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Extending a Theorem of van Aubel to the Simplex

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Abstract. We will extend an interesting theorem of van Aubel's for a triangle in the plane to a simplex in the n-dimensional Euclidean space. The barycentric coordinates over simplex and parallel projections in the n-dimensional Euclidean space will be used for the proof of these extensions.

Key Words: van Aubel's theorem, extension, simplex, n-dimensional Euclidean space

MSC 2020: 51M04 (primary), 51N20

1 Some terms and notations

Throughout this paper we use the terms and notations as follows:

- $\mathbb{R}^n$ is the Cartesian model of the n-dimensional Euclidean geometry and $\mathbb{E}^n$ is n-dimensional Euclidean space ($n \geq 2$). That means that each point P in $\mathbb{E}^n$ corresponds to an n-tuple of numbers $(p_1, p_2, \ldots, p_n)$ in $\mathbb{R}^n$; see [10, Chapter 1].
- The Euclidean vector connecting an initial point P with a terminal point Q (in $\mathbb{E}^n$) is denoted by $\overrightarrow{PQ} = -P + Q$ (see [10, pp. 5], [7]). The zero vector in $\mathbb{E}^n$ is denoted by $\overrightarrow{0}$.
- The notation of length segment XY adopts Newton's idea of directed line segments (see [4, p. 30]), it means $XY = -YX$;
- An n-simplex $\mathcal{A}$ in $\mathbb{E}^n$ is the set of $n+1$ points A_0, A_1, …, A_n such that n vectors $\overrightarrow{A_0A_1}$, $\overrightarrow{A_0A_2}$, …, $\overrightarrow{A_0A_n}$ are linearly independent; see [8, pp. 195–199] and [3, pp. 120–124]. For examples: 1-simplex is a segment, 2-simplex is a triangle, 3-simplex is a tetrahedron.
- A hyperplane passing through n points A_1, A_2, …, A_n in $\mathbb{E}^n$ is denoted by $(A_1A_2 \ldots A_n)$.
- A line passing through two points P and Q (in $\mathbb{E}^n$) is denoted by (PQ).

2 Introduction

Van Aubel's theorem is an interesting theorem in a triangle that describes the relationship between the length ratios related to the Cevian triangle. This theorem was mentioned in [2], [6, 547–548], or [11–13] as follows:

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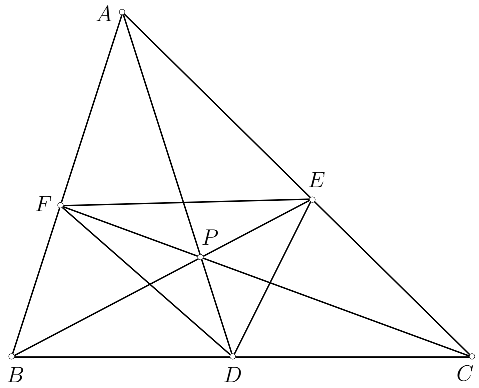

Figure 1: Illustration in plane of Theorem 1

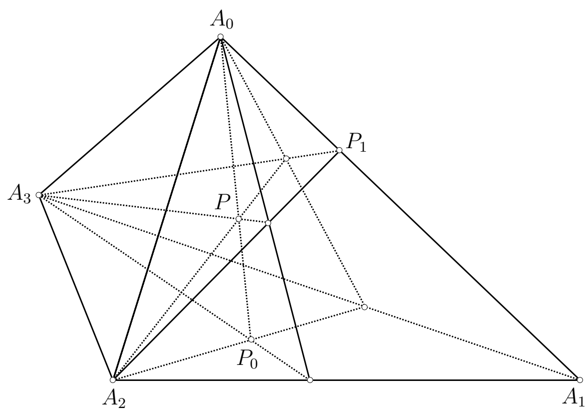

Figure 2: Illustration in 3D space of Theorem 2

Theorem 1 (van Aubel)**.** *Let ABC be a triangle with vertices A, B, and C in $\mathbb{E}^2$. Let P be an arbitrary point not lying on the lines (BC), (CA), and (AB). Assume that the lines (PA), (PB), and (PC) meet the lines (BC), (CA), and (AB) at D, E, and F, respectively. Then,*

$$\frac{AE}{EC} + \frac{AF}{FB} = \frac{AP}{PD}. \tag{1}$$

We now give two extensions to van Aubel's Theorem on the simplex like the idea in [9] as follows:

Theorem 2. *Let $\mathcal{A}$ be a n-simplex in the n-dimensional Euclidean space $\mathbb{E}^n$ with vertices A_0, A_1, ..., A_n. Let P be an arbitrary point in $\mathbb{E}^n$ such that P does not lie on any hyperplane containing facets of $\mathcal{A}$. For any $i = 1, \ldots, n$, let P_i be the intersection of the line (A_0A_i) with the hyperplane $\mathcal{P}_i = (PA_1A_2 \ldots A_{i-1}A_{i+1} \ldots A_n)$. Let P_0 be the intersection of the line (A_0P) with the hyperplane $\mathcal{P}_0 = (A_1A_2 \ldots A_n)$. Then,*

$$\sum_{i=1}^{n} \frac{A_0P_i}{P_iA_i} = \frac{A_0P}{PP_0}. \tag{2}$$

Theorem 3. *Let $\mathcal{A}$ be a n-simplex in the n-dimensional Euclidean space $\mathbb{E}^n$ with vertices A_0, A_1, ..., A_n. Let P be an arbitrary point in $\mathbb{E}^n$ such that P does not lie on any hyperplane containing facets of $\mathcal{A}$. For any $i = 1, \ldots, n$, let P_i be the intersection of the line (A_iP) with*

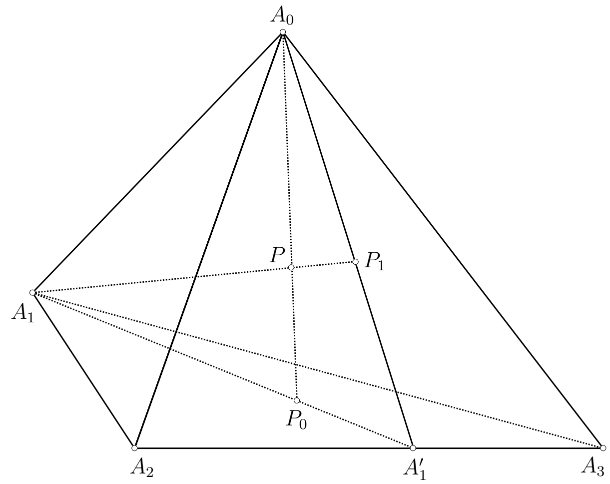

Figure 3: Illustration in 3D space of Theorem 3

the hyperplane $\mathcal{P}_i = (A_0A_1A_2 \dots A_{i-1}A_{i+1} \dots A_n)$*, let* A_i' *be the intersection of the line* (A_0P_i) *with the hyperplane* $\mathcal{P}_0 = (A_1A_2 \dots A_n)$*. Let* P_0 *be the intersection of the line* (A_0P) *with the hyperplane* $\mathcal{P}_0$*. Then,*

$$\sum_{i=1}^{n} \frac{A_0P_i}{P_iA_i'} = (n-1)\frac{A_0P}{PP_0}. \tag{3}$$

Remark 1. Where $n = 2$, we easily obtain the equation (1) from the Equations (2) and (3).

Using the two ideas of these extensions, we continue to extend a lemma of Blanchet's theorem in Section 3.

3 Proof of Theorems

We will recall the concept of barycentric coordinates over the simplex and some properties of parallel projection. The following definition comes from [8, pp. 195–199] and [10, pp. 9].

Definition 1 (Barycentric coordinates over simplex)**.** Let $\mathcal{A}$ be a n-simplex in the n-dimensional Euclidean space $\mathbb{E}^n$ with vertices $A_0, A_1, \dots, A_n$. Let P be any point in $\mathbb{E}^n$. Then, the $n+1$ real numbers $(x_0, x_1, x_2, \dots, x_n)$ satisfying

$$x_0 + x_1 + x_2 + \cdots + x_n \neq 0$$

and

$$x_0\overrightarrow{PA_0} + x_1\overrightarrow{PA_1} + \cdots + x_n\overrightarrow{PA_n} = \overrightarrow{0}$$

are called the barycentric coordinates of point P with respect to the simplex $\mathcal{A}$.

