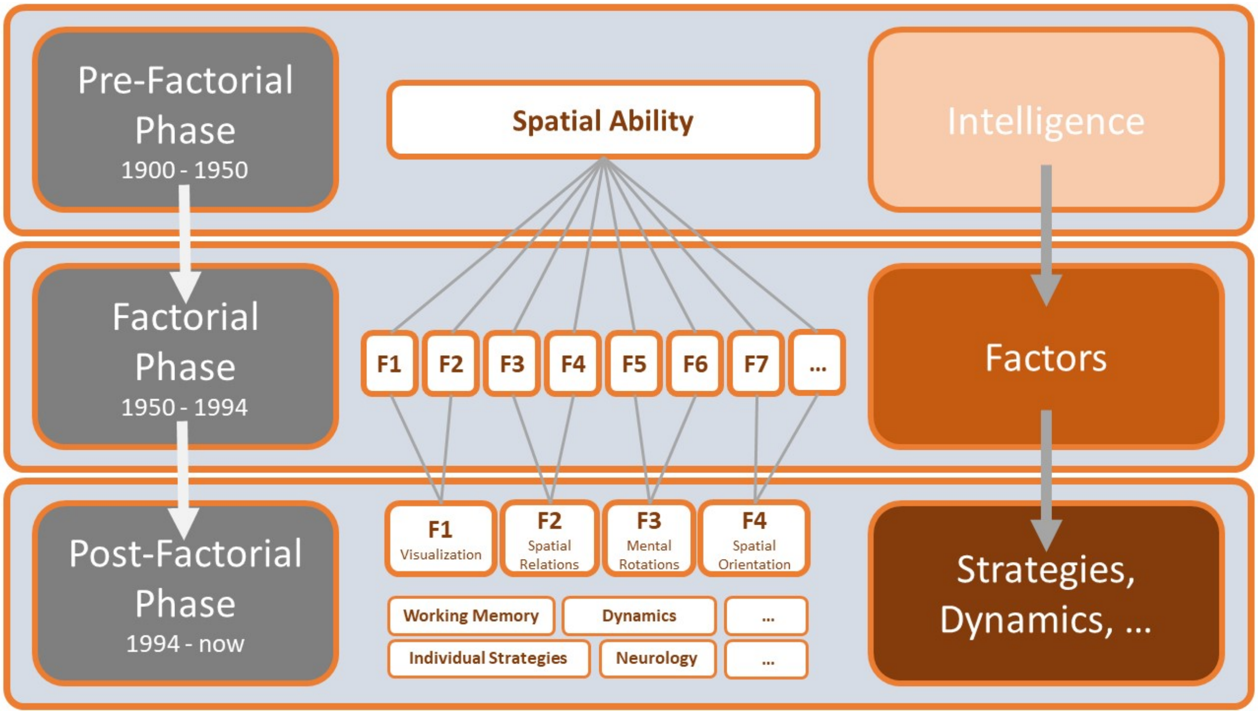

Figure 9: The three phases of spatial ability research

6.4. Neurology

Until well into the 20th century, it was difficult or even impossible for researchers to carry out investigations on the living human brain or to gain substantial knowledge from these experiments. In 1971, magnetic resonance imaging (MRI) was invented by Lauterbur, first published in 1973, and in 2003 he received the Nobel Prize in Medicine for this invention. Functional magnetic resonance imaging (fMRI) was developed 20 years after MRI. Belliveau and his colleagues made significant contributions to fMRI development. They first published results of functional magnetic resonance imaging in humans in 1991. Since that time, several other similar technics have been developed, all of which are now grouped under the umbrella term of "imaging techniques." All of these techniques share the common characteristic that they allow insights into the internal structures of the brain without having to surgically penetrate the skull. These technical developments have proven invaluable, as they have made it possible to gain substantial new scientific insights in this non-invasive way for nearly 50 years.

