Journal for Geometry and Graphics
Volume 25 (2021), No. 2, 265–270

A Curious Use of Extra Dimension in Classical Mechanics: Geometrization of Potential

Trung V. Phan[1], Anh Doan[2]

[1]*Department of Molecular, Cellular, and Developmental Biology, Yale University, New Haven, CT 06520, USA*
trung.phan@yale.edu

[2]*Department of Astronomy and Astrophysics, Pennsylvania State University, University Park, PA 16802, USA*
dnd5@psu.edu

Abstract. Extra dimensions can be utilized to simplify problems in classical mechanics, offering new insights. Here we show a simple example of how the motion of a test particle under the influence of an 1D inverse-quadratic potential is equivalent to that of another test particle moving freely in 2D Euclidean space and 3D Minkowskian space.

Key Words: particle motion, inverse-quadratic potential, Minkowskian space

MSC 2020: 70B05 (primary), 97M50

1 Introduction

Broadly speaking, physics questions are often defined by or complicated by their dimensions. On the surface, there seems to be a trend that physical questions become more difficult as the dimensionality increases. For example, collisions in two dimensions are harder to deal with than in one dimension, because the velocity becomes a two-dimensional vector not a scalar [13, 17]. Rigid body rotation in three dimensions is much more complicated than in two dimensions, since the angular velocity becomes a three-dimensional vector not a scalar [13, 17]. In quantum mechanics, any potential-well in one dimension and two dimensions has at least one bound state, but that claim is no longer correct in three dimensions [1, 24]. However, many findings in modern theoretical physics indicate that it is also possible to simplify problems by adding more dimensions. In the theory of general relativity, electromagnetism and gravity can be unified by adding an extra compact dimension [9, 10]. In condensed matter physics, quasicrystals can be treated as projections of a higher-dimensional lattice [8, 11, 14]. Finally,

ISSN 1433-8157/\$ 2.50

in string theory, a strongly interacting system can be more easily understood by considering a gravitational system in one more dimension [7, 21] via gauge/string duality [16].

Though typically used in more advanced topics, the addition of extra dimensions can help to simplify problems in classical physics. For example, the electrostatic problem of finding the charge distribution on a thin conducting circular disk, can be easily solved by an orthogonal projection of the charge distribution of a conducting sphere onto an equatorial plane [23]. But, those are rare. To students of classical and applied geometry, this curious use of extra dimension can appear in creative solutions of some well-known challenges, such as the proof of Desargues' theorem by Conway and Ryba, [2] and the cyclographic-projected solution of Apollonian circle problem by Stachel [22]. Some other interesting examples can also be found as exercises in *The Universe of Conics* [6, Exercise 4.1.1, pp. 132 and Exercise 4.4.4, pp. 167–168] and *The Universe of Quadrics* [20, Exercise 2.2.7 and 2.2.8, pp. 51].

In this note, we will concretely show how adding extra dimensions simplifies the classical mechanics problem of motion under the influence of an inverse-quadratic potential. To the best of our knowledge, this curious physical example has not been demonstrated elsewhere.

2 From 1D to 2D Euclidean Space

Consider a point particle of mass m in one-dimensional space moving under the influence of an inverse-quadratic potential $V_1(x) = \alpha/x^2$. This potential appears in experimental atomic physics [4, 15] and also is of many theoretical interests because it gives a scale-invariant Schrodinger's equation [3, 5, 18, 19]. For now, let us focus on a repulsive potential with $\alpha > 0$. Initially, when $t = 0$, the particle is at position x_0 with no velocity. The motion of the particle can be described by applying conservation of energy to arrive at the following integral:

$$\begin{aligned} & m(dx/dt)^2/2 + V_1(x) = V_1(x_0) \\ \Longrightarrow \quad & dx/dt = \Big(2\Big(V_1(x_0) - V_1(x)\Big)\Big/m\Big)^{1/2} \\ \Longrightarrow \quad & \int_{x_0}^{x(t)} dx\Big(2\alpha(x_0^{-2} - x^{-2})/m\Big)^{-1/2} = t\,. \end{aligned} \tag{1}$$

However, doing this integration is non-trivial. The solution requires changing variables to $y = (x_0^2 - x^2)^{1/2}$, at which point the integral becomes $\int dy y^{-1/2}$ up to a multiplication factor. We can get the equation of motion:

$$\begin{aligned} & \Big(mx_0^2\Big(x^2(t) - x_0^2\Big)\Big/2\alpha\Big)^{1/2} = t \\ \Longrightarrow \quad & x(t) = (x_0^2 + 2\alpha t^2/mx_0^2)^{1/2}\,. \end{aligned} \tag{2}$$

While this solution is tractable, there exists another way to describe the motion of the particle without the need of calculus. A "magic" from an extra dimension.