Definition 2. Let (α) be a hyperplane contained in $\mathbb{E}^n$. Let P be any point in $\mathbb{E}^n$. Let Δ be a line not parallel to (α). Mapping $\mathbf{pj}_{\Delta}^{(\alpha)}$ which maps P to the point P^* on (α) such that two lines (PP^*) and Δ are parallel is called a parallel projection with direction line Δ onto hyperplane (α).

We recall a theorem that is known as Thales' theorem in [1, pp.23].

Theorem 4 (Thales' theorem)**.** *The parallel projections are affine mappings.*

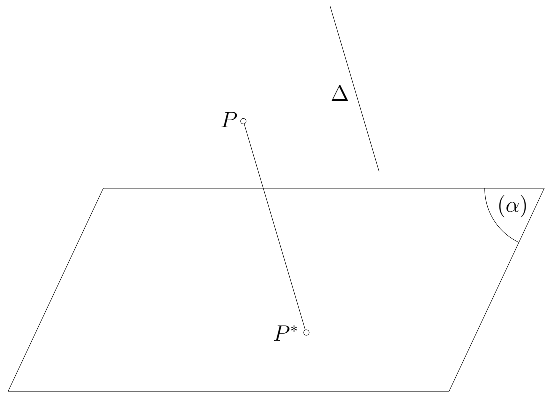

Figure 4: Illustration in 3D space of Definition 2

Remark 2. Since affine mappings preserve barycentric coordinates of points, we see that parallel projections also preserve barycentric coordinates of points.

We will first give the following lemma:

Lemma 1. *Let $\mathcal{A}$ be an n-simplex in the n-dimensional Euclidean space $\mathbb{E}^n$ with vertices A_0, A_1, ..., A_n. Let M be any point lying on the hyperplane $(A_1A_2\dots A_n)$. Assume that there are real numbers m, x_1, x_2 ..., x_n $(m \neq 0)$ satisfying the equation*

$$m\overrightarrow{A_0M} = x_1\overrightarrow{A_0A_1} + x_2\overrightarrow{A_0A_2} + \cdots + x_n\overrightarrow{A_0A_n}. \tag{4}$$

Then,

$$m = x_1 + x_2 + \cdots + x_n.$$

Proof. Since M lies on the hyperplane $\mathcal{P}_0 = (A_1A_2\dots A_n)$, n vectors $\overrightarrow{MA_1}$, $\overrightarrow{MA_2}$, ..., $\overrightarrow{MA_n}$ are linearly dependent or there are real numbers α_1, α_2, ..., α_n, not all zero, such that

$$\alpha_1\overrightarrow{MA_1} + \alpha_2\overrightarrow{MA_2} + \cdots + \alpha_n\overrightarrow{MA_n} = \overrightarrow{0} \tag{5}$$

which is equivalent to

$$\alpha_1\overrightarrow{A_0A_1} + \alpha_2\overrightarrow{A_0A_2} + \cdots + \alpha_n\overrightarrow{A_0A_n} = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)\overrightarrow{A_0M} \tag{6}$$

or

$$\overrightarrow{A_0M} = \sum_{i=1}^{n} \frac{\alpha_i}{\sum_{i=1}^{n}\alpha_i}\overrightarrow{A_0A_i}. \tag{7}$$

It follows from (4) that we have

$$\overrightarrow{A_0M} = \frac{x_1}{m}\overrightarrow{A_0A_1} + \frac{x_2}{m}\overrightarrow{A_0A_2} + \cdots + \frac{x_n}{m}\overrightarrow{A_0A_n}. \tag{8}$$

Since $A_0A_1\dots A_n$ is a simplex in $\mathbb{E}^n$, n vectors $\overrightarrow{A_0A_1}$, $\overrightarrow{A_0A_2}$, ..., $\overrightarrow{A_0A_n}$ are linearly independent. So that $\overrightarrow{A_0M}$ can be represented as a linear combination of the remaining vectors in a unique way. Combining with (7) and (8), we obtain

$$\frac{x_i}{m} = \frac{\alpha_i}{\sum_{i=1}^{n}\alpha_i} \tag{9}$$

for any integer $i = 1, \dots, n$.