In 1982, Ungerleider and Mishkin formulated the groundbreaking concept of two cortical visual systems in the brain (Ungerleider and Mishkin [43]). This concept describes how visual stimuli are processed from the primary visual cortex (V1; the orange area in Figure 10), which is located at the back of the brain. Ungerleider and Mishkin recognized that the processing proceeds in two different pathways. On the one hand, spatial objects per se (color, texture, shape, size, . . .) are stored and recognized in areas that run in the so-called ventral pathway. This ventral visual pathway (also called ventral stream, see Figure10) starts in the primary visual cortex and runs towards the temporal lobe. This visual pathway is often referred to as the "what" pathway [30], because spatial objects (such as faces, houses, trees, cubes, cones, spheres, . . .), i.e., the "what", are stored and recognized in the corresponding brain areas. In contrast, the dorsal visual pathway runs from V1 to the posterior top of the brain and is

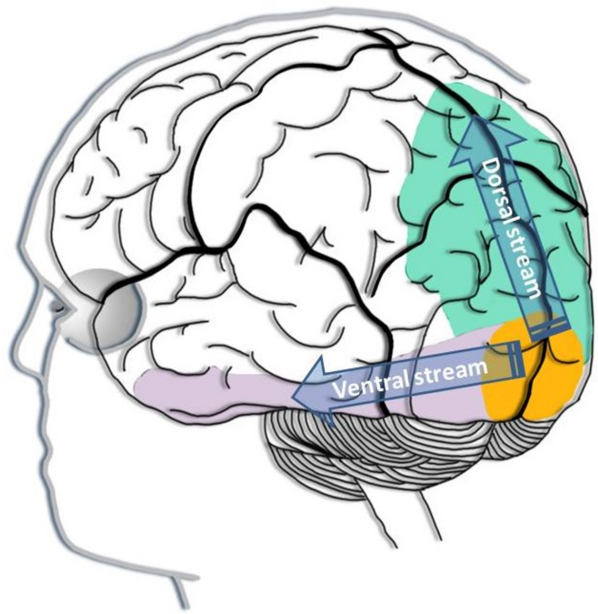

Figure 10: The two visual pathways (ventral stream and dorsal stream) [https://commons.wikimedia.org/wiki/File:Ventral_and_dorsal_stream_in_visual_information_processing.png Andrey Vyshedskiy, Shreyas Mahapatra, Rita Dunn, CC BY 4.0 https://creativecommons.org/licenses/by/4.0>]

called the "where and when" pathway. In these brain areas, the location of spatial objects is recognized. It is here, also, where the temporal sequences of moving objects are detected and interpreted. Instructions for actions also emerge from these brain areas. Here there is also a close connection to the motor control areas, which is involved in the planning, control, and execution of voluntary movements.

The theory of two distinct visual pathways in which optical stimuli are processed more or less separately after being transmitted from the eyes to the primary visual cortex was for many years an extremely fruitful starting point for investigating the concrete methods of processing optical stimuli in the brain. Modern imaging techniques have played a major role in providing significant insights in this field in recent years. Within the current state of research, it has now become established that the concept of two visual pathways, and thus the clear separation of the processing of optical stimuli in the brain into the "what" pathway and "when/where" pathway, cannot be confirmed in its original form. Instead, it has been shown that all brain areas involved in the processing of visual stimuli are interconnected to a high degree and they exchange information continuously. The theory of separating visual stimuli through the primary visual cortex into separate visual pathways is now considered outdated and has given way to the view that all brain areas involved in the processing of visual stimuli exchange information in a manifold neuronal continuous manner [9].

In addition to cognitive recognition and operation of spatial objects, spatial thinking also involves the mental switching of spatial perspectives termed spatial orientation. Spatial orientation processing is located within two mutually adjacent brain areas, in the enthorinal cortex and in the hippocampus. In the enthorinal cortex the so-called grid cells are located, which are arranged in a hexagonal shape and which have the property that they can recognize an exact spatial position. In the hippocampus, the place cells are located, each of which can remember the *characteristics* of a particular place (but do not know where this place is located). Through the cooperation of grid cells and place cells it is possible to know where one is located and what properties this place has. These two brain regions form our internal navigation system due to the fascinating cooperation of grid cells and place cells. Research into these structures of spatial orientation was significantly influenced by the British-American researcher O'Keefe and the Norwegian research couple Moser and Moser [52] and [29], for which all three received the Nobel Prize in Medicine in 2014.

Vision and spatial thinking make up more than half of the total brain area [30]. When thinking about spatial objects, the same brain areas are activated as when actually looking at these objects. The findings from neurology show that spatial thinking is an extraordinarily complex procedure that is processed in numerous different brain areas, each of which is specialized for specific tasks. Thoroughly understanding brain function during spatial processing will likely be an area of active research far into the future. In addition, those areas that are involved in the processing of optical stimuli in the brain, or even at the intermediate stations between the eye and the brain (such as the geniculate nucleus), are constantly exchanging information with those parts of the brain where related experiences of different senses are stored. With the help of this continuous exchange, at all stations of the processing of optical stimuli, information is filtered, interpreted, compared with experiences, and compressed. Only the compression of optical stimuli enables a human brain to cognitively process the incredibly high number of optical impressions received throughout the day.