Consider a general central potential V_2 in a two-dimensional space. In the polar coordinates $\vec{r} = (r, \theta)$ where the origin is the center of the potential, we have rotational symmetry. The kinetic energy K_θ stored in the compact angular dimension depends on the angular momentum p_θ and the moment of inertia $m_\theta = mr^2$:

$$K_\theta = p_\theta^2/2m_\theta = p_\theta^2/2mr^2 = K_\theta(r)\,. \tag{3}$$

The effective potential in the radial dimension [13, 17] is just the sum of the central potential $V_2(r)$ and the kinetic energy $K_\theta(r)$. It should be noted that $K_\theta(r)$ and $V_1(x)$ are both inverse-quadratic functions.

We note now an exact correspondence between this 2D scenario and the 1D problem considered above. The motion of the point particle in one-dimensional space under V_1 potential is dual to the radial motion of its counterpart moving freely in two-dimensional space (no potential $V_2 = 0$), given that the angular momentum is exactly $p_\theta = (2m\alpha)^{1/2}$:

$$x(t)\Big|_{V_1=\alpha/x^2} \quad \Longleftrightarrow \quad r(t)\Big|_{V_2=0, p_\theta=(2m\alpha)^{1/2}} . \tag{4}$$

In other words, with an extra angular dimension we can eliminate the potential.

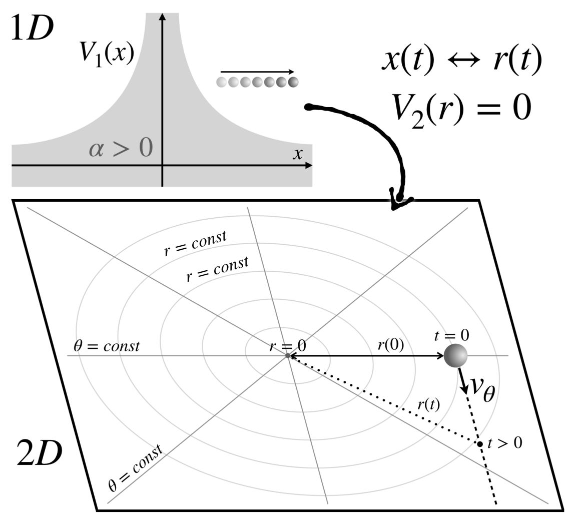

Figure 1: The 1D/2D duality. A repulsive inverse-quadratic potential in an 1D space is dual to no potential in a 2D space.

Using $x(0) = x_0$ and $dx/dt(0) = 0$, we have the corresponding radial position $r(0) = x_0$ and radial velocity $dr/dt(0) = 0$. The tangent velocity is given by:

$$v_\theta \equiv r d\theta/dt(0) = p_\theta/mr(0) = (2\alpha/mx_0^2)^{1/2} . \tag{5}$$

See Fig. 1 for the detail of this 1D/2D duality. We can arrive at the same answer (2) with the Pythagorean theorem:

$$r(t) = \left(r^2(0) + (v_\theta t)^2\right)^{1/2} = (x_0^2 + 2\alpha t^2/mx_0^2)^{1/2} . \tag{6}$$

While we arrived at the same answer, the solution this time is purely geometric and does not involve any calculus.

3 From 1D to 3D Minkowskian Space

The tools developed here can also be used for an arbitrary inverse-quadratic potential. However, it is more complicated and requires generalization to three dimensions. For an attractive potential, we can directly use $\alpha = -|\alpha| < 0$ to get the equation of motion $x(t)$ and also the lifetime τ until the point particle meets the singularity at position $x = 0$:

$$\begin{aligned} x(t) &= \left(x_0^2 - 2|\alpha|t^2/mx_0^2\right)^{1/2}, \\ x(\tau) &= 0 \quad \Longrightarrow \quad \tau = \left(mx_0^4/2|\alpha|\right)^{1/2}. \end{aligned} \tag{7}$$