Summing up (9) for any $1 = 1, \dots, n$, we have

$$\sum_{i=1}^{n} \frac{x_i}{m} = \sum_{i=1}^{n} \frac{\alpha_i}{\sum_{i=1}^{n} \alpha_i} = 1 \tag{10}$$

or

$$m = x_1 + x_2 + \cdots + x_n.$$

This completes the proof of Lemma 1. □

Lemma 2. *Let $\mathcal{A}$ be a n-simplex in the n-dimensional Euclidean space $\mathbb{E}^n$ with vertices A_0, A_1, ..., A_n. Let P be an arbitrary point in $\mathbb{E}^n$ such that P does not lie on any hyperplane containing facets of $\mathcal{A}$. Let P_0 be the intersection of the line (A_0P) with the hyperplane $(A_1A_2 \dots A_n)$. Assume that P has barycentric coordinates $(x_0, x_1, \dots, x_n)$ with respect to the simplex $\mathcal{A}$. Then,*

i) $$x_1\overrightarrow{P_0A_1} + x_2\overrightarrow{P_0A_2} + \cdots + x_n\overrightarrow{P_0A_n} = \vec{0},$$

ii) $$\frac{A_0P}{PP_0} = \frac{x_1 + x_2 + \cdots + x_n}{x_0}.$$

Proof. Since P has barycentric coordinates $(x_0, x_1, \dots, x_n)$ with respect to $\mathcal{A}$, we have

$$\sum_{i=0}^{n} x_i\overrightarrow{PA_i} = \vec{0}. \tag{11}$$

Considering the projection $\mathbf{pj}$ parallel to the direction line (A_0P) onto the hyperplane $\mathcal{P}_0 = (A_1A_2 \dots A_n)$, we see that $\mathbf{pj}(A_0) = \mathbf{pj}(P) = P_0$, $\mathbf{pj}(A_1) = A_1$, $\mathbf{pj}(A_2) = A_2$, ..., $\mathbf{pj}(A_n) = A_n$. Since barycentric coordinates are invariant under an affine transformation (Remark 2), it follows from equation (11) that we have

$$\sum_{i=0}^{n} x_i\overrightarrow{\mathbf{pj}(P)\,\mathbf{pj}(A_i)} = \vec{0} \tag{12}$$

This proves Part i). Also from this,

$$x_1\overrightarrow{P_0A_1} + x_2\overrightarrow{P_0A_2} + \cdots + x_n\overrightarrow{P_0A_n} = \vec{0}. \tag{13}$$

Therefore,

$$-x_0\overrightarrow{PA_0} = x_1\overrightarrow{PA_1} + x_2\overrightarrow{PA_2} + \cdots + x_n\overrightarrow{PA_n} = (x_1 + x_2 + \cdots + x_n)\overrightarrow{PP_0}. \tag{14}$$

From this and using directed line segments, we deduce Part ii). This completes the proof of Lemma 2. □

Coming back to the main theorems.

Proof of Theorem 2. (See illustration in 3D space in Figure 2). We assume that P has barycentric coordinates $(x_0, x_1, \dots, x_n)$ with respect to the simplex $\mathcal{A}$. Then,

$$\sum_{i=0}^{n} x_i\overrightarrow{PA_i} = \vec{0} \tag{15}$$

or

$$x_1\overrightarrow{A_0A_1} + x_2\overrightarrow{A_0A_2} + \cdots + x_n\overrightarrow{A_0A_n} = (x_0 + x_1 + \cdots + x_n)\overrightarrow{A_0P}. \tag{16}$$

Since P_1 lies on the line (A_0A_1), using directed line segments as one-dimensional vector algebra [4, p. 30], we have

$$\overrightarrow{A_0A_1} = \frac{A_0A_1}{A_0P_1} \cdot \overrightarrow{A_0P_1}. \tag{17}$$

From (16) and (17), we obtain

$$x_1\frac{A_0A_1}{A_0P_1} \cdot \overrightarrow{A_0P_1} + x_2\overrightarrow{A_0A_2} + \cdots + x_n\overrightarrow{A_0A_n} = (x_0 + x_1 + \cdots + x_n)\overrightarrow{A_0P}. \tag{18}$$

