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# Parametric Equations of a Spatial Curve as a Function of Length of the Arc with Given Dependences of Curvature and Angle of Ascent

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**Abstract.** The shape of a flat curve is completely determined by its natural equation that is the dependence of the curvature on the length of the arc. From the natural equation it is possible to pass to parametric equations of a curve which allow constructing a curve on the plane. The shape of the spatial curve is determined by two natural equations: the dependence of the torsion of the curvature at the length of the arc is combined with the dependence of the curvature. However, there is no simple transition from the natural equations of the spatial curve to the parametric ones. To reproduce the curve in space, it is necessary to solve a system of differential equations using numerical methods. We propose to replace the dependence of the torsion with the dependence of the angle of ascent on the length of the arc. In this case similarly to a flat curve, it is possible to write the parametric equations of the spatial curve. Examples are given, spatial curves are constructed, and partial case for a slope curve is considered.

*Key Words:* spatial curve, natural and parametric equations, angle of ascent, curvature, torsion, slope curve

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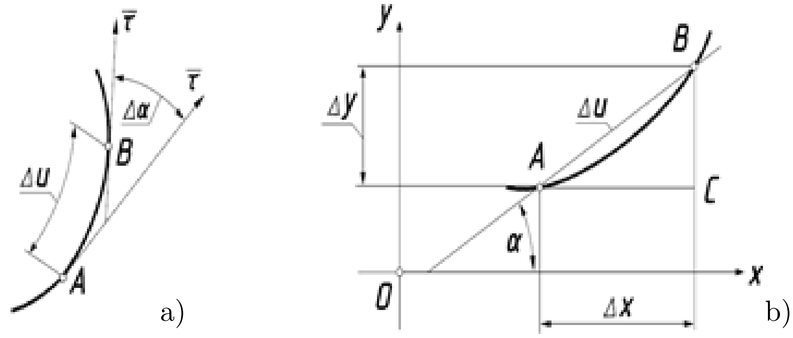



Figure 1: Graphic illustrations to the transition from natural to parametric equations of the curve: a) to determine the angle $\alpha$ of rotation of the tangent; b) to determine the association of the angle $\alpha$ with the coordinates of the point of the curve

# 1 Introduction

The tasks of forming flat curves are mainly ensuring their passage through given points, with a given angle of tangent, in the formation of contours and compliance with other conditions. Research in this direction is conducted using splines and polynomials [5]. Among the variety of formation of spatial curves a special place is occupied by the curves described by the equations as a function of the arc length. Such equations are convenient to use when the length of the arc of a curve is an invariant of its transformation. The construction of spatial curves as a function of the arc length is considered in the works [7, 8]. In [2], spatial curves are constructed on the basis of polynomials using hodographs. The article considers another way of constructing a spatial curve as a function of the arc length according to the given dependences of the curvature and the angle of ascent with respect to the horizontal plane.

# 2 Flat curve construction

A flat curve can be given by the natural equation of the dependence of its curvature $k_f$ on the arc length $u$: $k_f = k_f(u)$. It completely defines the curve on the plane and does not change when it is rotated or translated [4]. The average value of the curvature in the section of the arc $AB$ (Fig. 1.a) is found as the ratio of the increment of the angle $\Delta\alpha$, which is determined by the rotation of the tangent, to the corresponding arc $\Delta u$ of the curve. The value of the curvature at the point of the curve is obtained as the limiting value of this ratio, when the arc $AB$ shrinks to the point, that is when $\Delta u$ goes to zero: $k_f = d\alpha/du$. From the definition of curvature, we can find the value of the angle $\alpha$ of rotation of tangent to the curve when it moves along the arc at the length $u$:

$$\alpha = \int k_f \, du + \alpha_0, \tag{1}$$

where $\alpha_0$ is the constant of integration (initial angle).