However, as we re-examine the problem from a two-dimensional perspective as explained above, this indicates an imaginary value of angular momentum $p_\theta = i\left(2m|\alpha|\right)^{1/2}$. To generalize this extra-dimensional trick for all real values of α, we need to complexify the angular dimension $\theta = \theta_R + i\theta_I$ (with θ_R and θ_I are real). Thus the corresponding space will be three-dimensional with $(+,+,-)$ metric signature [12]. Note that there are now two extra dimensions instead of one: while θ_R is a compact dimension, θ_I is an open one. The particle moves in the $(+,+)$ Euclidean plane when $\alpha > 0$, and in the $(+,-)$ Minkowskian plane when $\alpha < 0$. See Fig. 2 for the detail of this 1D/3D duality.

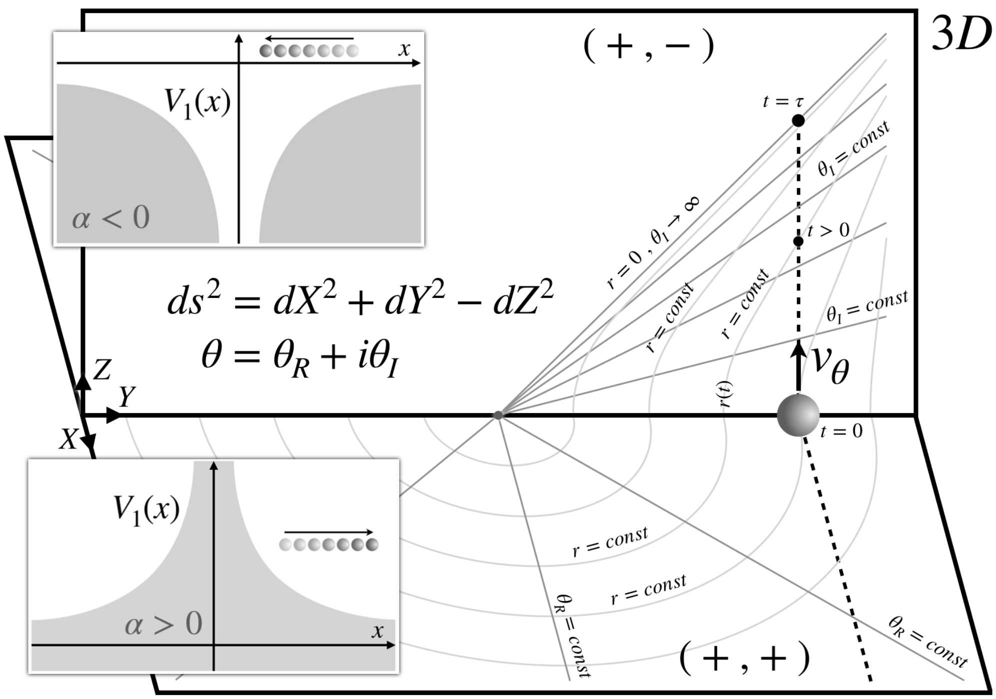

Figure 2: The 1D/3D duality. An inverse-quadratic potential in an 1D space is dual to no potential in a 3D space with $(+,+,-)$ metric signature. Different signs of the potential correspond to different planes of motion: $(+,+)$ plane when the potential is repulsive and $(+,-)$ plane when the potential is attractive.

Acknowledgments

This note comes from our final projects for MIT's course 8.223 in 2012 and 2013. We thank Duy V. Nguyen and the xPhO club for their support to share this finding to a wider audience. We thank Qiantan Hong and Leonid Levitov for examples in physics using extra dimensions. We also thank Oak Nelson and Todd Springer for many useful comments to make this note more approachable for general readers.

References

[1] K. Chadan, N. N. Khuri, A. Martin, and T. T. Wu: *Bound states in one and two spatial dimensions*. J. Math. Phys. **44**(2), 406–422, 2003. doi: 10.1063/1.1532538.

[2] J. Conway and A. Ryba: *The Pascal Mysticum Demystified*. Math. Intelligencer **34**, 4–8, 2012. doi: 10.1007/s00283-012-9301-4.

[3] S. A. Coon and B. R. Holstein: *Anomalies in quantum mechanics: The $1/r^2$ potential*. Amer. J. Phys. **70**(5), 513–519, 2002. doi: 10.1119/1.1456071.