Since P_1 lies on the line (A_0A_1) (P_1 does not coincide with A_0 and A_1 because P does not lie on the any hyperplane containing facets of $\mathcal{A}$), we easily see that $A_0P_1A_2 \ldots A_n$ is also a simplex in $\mathbb{E}^n$. Now using Lemma 1, we get that

$$x_0 + x_1 + \cdots x_n = x_1\frac{A_0A_1}{A_0P_1} + x_2 + \cdots + x_n \tag{19}$$

which is equivalent to

$$\frac{A_0A_1}{A_0P_1} - 1 = \frac{x_0}{x_1} \tag{20}$$

or

$$\frac{P_1A_1}{A_0P_1} = \frac{x_0}{x_1} \tag{21}$$

or

$$\frac{A_0P_1}{P_1A_1} = \frac{x_1}{x_0}. \tag{22}$$

Similarly, for any $n = 2, \ldots, n$, we have

$$\frac{A_0P_i}{P_iA_i} = \frac{x_i}{x_0}. \tag{23}$$

Summing up the Equations (22) and (23) for any $i = 1, \ldots, n$ and combining with Lemma 2, we have

$$\sum_{i=1}^{n} \frac{A_0P_i}{P_iA_i} = \frac{x_1 + x_2 + \cdots + x_n}{x_0} = \frac{A_0P}{PP_0}. \tag{24}$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. (See illustration in 3D space in Figure 3). We assume that P has barycentric coordinates (see [5]) $(x_0, x_1, \ldots, x_n)$ with respect to the simplex $\mathcal{A}$. Then,

$$\sum_{i=1}^{n} x_i\overrightarrow{PA_i} = \vec{0}. \tag{25}$$

It follows from Part i) of Lemma 2 that we have

$$x_0\overrightarrow{P_1A_0} + x_2\overrightarrow{P_1A_2} + \cdots + x_n\overrightarrow{P_1A_n} = \vec{0}. \tag{26}$$

Considering the projection $\mathbf{pj}_1$ parallel to the direction line (A_0P_1) onto the hyperplane $\mathcal{P}_0 = (A_1A_2 \ldots A_n)$, we see that $\mathbf{pj}_1(A_0) = \mathbf{pj}_1(P_1) = A_1'$, $\mathbf{pj}_1(A_2) = A_2$, …, $\mathbf{pj}_1(A_n) = A_n$. Since barycentric coordinates are invariant under an affine transformation (Remark 2), it follows from equation (26) that we have

$$x_0\overrightarrow{\mathbf{pj}_1(P_1)\,\mathbf{pj}_1(A_0)} + x_2\overrightarrow{\mathbf{pj}_1(P_1)\,\mathbf{pj}_1(A_2)} + \cdots + x_n\overrightarrow{\mathbf{pj}_1(P_1)\,\mathbf{pj}_1(A_n)} = \vec{0} \tag{27}$$

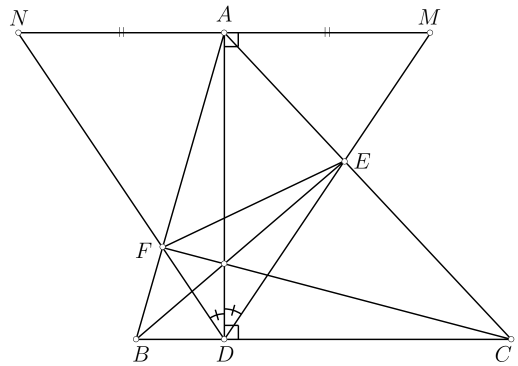

Figure 5: Illustration in plane of Blanchet's theorem

or

$$x_2\overrightarrow{A_1'A_2} + x_3\overrightarrow{A_1'A_3} + \cdots + x_n\overrightarrow{A_1'A_n} = \vec{0} \tag{28}$$

or

$$x_2\overrightarrow{P_1A_2} + x_3\overrightarrow{P_1A_3} + \cdots + x_n\overrightarrow{P_1A_n} = (x_2 + x_3 + \cdots + x_n)\overrightarrow{P_1A_1'}. \tag{29}$$