For nearly 50 years, neurology has provided striking evidence about spatial thinking. Significant findings include the fact that optical stimuli are processed in different specified brain areas that are in constant neuronal exchange with one another. Based on our knowledge of brain function, the recommendation can be made that the different facets of spatial thinking should be trained in a continuous, structured, and differentiated way. Through the systematic training of spatial thinking skills, which aims to address all brain areas involved in vision and spatial thinking in a balanced ratio, and which trains all individual areas and promotes the neuronal exchange between the areas, therefore seems to be an obvious and promising approach for intervention.

7. The generic perspectives on spatial thinking at a glance

The ability to think spatially has been studied by different scientific fields for about 140 years. Four perspectives can be identified: (1) developmental psychology, (2) visual perception, (3) factor-analytic models, and (4) neurology. Each of these perspectives employs different methods, traditions, and objectives in examining spatial thinking skills. The statements, findings, models developed, and approaches taken by each research discipline reflect this diversity. Figure 11 concisely summarizes relevant terms of the respective research areas in a keyword-like manner and shows which age group has been the primary focus of each.

Looking at this heterogeneous landscape of the four fields of science that are conducting significant research on spatial thinking, it is clear that there have been, and continue to be, numerous efforts and great strides in exploring the foundations of spatial thinking. By discussing the fundamentals of these research traditions and their models of the four perspectives and the age groups that are addressed by each, it is clear that some common ground and overlaps between the fields can be recognized, but individual areas of research in these fields have yet to develop a common model of spatial thinking. In the subsequent section of this paper, a unifying model is proposed, that incorporates elements from each of the previously described research perspectives.

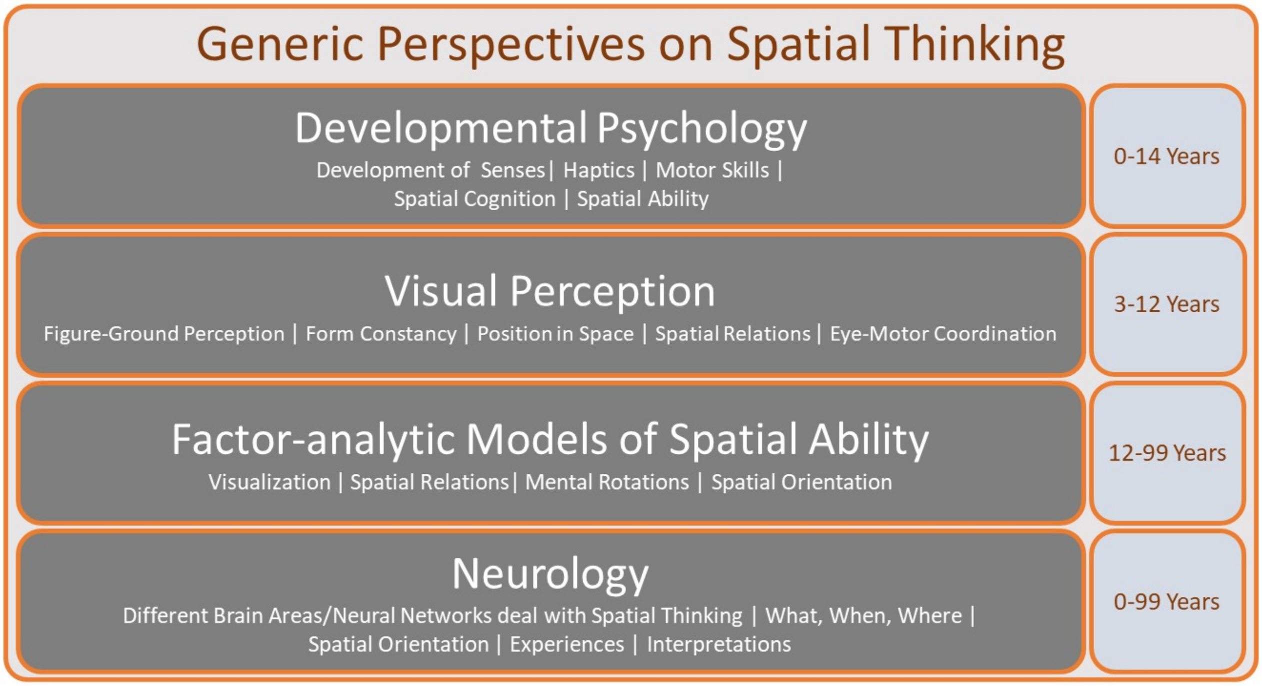

Figure 11: Generic perspectives on spatial thinking

8. A Holistic model of the basic practices of spatial thinking

A particularly attractive direction for future development and progress in the field of spatial thinking is now to build bridges between the four perspectives and to unite the four areas under a common umbrella by extracting the relevant insights and facets of the individual perspectives and bringing them together into a common model for spatial thinking. The model of the basic practices of spatial thinking was developed by [26], which proposes a total of eight practices for spatial thinking (Figure 12).