Fig. 1.b shows the chord $\Delta u$, which joins the points $A$ and $B$ of the curve and the increments of the coordinates $\Delta x$ and $\Delta y$ correspond to this chord. Due to the aspect ratio of the right triangle ABC, we can find the sine and cosine of the angle $\alpha$. When the arc AB shrinks to the point that is as the limiting value, the right triangle becomes infinitesimal and from it we can write:

$$\frac{dx}{du} = \cos\alpha; \qquad \frac{dy}{du} = \sin\alpha\,. \tag{2}$$

Let's substitute the expression of the angle $\alpha$ from (1) to (2), assuming that $\alpha_0$ is equal to zero, since its value has no impact on the shape of the curve, but only the rotation with respect to the coordinate system $Oxy$. The parametric equations of the curve given by the natural equation $k_f = k_f(u)$ read as:

$$x = \int \cos\Big(\int k_f\, du\Big) du; \qquad y = \int \sin\Big(\int k_f\, du\Big) du. \tag{3}$$

## 3 Spatial curve construction

The shape of the spatial curve is determined by two natural equations: in addition to the dependence of the curvature on the arc length, it is also necessary specify the dependence of the torsion. However, the parametric equations of the spatial curve can‘t be written due to the curvature and torsion in the form (3). The parametric equations of the spatial curve include three unknown Euler angles, which determine the conformity of the movement of the accompanying Frenet trihedral in space [3]. They may be found by numerical methods from the system of three differential equations, which include expressions of curvature and torsion [1].

We propose to elevate a flat curve into a spatial one, setting the dependence of the angle of ascent of the spatial curve instead of the dependence on torsion. In this case, the flat curve (3) will be a horizontal projection of the spatial curve. We denote the length of the arc of the spatial curve by $s$, and the curvature by $k$. It is obvious that there is a certain relationship between the curvature of the flat curve $k_f = k_f(u)$ and the spatial curve $k = k(s)$.

We set the dependence of the angle of ascent $\beta$ as a function of the arc length of the spatial curve: $\beta = \beta(s)$. To each current point $A_f$ will correspond the point $A_s$ of the spatial curve located on the cylinder, the horizontal projection of which is a flat curve (Fig. 2.a). On the development of such cylinder, the spatial curve will turn into a plane curve (Fig. 2.b).

At the limiting passage from the chord to the tangent at the point $A_s$, we obtain an infinitesimal right triangle $CA_sA_f$ (Fig. 2.b). From it we write down the obvious relations:

$$\frac{dz}{du} = \operatorname{tg}\beta; \qquad \frac{du}{ds} = \cos\beta. \tag{4}$$

To the parametric equations (3) we add the third equation based on the first equation in (4):

$$z = \int \operatorname{tg}\beta\, du. \tag{5}$$

In equations (3) and (5) we substitute $du = \cos\beta\, ds$ on the basis of the second dependence (4) we get:

$$\begin{aligned} x &= \int \cos\Big(\int k_f \cos\beta\, ds\Big) \cos\beta\, ds; \\ y &= \int \sin\Big(\int k_f \cos\beta\, ds\Big) \cos\beta\, ds; \\ z &= \int \sin\beta\, ds. \end{aligned} \tag{6}$$

The parametric equations (6) of the spatial curve include the curvature $k_f$ of its horizontal projection. Let's express it in terms of the curvature $k$ of the spatial curve. For a curve given by equations as a function of the arc length, the curvature is given by the expression:

$$k = \sqrt{x''^2 + y''^2 + z''^2}. \tag{7}$$

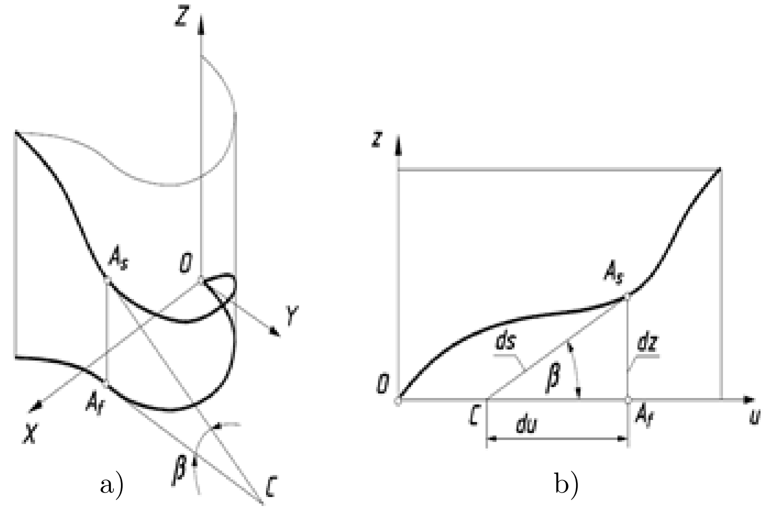



Figure 2: Graphic illustrations to determine the parametric equations of the spatial curve: a) relationship between plane and spatial curves due to the dependence $\beta = \beta(s)$; b) converting a spatial curve to a flat one on the net of a cylinder

Since the first derivatives of equations (6) are defined very simply, we write the second derivatives:

$$\begin{aligned} x'' &= -k_f \cos^2\beta \sin\Big(\int k_f \cos\beta\, ds\Big) - \beta' \sin\beta \cos\Big(\int k_f \cos\beta\, ds\Big); \\ y'' &= k_f \cos^2\beta \cos\Big(\int k_f \cos\beta\, ds\Big) - \beta' \sin\beta \sin\Big(\int k_f \cos\beta\, ds\Big); \\ z'' &= \beta' \cos\beta. \end{aligned} \tag{8}$$

After substitution (8) in (7) and simplifications we get:

$$k = \sqrt{\beta'^2 + k_f^2 \cos^4\beta}. \tag{9}$$

Solving (9) with respect to $k_f$, we find the expression of the curvature of the flat curve – the horizontal projection of the spatial curve:

$$k_f = \frac{\sqrt{k^2 - \beta'^2}}{\cos^2\beta}. \tag{10}$$

By substituting (10) into (6) we finally get the parametric equations of the spatial curve given by two dependencies: the curvature $k = k(s)$ and the angle of ascent $\beta = \beta(s)$:

$$\begin{aligned} x &= \int \cos\Big(\int \frac{\sqrt{k^2 - \beta'^2}}{\cos\beta}\, ds\Big) \cos\beta\, ds; \\ y &= \int \sin\Big(\int \frac{\sqrt{k^2 - \beta'^2}}{\cos\beta}\, ds\Big) \cos\beta\, ds; \\ z &= \int \sin\beta\, ds. \end{aligned} \tag{11}$$

The availability of integrals does not allow to obtain parametric equations of the spatial curve for arbitrary dependencies $k = k(s)$ and $\beta = \beta(s)$. Let's consider examples when this can be done.

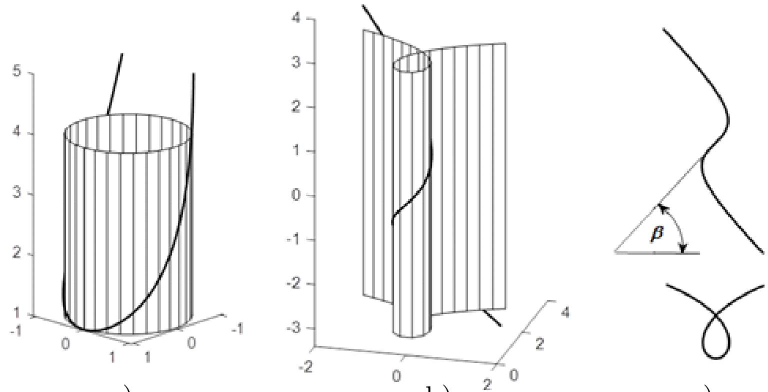



Figure 3: Spatial curves of equal length: a) the curve constructed by equations (14); b) the slope curve constructed by equations (17); c) projection of the slope curve indicating the angle $\beta$

## 4 Examples

The examples consider the general case of constructing a spatial curve, its transformation into a slope curve and the transformation of the slope curve into a flat curve. At the specified transformations dependence of curvature on arc length does not change.

### Example 1

A general case of constructing a curve. For the horizontal projection of the spatial curve we take a circle of unit radius, for which $k_f = 1$. Let the tangent of the angle $\beta$ changes according to the linear law: $\operatorname{tg}\beta = s$.