From Equations (26) and (29), we obtain

$$x_0\overrightarrow{P_1A_0} + (x_2 + x_3 + \cdots + x_n)\overrightarrow{P_1A_1'} = \vec{0} \tag{30}$$

or

$$\frac{A_0P_1}{P_1A_1'} = \frac{x_2 + x_3 + \cdots + x_n}{x_0}. \tag{31}$$

Similarly, for any $i = 2, \ldots, n$, we get that

$$\frac{A_0P_i}{P_iA_i'} = \frac{x_1 + x_2 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n}{x_0}. \tag{32}$$

Summing up equation (31) and (32) for any $i = 1, \ldots, n$ and using Lemma 2 Part ii), we deduce that

$$\sum_{i=1}^{n} \frac{A_0P_i}{P_iA_i'} = (n-1)\frac{x_1 + x_2 + \cdots + x_n}{x_0} = (n-1)\frac{A_0P}{PP_0}. \tag{33}$$

This completes the proof of Theorem 3. □

4 Extending a lemma leading to Blanchet's theorem

The book [6, pp. 471–472] mentioned the following theorem as Blanchet's theorem:

Theorem 5 (Blanchet)**.** *Let ABC be a triangle with altitude AD (D lies on the line (BC)). Let P be any point on the line (AD). Lines (PB) and (PC) meet the lines (CA) and (AB) at E and F, respectively. Then, line (DA) bisects angle $\angle EDF$. (See Figure 5).*

A lemma related to Cevian triangle that has been used to lead to this theorem is

Lemma 3. *Let ABC be a triangle. Let P be an arbitrary point not lying on the lines (BC), (CA) and (AB). Assume that the lines (PA), (PB), and (PC) meet the lines (BC), (CA), and (AB) at D, E, and F, respectively. The line passes through A and is parallel to the line (BC) meets the lines (DE) and (DF) at M and N, respectively. Then, A is the midpoint of MN. (See Figure 6).*

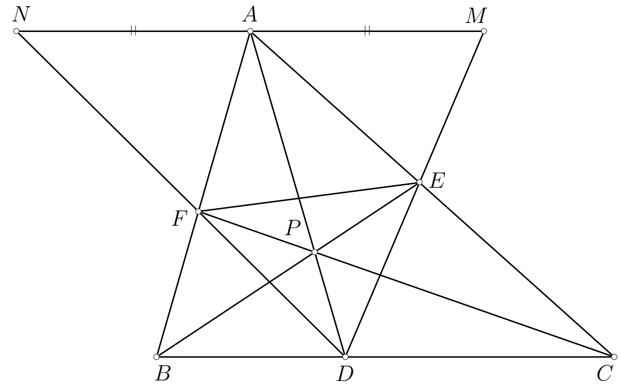

Figure 6: Illustration in plane of Blanchet's lemma

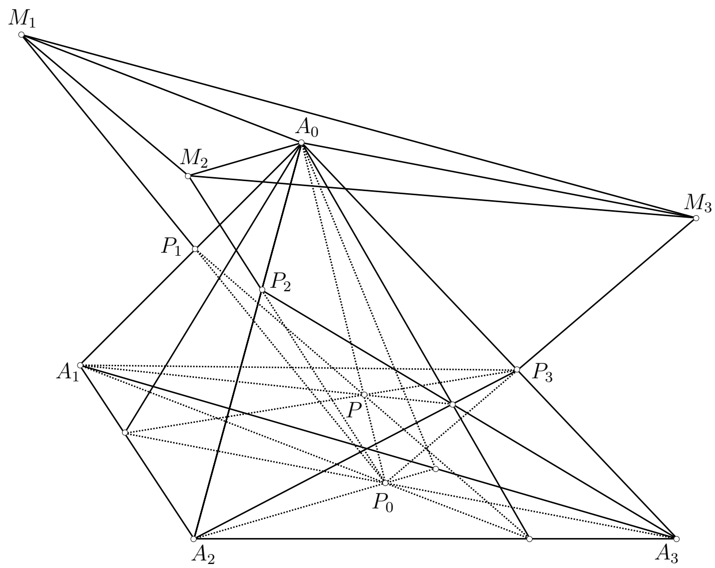

Figure 7: Illustration in 3D space of Theorem 6

Using the idea in extending van Aubel's theorem, we can easily extend this Lemma to the two following theorems:

Theorem 6. *Let $\mathcal{A}$ be an n-simplex in the n-dimensional Euclidean space $\mathbb{E}^n$ with vertices A_0, A_1, $\ldots$, A_n. Let P be an arbitrary point in $\mathbb{E}^n$ such that P does not lie on any hyperplane containing facets of $\mathcal{A}$. For any $i = 1, \ldots, n$, let P_i be the intersection of the line (A_0A_i) with the hyperplane $\mathcal{P}_i = (PA_1A_2 \ldots A_{i-1}A_{i+1} \ldots A_n)$. Let P_0 be the intersection of the line (A_0P) with the hyperplane $\mathcal{P}_0 = (A_1A_2 \ldots A_n)$. Denote by (α) the hyperplane passing through A_0 which is parallel to the hyperplane $\mathcal{P}_0$. For any $i = 1, \ldots, n$, the line (A_0P_i) intersects the hyperplane (α) at M_i. Then, A_0 is the centroid of set of points $\{M_1, M_2, \ldots, M_n\}$.*

Theorem 7. *Let $\mathcal{A}$ be an n-simplex in the n-dimensional Euclidean space $\mathbb{E}^n$ with vertices A_0, A_1, $\ldots$, A_n. Let P be an arbitrary point in $\mathbb{E}^n$ such that P does not lie on any hyperplane containing facets of $\mathcal{A}$. For any $i = 1, \ldots, n$, let P_i be the intersection of the line (A_iP) with the hyperplane $\mathcal{P}_i = (A_0A_1A_2 \ldots A_{i-1}A_{i+1} \ldots A_n)$. Let P_0 be the intersection of the line (A_0P) with the hyperplane $\mathcal{P}_0 = (A_1A_2 \ldots A_n)$. Denote by (α) the hyperplane passing through A_0 and is parallel to the hyperplane $\mathcal{P}_0$. For any $i = 1, \ldots, n$, the line (P_0P_i) intersects the hyperplane (α) at M_i. Then, A_0 is the centroid of the set of points $\{M_1, M_2, \ldots, M_n\}$.*

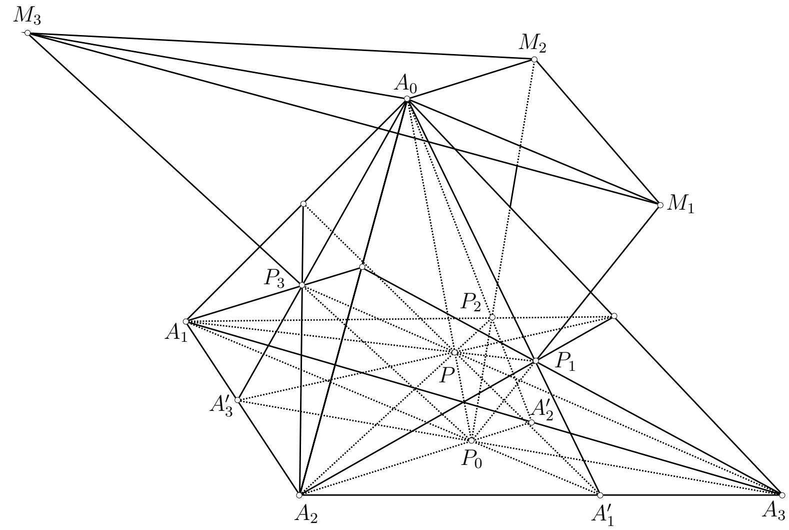

Figure 8: Illustration in 3D space of Theorem 7

Proof of Theorem 6. (See illustration in 3D space in Figure 7.) Because the hyperplane (α) is parallel to hyperplane $\mathcal{P}_0$ and the two lines (P_1A_1) and (P_0P_1) meet (α) at A_0 and M_1, respectively, (noting that A_1 and P_0 belong to $\mathcal{P}_0$) we deduce that the two lines (A_0M_1) and (A_1P_0) are parallel. We get