The first five stages in the spatial thinking model (visualization, form constancy, spatial position, spatial transformations, and object combinations) are understood as successive stages. Children and adolescents generally develop these stages sequentially, with the ability to visualize being the most basic stage. Building on this, the abilities to recognize form constancy, spatial positions, spatial transformations, and finally object combinations are developed in this ascending order. Study results by Hofer [17] and Wallinger [49] confirm this structure and the consistency of the content of the first five basic practices. The other three basic practices of spatial thinking are dynamics, motor skills, and spatial orientation.

The individual stages of the model of the basic practices describe concrete spatial skills, are content oriented, and are therefore independent of the individual strategy learners use to complete the tasks at each stage. The primary concern in the conception of the model was to extract the basic statements, findings, factors, and facets of the four generic-historical perspectives of spatial thinking research and to unite them coherently in a new overarching and holistic structure. The model should also provide orientation for learners at all developmental levels and ages, to see which facets of spatial thinking have already been developed and to what depth. The model of the eight basic practices of spatial thinking was first published in 2020 [26]. In this publication, all stages are described in detail, their basic functions are discussed, partial abilities are listed, partial aspects are named, exemplary exercises are pre-

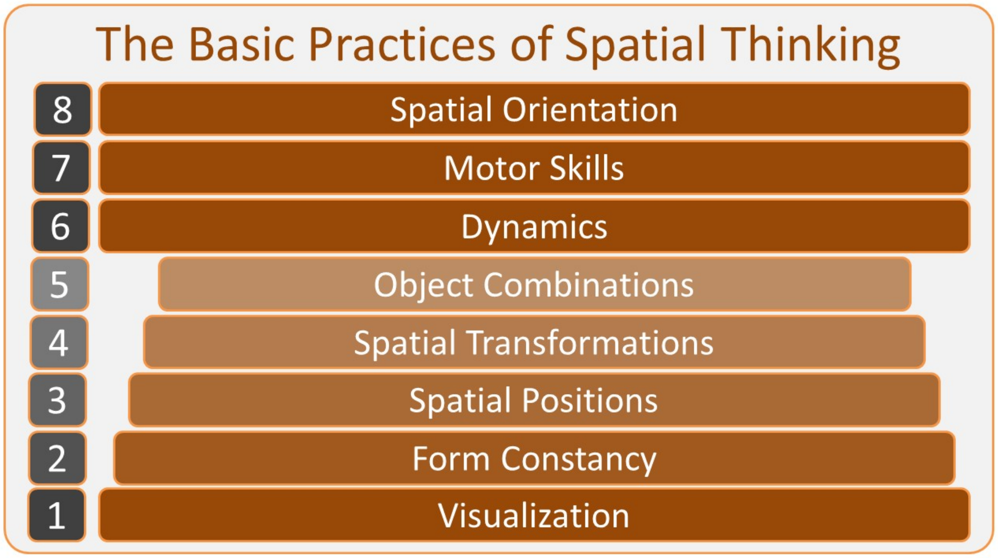

Figure 12: The eight basic practices of spatial thinking

sented, references to the four generic-historical perspectives of spatial thinking are described, and, among other things, cross-references to writing and reading are made.

9. Summary

In 1879, Galton described the intelligence facet of spatial thinking as "visualising faculty", writing: "Memories that are extremely vivid may at the same time be very mobile, and capable of blending together. Much instruction on these matters can be derived from those who possess the power of what is called the visualising faculty, in a high degree. The objects of their memory are conspicuous images". The end of the 19th century is considered as the starting point in spatial thinking research not only because of this description from Galton but also because this was the time where human intelligence became the focus of scientific investigations. Various one-dimensional and multidimensional intelligence models have been developed since that time, in which the different parts of intelligence have been formulated or postulated. Almost all intelligence models recognize that spatial thinking (space) is a fundamental facet of intelligence and space is included as a factor in the various structural models.