[4] C. Desfrançois, H. Abdoul-Carime, N. Khelifa, and J. P. Schermann: *From $1/r$ to $1/r^2$ Potentials: Electron Exchange between Rydberg Atoms and Polar Molecules*. Phys. Rev. Lett. **73**(18), 2436–2439, 1994. doi: 10.1103/physrevlett.73.2436.

[5] A. M. Essin and D. J. Griffiths: *Quantum mechanics of the $1/x^2$ potential*. Amer. J. Phys. **74**(2), 109–117, 2006. doi: 10.1119/1.2165248.

[6] G. Glaeser, H. Stachel, and B. Odehnal: *The Universe of Conics. From the ancient Greeks to 21st century developments*. Springer Spektrum, Berlin, Heidelberg, 2016. ISBN 978-3-662-45449-7. doi: 10.1007/978-3-662-45450-3.

[7] S. A. Hartnoll, P. K. Kovtun, M. Müller, and S. Sachdev: *Theory of the Nernst effect near quantum phase transitions in condensed matter and in dyonic black holes*. Phys. Rev. B **76**(14), 2007. doi: 10.1103/physrevb.76.144502.

[8] P. A. Kalugin, A. Y. Kitayev, and L. S. Levitov: *6-dimensional properties of Al0.86Mn0.14 alloy*. J. Phys. Lett. **46**(13), 601–607, 1985. doi: 10.1051/jphyslet:019850046013060100.

[9] T. Kaluza: *Zum Unitätsproblem der Physik*. Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 966–972, 1921.

[10] O. Klein: *Quantentheorie und fünfdimensionale Relativitätstheorie*. Z. Phys. **37**(12), 895–906, 1926.

[11] P. Kramer and R. Neri: *On periodic and non-periodic space fillings of E^m obtained by projection*. Acta Crystallogr. A **40**(5), 580–587, 1984. doi: 10.1107/s0108767384001203.

[12] L. D. Landau and E. M. Lifshitz: *The classical theory of fields*. Elsevier, 2013.

[13] L. D. Landau and L. E. M.: *Mechanics*. Elsevier, 2013.

[14] D. Levine and P. J. Steinhardt: *Quasicrystals: a new class of ordered structures*. Phys. Rev. Lett **53**(26), 2477–2480, 1984. doi: 10.1103/PhysRevLett.53.2477.

[15] J.-M. Lévy-Leblond: *Electron Capture by Polar Molecules*. Phys. Rev. **153**(1), 1–4, 1967. doi: 10.1103/physrev.153.1.

[16] J. Maldacena: *The large-N limit of superconformal field theories and supergravity*. Internat. J. Theoret. Phys. **38**(4), 1113, 1999. doi: 10.1023/a:1026654312961.

[17] D. Morin: *Introduction to classical mechanics: with problems and solutions*. Cambridge University Press, 2008.

[18] T. X. Nguyen and F. Marsiglio: *Numerical and analytical study of the bound states of the $-\alpha/x^2$ potential*. Amer. J. Phys. **88**(9), 746–752, 2020. doi: 10.1119/10.0001533.

[19] A. F. Nicholson: *Bound States and Scattering in an r^{-2} Potential*. Aust. J. Phys. **15**(2), 174–179, 1962.

[20] B. Odehnal, H. Stachel, and G. Glaeser: *The Universe of Quadrics*. Springer-Verlag, Berlin, Heidelberg, 2020. ISBN 978-3-662-61052-7. doi: 10.1007/978-3-662-61053-4.

[21] D. T. Son and A. O. Starinets: *Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications*. J. High Energy Physics **2002**(09), 042, 2002. doi: 10.1088/1126-6708/2002/09/042.

[22] H. Stachel: *Why shall we also teach the theory behind Engineering Graphics*. In *7th International Conference on Engineering Computer Graphics and Descriptive Geometry, Proceedings*, vol. 1, 7–9. 1996.

[23] W. Thomson: *Reprint of papers on electrostatics and magnetism*. Cambridge University Press, 2011.

[24] K. Yang and M. de Llano: *Simple variational proof that any two-dimensional potential well supports at least one bound state*. Amer. J. Phys. **57**(1), 85–86, 1989. doi: 10.1119/1.15878.

Received November 4, 2021; final form November 10, 2021.