$$\frac{A_0M_1}{P_0A_1} = \frac{A_0P_1}{P_1A_1}. \tag{34}$$

Combining with equation (22), this implies

$$\frac{A_0M_1}{P_0A_1} = \frac{A_0P_1}{P_1A_1} = \frac{x_1}{x_0}, \tag{35}$$

this means

$$x_0\overrightarrow{A_0M_1} = x_1\overrightarrow{P_0A_1}. \tag{36}$$

Similarly, for any $i = 2, \ldots, n$

$$x_0\overrightarrow{A_0M_i} = x_i\overrightarrow{P_0A_i}. \tag{37}$$

Summing up the Equations (36) and (37) for any $i = 1, \ldots, n$ and using Part i) of Lemma 2, we obtain

$$x_0\left(\sum_{i=1}^{n}\overrightarrow{A_0M_i}\right) = \sum_{i=1}^{n} x_i\overrightarrow{P_0A_i} = \vec{0}. \tag{38}$$

Note that since P does not lie in the hyperplane $\mathcal{P}_0$, x_0 is different from 0. From (38) we have

$$\sum_{i=1}^{n}\overrightarrow{A_0M_i} = \vec{0}$$

or A_0 is the centroid of the set of points $\{M_1, M_2, \ldots, M_n\}$. This completes the proof of Theorem 6. □

Proof of Theorem 7. (See illustration in Figure 8.) Let A'_i be the intersection of line (A_0P_i) with the hyperplane $\mathcal{P}_0$. Because the hyperplane (α) is parallel to the hyperplane $\mathcal{P}_0$ and the two lines (P_1A_1') and (P_0P_1) meet (α) at A_0 and M_1, respectively, (noting that A_1' and P_0 belong to $\mathcal{P}_0$), we deduce that two lines (A_0M_1) and $(A_1'P_0)$ are parallel. We get

$$\frac{A_0M_1}{P_0A_1'} = \frac{A_0P_1}{P_1A_1'}. \tag{39}$$

Combining with equation (31), this implies

$$\frac{A_0M_1}{P_0A_1'} = \frac{A_0P_1}{P_1A_1'} = \frac{x_2 + x_3 + \cdots + x_n}{x_0} \tag{40}$$

or

$$x_0\overrightarrow{A_0M_1} = (x_2 + x_3 + \cdots + x_n)\overrightarrow{P_0A_1'}. \tag{41}$$

It follows from (28) that we have

$$x_2\overrightarrow{P_0A_2} + x_3\overrightarrow{P_0A_3} + \cdots + x_n\overrightarrow{P_0A_n} = (x_2 + x_3 + \cdots + x_n)\overrightarrow{P_0A_1'}. \tag{42}$$

From (41) and (42), we obtain

$$x_0\overrightarrow{A_0M_1} = x_2\overrightarrow{P_0A_2} + x_3\overrightarrow{P_0A_3} + \cdots + x_n\overrightarrow{P_0A_n}. \tag{43}$$

Similarly, for any $i = 2, \ldots, n$ we have

$$x_0\overrightarrow{A_0M_i} = x_1\overrightarrow{P_0A_1} + x_2\overrightarrow{P_0A_2} + \cdots + x_{i-1}\overrightarrow{P_0A_{i-1}} + x_{i+1}\overrightarrow{P_0A_{i+1}} + \cdots + x_n\overrightarrow{P_0A_n}. \tag{44}$$

Summing up the Equations (43) and (44) for any $i = 1, \ldots, n$ and using Part i) of Lemma 2, we obtain

$$x_0\left(\sum_{i=1}^{n}\overrightarrow{A_0M_i}\right) = (n-1)\left(\sum_{i=1}^{n}x_i\overrightarrow{P_0A_i}\right) = \vec{0}. \tag{45}$$

Note that since P does not lie in the hyperplane $\mathcal{P}_0$, x_0 is different from 0. From (45) we have

$$\sum_{i=1}^{n}\overrightarrow{A_0M_i} = \vec{0}$$

or A_0 is the centroid of the set of points $\{M_1, M_2, \ldots, M_n\}$. This completes the proof of Theorem 7. □

5 Conclusion

We have extended van Aubel's theorem on the ratio of the lengths involved in the Cevian triangle of plane geometry to the simplex. The important tool we have used here is the barycentric coordinates system over simplex and parallel projection. Moreover, by this method, we can also extend the lemma of Blanchet's Theorem. We see the usefulness of the barycentric coordinate system over simplex.

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