The ability to think spatially varies greatly from person to person. Galton confirmed that with the words "I do not know any faculty that varies so much as this in different persons. None can vary more, because its range lies between perfection and nothingness." [11]. This circumstance and the fact that spatial thinking is a fundamental aspect of numerous models of intelligence provided sufficient motivation for many researchers to devote detailed attention to this specific part of human intelligence. At least four strikingly different approaches to research surrounding spatial thinking can be discerned. These four generic-historical perspectives of spatial thinking are: developmental psychology, visual perception, factor-analytic models, and finally neurology. Each looks at spatial thinking research from different perspectives, focuses their studies on certain age groups, and therefore formulates different models, approaches, and hypotheses. In this paper, the four major approaches, their representatives and core statements are discussed in a succinct way, providing a more holistic view of the field.

A meaningful development in the area of spatial thinking research is to extract the essence of the four perspectives and to unify them in a holistic model. A proposal for such a model is referred to in the conclusion of the paper, as the model of the basic practices of spatial thinking. The model consists of eight different stages. The first five stages are understood as successive stages. The ability to visualize is the most basic stage. This is followed by the stages of recognizing form constancies, spatial positions, spatial transformations, and object combinations. Results of studies [17, 49] show that the skills associated with each stage show increasing levels of complexity and that children and adolescents progress through these skills in the order shown in the model. The other three stages of the model (dynamics, motor skills, and spatial orientation) were identified as essential spatial skills in all four generic-historical perspectives and were therefore included for coherency in the model of the basic practices of spatial thinking. The individual basic practices of the model are described and argued in detail in the initial publication by [26]. Investigations show that the model can be used as a coherent structural basis for the development of learning materials across all age levels and, can therefore also be used for the comprehensive diagnosis of spatial thinking skills.

A. Appendix: Explanation of terms

In the literature, numerous different terms are used regarding spatial thinking. Each of these terms describes a certain facet (specific ability, concrete phenomenon, etc.). Often it can be noted that these terms are used casually, carelessly, and imprecisely. The following explanations of some of the frequently used terms related to spatial thinking are intended to highlight the specifics of each term and describes how the terms should be understood in the context of this paper.

Spatial Ability or Visuo-Spatial Ability Spatial ability is the ability of an individual to mentally imagine spatial objects, to manipulate them mentally (move, scale, rotate, mirror, ...) to recognize relations between several spatial objects and to move to different positions in space solely in the imagination (i.e. to imagine space from different perspectives, also known as perspective taking). The description fits with Sorby's definition "The ability to visualize objects and situations in one's mind and to manipulate those images" [36] and Lohman's definition of spatial ability "Spatial ability may be defined as the ability to generate, retain, retrieve, and transform well-structured visual images" [23], and to the definition of visuo-spatial ability of many other researchers. Spatial ability is therefore a cognitive human ability and does not include any real activities (such as real viewing of spatial objects or performing (fine)motor movements). The term spatial ability is often used broadly. The clarification in this article should help to classify this term precisely. The term spatial ability is often used in the context of the factor analytic models of spatial thinking research. Sometimes, however, the term is seen somewhat diluted and is extended by facets of (anatomical) vision, visual perception or even motor movements. Here it should be clarified that spatial ability is a purely cognitive "mind game" (see Figure 5).

Spatial Cognition Spatial cognition is defined as the ability to determine spatial relationships in relation to one's body. The vertical and horizontal have a special position in this context. Linn and Petersen (1985) write: "... subjects are required to determine spatial relationships with respect to the orientation of their own bodies ...". Guilford [14] states, "A third factor 'Space III' [. . .] seems to suggest a capacity for orientation in the direction of gravity (the vertical)." and further, that ". . . the Space III factor seems to depend on the effect of gravity on the organism and the learned ability to interpret it correctly." and can be interpreted as a "kinesthetic system factor." Spatial perception is therefore to be seen as the anatomical-kinesthetic ability of an individual to be able to recognize specifically the vertical in relation to the position of his or her own body. It therefore represents the ability of an individual to be able to recognize the position of her/his own body in relation to the environment.

Spatial Intelligence The term is synonymous with the term spatial thinking (see below). According to the descriptions, the definitions and use of the term in context, the term spatial intelligence very generously includes numerous facets of spatial thinking and that it is therefore also understood to be synonymous with the term spatial thinking.

Spatial Perception The brain has numerous processes to be able to create a spatial, three-dimensional image from the two-dimensional image of our environment on the retina (a layer of nerve tissue that covers the inside of the eyes). This ability, called spatial perception, arises from the interaction of a multitude of different interpretations of visual perception, such as monocular/binocular vision, color interpretations, familiar size, overlaps, shadows, texture density, transverse disparity, muscle movements of the lens, motion parallax, and also includes the stimuli and impressions of other sensory sources (e.g., kinesthetics, hearing).