The necessary expressions for further calculations are:

$$\beta = \operatorname{Arctg} s; \qquad \beta' = \frac{1}{1+s^2}; \qquad \cos\beta = \frac{1}{\sqrt{1+s^2}}. \tag{12}$$

We substitute expressions (12) into (9) and after simplifications we get:

$$k = \sqrt{\beta'^2 + k_f^2 \cos^4\beta} = \frac{\sqrt{2}}{1+s^2}. \tag{13}$$

The following substitution (12) and (13) into (11) leads to the integration of the following expressions:

$$\begin{aligned} x &= \int \frac{\cos(\operatorname{Arcsinh} s)}{\sqrt{1+s^2}}\,ds = \sin(\operatorname{Arcsinh} s); \\ y &= \int \frac{\sin(\operatorname{Arcsinh} s)}{\sqrt{1+s^2}}\,ds = -\cos(\operatorname{Arcsinh} s); \\ z &= \int \sin(\operatorname{Arctg} s)\,ds = \sqrt{1+s^2}. \end{aligned} \tag{14}$$

According to the parametric equations (14) in Fig. 3.a a curve is constructed on a cylinder of unit radius when the arc $s$ belongs to the interval $[-5, 5]$.

It is known from differential geometry of bendings of developable surfaces, the regularity of the change in the curvature of its edge of regression remains unchanged, and only the torsion changes. In our case, to control the bending of the developable surface, for which the edge of regression is the curve (11), is possible by changing the dependence $\beta = \beta(s)$. In this sense, our article echoes the work [6], which also considers the relationship between the curve and the surface.

## Example 2

Here we show the transformation of the curve (14) by changing the dependence $\beta = \beta(s)$. So, we consider the particular case when $\beta = \text{const}$.

In this case, the dependence $k = k(s)$ is preserved. The necessary expressions for further calculations are:

$$k = \frac{\sqrt{2}}{1+s^2}; \qquad \beta = \text{const}; \qquad \beta' = 0. \tag{15}$$

Substitution of (15) into (11) leads to expressions that can be integrated only partially:

$$\begin{aligned} x &= \cos\beta \int \cos\Bigl(\frac{\sqrt{2}\,\mathrm{Arctg}\, s}{\cos\beta}\Bigr)\, ds; \\ y &= \cos\beta \int \sin\Bigl(\frac{\sqrt{2}\,\mathrm{Arctg}\, s}{\cos\beta}\Bigr)\, ds; \\ z &= s\sin\beta. \end{aligned} \tag{16}$$

For some values of the angle $\beta$, expressions (16) can be integrated. For example, for $\beta = \pi/4$ we obtain the following equations of the slope curve:

$$x = \sqrt{2}\,\mathrm{Arctg}\, s - \frac{s}{\sqrt{2}}; \qquad y = -\sqrt{2}\ln\frac{1}{\sqrt{1+s^2}}; \qquad z = \frac{s}{\sqrt{2}}. \tag{17}$$

The curve constructed according to equations (16) is depicted on the corresponding cylinder (Fig. 3.b) and in the projections (Fig. 3.c).

## Example 3

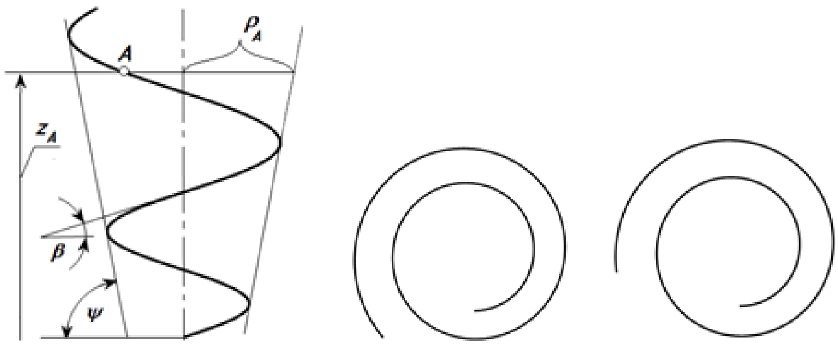
The slope curve is given by the angle $\beta = \text{const}$ and the dependence of the curvature $k = a/s$.