Spatial Thinking The term "spatial thinking" is used as a unifying, common umbrella term for the human ability to direct optical stimuli received by the eye into the brain, to interpret these stimuli and thus to recognize spatial objects, to mentally imagine spatial objects, to mentally manipulate these objects, to imagine different positions in space, to take different spatial perspectives, to perceive and interpret motion sequences, and to execute (fine)motor spatial movements. Spatial thinking is often based on optical sensory impressions and includes the anatomical and basal cognitive ability for visual perception, the purely mental ability to think spatially and finally the corresponding spatial motor processes.

Visual Perception Visual perception is a preliminary stage (to the purely cognitive-mental operation with spatial objects) of the anatomical reception and basal-cognitive processing of optical stimuli. The optical stimuli received via the eye are filtered by the brain and its upstream areas and relevant information is extracted and interpreted. These anatomical and basal cognitive steps finally enable the brain to "recognize" spatial objects by matching them with memories or to "learn" spatial objects that are still unknown.

References

[1] I. Al Haytham: *The optics of Ibn al-Haytham: Books I - III "On direct vision"*. Studies of the Warburg Institute. Warburg Institute, London, 1989. ISBN 0854810722.

[2] O. Blazhenkova and M. Kozhevnikov: *Visual-object ability: A new dimension of non-verbal intelligence*. Cognition **117**(3), 276–301, 2010. ISSN 1873-7838. doi: 10.1016/j.cognition.2010.08.021.

[3] J. S. Bruner: *Der Prozess der Erziehung*, vol. Bd. 4 of *Sprache und Lernen*. Berlin-Verl. and Pädagogischer Verlag Schwann, Berlin and Düsseldorf, 1970.

[4] J. S. Bruner: *The process of education*. Harvard University Press, Cambridge, 1977. ISBN 0674710010.

[5] J. B. Carroll: *Human cognitive abilities: A survey of factor-analytic studies*. Cambridge Univ. Press, Cambridge, New York, 1993. ISBN 0521387124.

[6] R. B. Cattell: *Theory of fluid and crystallized intelligence: A critical experiment*. J. Educ. Psychol. **54**(1), 1–22, 1963. ISSN 1939-2176. doi: 10.1037/h0046743.

[7] Y.-L. Cheng and K. S. Mix: *Spatial training improves children's mathematics ability*. J. Cogn. Dev. **15**(1), 2–11, 2014. ISSN 1532-7647. doi: 10.1080/15248372.2012.725186.

[8] J. W. French: *The description of aptitude and achievement tests in terms of rotated factors*, vol. monograph no. 5 of *Psychometric*. University of Chicago Press, Chicago, 1951.

[9] E. Freud, D. C. Plaut, and M. Behrmann: *'What' is happening in the dorsal visual pathway*. Trends Cogn. Sci. **20**(10), 773–784, 2016. ISSN 1879-307X. doi: 10.1016/j.tics.2016.08.003.

[10] M. Frostig and D. Horne: *The Frostig program for the development of visual perception: Teachers' guide*. Follett Educational Corp, Chicago, 1964.

[11] F. Galton: *Generic images*. The Nineteenth Century **6**(1), 157–169, 1879.

[12] G. Gittler and J. Glück: *Differential Transfer of Learning: Effects of Instruction in Descriptive Geometry on Spatial Test Performance*. J. Geom. Graph. **2**(1), 71–84, 1998. ISSN 1433-8157.

[13] J. Glück, H. Kaufmann, A. Dünser, and K. Steigbügl: *Geometrie und Raumvorstellung: Psychologische Perspektiven*. Informationsblätter der Geometrie **24**(1), 4–11, 2005.

[14] J. P. Guilford: *The structure of intellect*. Psychol. Bull. **53**(4), 267–293, 1956. ISSN 1939-1455. doi: 10.1037/h0040755.

[15] Hawes, Z., Tepylo, D., & Moss, J.: *Developing spatial thinking: Implications for early mathematics education*. In B. Davis, ed., *Spatial reasoning in the early years*, 29–44. Taylor and Francis, Hoboken, 2015. ISBN 9781138792036.

[16] M. Hegarty: *Components of spatial intelligence*. In *Psychology of Learning and Motivation*, vol. 52, 265–297. Academic Press, 2010. doi: 10.1016/S0079-7421(10)52007-3.

[17] M. Hofer: *Untersuchung zu den Grundroutinen des räumlichen Denkens und Handelns*. Master thesis, Paris Lodron University Salzburg, Salzburg, 2020.

[18] G. Hüther: *Was wir sind und was wir sein könnten: Ein neurobiologischer Mutmacher*, vol. 18850. Fischer Taschenbuch, Frankfurt, M, 2 ed., 2013. ISBN 9783596188505.

[19] M. Kozhevnikov, M. Kozhevnikov, C. J. Yu, and O. Blazhenkova: *Creativity, Visualization Abilities, and Visual Cognitive Style*. Br. J. Educ. Psychol. **83**(2), 196–209, 2013. doi: 10.1111/bjep.12013.

[20] M. C. Linn and A. C. Petersen: *Emergence and Characterization of Sex: Differences in Spatial Ability: A Meta-Analysis.* Child Dev. **56**(6), 1479–1498, 1985. doi: 10.2307/1130467.

[21] D. F. Lohman: *Spatial Ability: A Review and Reanalysis of the Correlational Literature.* Tech. Rep. 8, Standford University School of Education, 1979. https://apps.dtic.mil/sti/citations/ADA075972.

[22] D. F. Lohman: *Spatial abilities as traits, processes, and knowledge.* In R. J. Sternberg, ed., *Advances in the Psychology of Human Intelligence*, 181–249. Routledge Taylor & Francis Group, New York, 1988. ISBN 9781315808246. doi: 10.4324/9781315808246.

[23] D. F. Lohman: *Spatial Ability and G.* Paper presented at the first Spearman Seminar, 1993. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.111.7385&rep=rep1&type=pdf.

[24] P. H. Maier: *Räumliches Vorstellungsvermögen: Komponenten, geschlechtsspezifische Differenzen, Relevanz, Entwicklung und Realisierung in der Realschule*, vol. Bd. 493 of *Europäische Hochschulschriften. Reihe 6, Psychologie.* P. Lang, Frankfurt a. M., Berlin, Bern etc., 1994. ISBN 3631478259.

[25] G. Maresch: *Spatial Ability: The Phases of Spatial Ability Research.* J. Geom. Graph. **17**(2), 237–250, 2013. ISSN 1433-8157.

[26] G. Maresch: *Die Grundroutinen des räumlichen Denkens und Handelns.* In J. Zumbach, G. Maresch, A. Strahl, and T. Fleischer, eds., *Neue Impulse in der Naturwissenschaftsdidaktik*, 121–133. Waxmann, 2020. ISBN 978-3-8309-4178-1.

[27] M. G. McGee: *Human Spatial Abilities Psychometric Studies and Environmental, Genetic, Hormonal, and Neurological Influences.* Psychol. Bull. **86**(5), 889–918, 1979. ISSN 1939-1455.

[28] D. R. Montello, K. E. Grossner, and D. G. Janelle: *Space in Mind: Concepts for Spatial Learning and Education.* The MIT Press, Cambridge (Massachusetts), 2014. ISBN 9780262028295.

[29] E. I. Moser, Y. Roudi, M. P. Witter, C. Kentros, T. Bonhoeffer, and M. B. Moser: *Grid cells and cortical representation.* Nat. Rev. Neurosci. **15**(7), 466–481, 2014 Jul.

[30] J. Nänni: *Visuelle Wahrnehmung: Eine interaktive Entdeckungsreise durch unser Sehsystem.* Niggli, Salenstein, 2008. ISBN 3721206185.

[31] J. Piaget: *La naissance de l'intelligence chez l'enfant.* Collection d'actualités pédagogiques publiée sous les auspices de l'Institut J. J. Rousseau et de la Société belge de pédotechnie. Delachaux & Niestlé S.A., Neuchâtel and Paris, 1936.

[32] J. Piaget and B. Inhelder: *The child's conception of space.* International library of psychology, philosophy and scientific method. Routledge & Keagan Paul, London, 1956.

[33] J. Piaget and B. Inhelder: *Die Entwicklung des räumlichen Denkens beim Kinde.* Klett, Stuttgart, 1 ed., 1971. ISBN 3129263209.

[34] D. H. Rost: *Raumvorstellung: Psychologische und pädagogische Aspekte.* Beltz-Forschungsberichte. Beltz, Weinheim, 1 ed., 1977. ISBN 3407580126.