When $\beta = \text{const}$ the equations (11) simplify as follows:

$$\begin{aligned} x &= \cos\beta \int \cos\Bigl(\int \frac{k}{\cos\beta}\, ds\Bigr)\, ds; \\ y &= \cos\beta \int \sin\Bigl(\int \frac{k}{\cos\beta}\, ds\Bigr)\, ds; \\ z &= s\sin\beta. \end{aligned} \tag{18}$$

By substituting the curvature $k = a/s$ into (18) and after integration we obtain the parametric equations of the slope curve:

$$\begin{aligned} x &= \frac{s\cos^2\beta}{a^2+\cos^2\beta}\Bigl(\cos\beta\cos\frac{a\ln s}{\cos\beta} + a\sin\frac{a\ln s}{\cos\beta}\Bigr); \\ y &= \frac{s\cos^2\beta}{a^2+\cos^2\beta}\Bigl(\cos\beta\sin\frac{a\ln s}{\cos\beta} - a\cos\frac{a\ln s}{\cos\beta}\Bigr); \\ z &= s\sin\beta. \end{aligned} \tag{19}$$




a) b) c)

Figure 4: Projections of the slope curve with a given natural equation $k = a/s$: a) frontal projection ($\beta = \pi/10$); b) horizontal projection ($\beta = \pi/10$); c) horizontal projection ($\beta = 0$, flat curve)

By computing the second derivatives of $x$, $y$ and $z$ from (19) and substituting into (7), we obtain the expression for curvature, which was given, that is $k = a/s$. Therefore, the dependence for the curvature on the length of the arc does not change for any angle of ascent $\beta$, including $\beta = 0$, that is for the flat curve.

In Fig. 4.a, b, the curve is constructed in the projections of equations (19) for $a = 15$, $\beta = \pi/10$. It is a conic slope curve, its horizontal projection is a logarithmic spiral, the natural equation of which can be found by formula (10). As the angle $\beta$ decreases, the slope of generators of the cone $\psi$ decreases, and when $\beta = 0$, the cone turns into a plane, and the spatial curve turns into a plane curve (Fig. 3.c). Thus, when the conical slope line is transformed while preserving its natural equation $k = a/s$, its horizontal projection is also transformed, remaining a logarithmic spiral.

Let's find the expression for the angle $\psi$. The point $A$ on the slope curve corresponds to a certain value of the variable $s$, the distance from which to the axis of the cone is denoted by $\rho_A$. As the $z$ coordinate increases, the point $A$ of the curve moves upwards, and the distance $\rho$ from it to the cone axis increases linearly. The distance $\rho$ is given by the following formula:

$$\rho = \sqrt{x^2 + y^2} = \frac{\sqrt{2}s\cos^2\beta}{\sqrt{1 + 2a^2 + \cos 2\beta}}. \tag{20}$$

The increment of the radius $\rho$ corresponds to the increment of the $z$ coordinate. If we assume that the measuring of these distances is from the top of the cone, we can write:

$$\operatorname{tg}\beta = \frac{z}{\rho}, \quad \text{where} \quad \beta = \operatorname{Arctg}\frac{\sin\beta\sqrt{1 + 2a^2 + \cos 2\beta}}{\sqrt{2}\cos^2\beta}. \tag{21}$$

When $k = \text{const}$ formulas (18) give a cylindrical slope line (screw line). With similar transformation, the horizontal projection is the circle which curvature changes and according to formula (10) is determined from the expression: $k_f = k/cos^2\beta$.

## 5 Conclusion

The parametric equations obtained in this paper allow us to construct spatial curves according to the given conformities of change of curvature and angle of ascent with respect to to the horizontal plane. Since they involve integrals, it is necessary to apply numerical methods of integration except in rare cases. For slope curves, that is for a constant angle of ascent, the parametric equations are simplified. The equations make it possible to transform the curve while preserving the conformity of change of the curvature. To do this, we need to set a new conformity of change of the angle of ascent. At a constant value of the angle of ascent, the line is transformed into a slope curve, and at zero value into a flat curve.

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