[35] N. G. Shakeshaft, K. Rimfeld, K. Schofield, S. Selzam, M. Malanchini, M. Rodic, Y. Kovas, and R. Plomin: *Rotation is visualisation, 3D is 2D: using a novel measure to investigate the genetics of spatial ability.* Sci. Rep. **6**, 1–10, 2016. doi: 10.1038/srep30545.

[36] S. A. Sorby, T. Drummer, K. Hungwe, and P. Charlesworth: *Developing 3-D Spatial Visualization Skills for Non-Engineering Students.* In *Proceedings of the 2005 American Society for Engineering Education Annual Conference & Exposition.* 2005.

[37] E. Souvignier: *Förderung räumlicher Fähigkeiten: Trainingsstudien mit lernbeeinträchtigten Schülern*, vol. 22 of *Pädagogische Psychologie und Entwicklungspsychologie.* Waxmann, Münster, 2000. ISBN 3893258973.

[38] C. Spearman: *'General intelligence,' objectively determined and measured.* Am. J. Psychol. **15**(2), 201–293, 1904. ISSN 1939-8298. doi: 10.2307/1412107.

[39] F. Stückrath: *Kind und Raum: Psychologische Voraussetzungen der Raumlehre in der Volksschule.* Psychologie der Unterrichtsfächer der Volksschule. Kösel, München, 3 ed., 1968.

[40] L. L. Thurstone: *Primary Mental Abilities.* University of Chicago Press, Chicago, Illinois, 1938.

[41] L. L. Thurstone: *Some prime abilities in visual thinking.* Proc. Am. Philos. Soc. **94**(6), 517–521, 1950.

[42] L. L. Thurstone: *The differential growth of mental abilities.* University of North Carolina Psychometrie Laboratory, No. 14, Chapel Hill, North Carolina, 1955.

[43] L. G. Ungerleider and M. Mishkin: *Two cortical visual systems.* In D. J. Ingle, A. Goodale, Melvyn, and R. J. W. Mansfield, eds., *Analysis of Visual Behavior*, 549–586. The MIT Press, Cambridge (Massachusetts), London (England), 1982. ISBN 0-262-09022-8.

[44] D. H. Uttal and C. A. Cohen: *Spatial thinking and STEM education: When, why, and how?* In *Psychology of Learning and Motivation*, vol. 57, 147–181. Academic Press, 2012. doi: 10.1016/B978-0-12-394293-7.00004-2.

[45] D. H. Uttal, N. G. Meadow, E. Tipton, L. L. Hand, A. R. Alden, C. Warren, and N. S. Newcombe: *The malleability of spatial skills: A meta-analysis of training studies.* Psychol. Bull. **139**(2), 352–402, 2013. ISSN 1939-1455. doi: 10.1037/a0028446.

[46] P. M. van Hiele and D. van Hiele-Geldorf: *Die Bedeutung der Denkebenen im Unterrichtssystem nach der deduktiven Methode*. In H. G. Steiner, ed., *Didaktik der Mathematik*, 127–139. Wissenschaftliche Buchgesellschaft, Darmstadt, 1978. ISBN 3-534-06005-9.

[47] S. G. Vandenberg and A. R. Kuse: *Mental rotations, a group test of three-dimensional spatial visualization*. Percept. Mot. Skills. **47**(2), 599–604, 1978. ISSN 1558-688X. doi: `10.2466/pms.1978.47.2.599`.

[48] J. Wai, D. Lubinski, and C. P. Benbow: *Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance*. J. Educ. Psychol. **101**(4), 817–835, 2009. ISSN 1939-2176. doi: `10.1037/a0016127`.

[49] S. Wallinger: *Untersuchungen zu fünf Grundroutinen des räumlichen Denkens*. Master thesis, Paris Lordon Universität Salzburg, Salzburg, 2020.

[50] M. Wilhelm and G. Maresch: *Wie ausgeprägt ist meine räumliche Orientierungsfähigkeit? Entwicklung des neuen Messverfahrens SMART durch reale Bewegung im Raum*. Informationsblätter der Geometrie **36**(1), 21–23, 2017.

Internet Sources

[51] G. Maresch, K. Scheiber, and H. Slepcevic: *Spatial Thinking Training: Online Training- and Diagnosis Platform*, 2021. `https://adi3d.at/rif20/en/`. Last Accessed: September 29, 2021.

[52] E. I. Moser: *Grid Cells and the Entorhinal Map of Space: Nobel Lecture 071214*, 2014. `https://www.nobelprize.org/prizes/medicine/2014/edvard-moser/lecture/`. Last Accessed: September 29, 2021.

Received July 8, 2021; final form September 17, 2021.