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On a Proof of the Thābit Ibn Qurra's Generalization of the Pythagorean Theorem

Giovanni Vincenzi

Dipartimento di Matematica, Università di Salerno, Fisciano, Italy
vincenzi@unisa.it

Abstract. One of the most interesting generalizations of the Pythagorean theorem was stated by Thābit in the IX century. However, as claimed an expert of history of mathematics the Arab mathematician did not present the proof, as it can likely be obtained by elementary properties regarding similar triangles.

According to historical documents, it is challenging to establish whether a proof of Thābit's theorem exists based exclusively on equidecomposibility, as in the case of the Pythagorean and Pappus theorems. This article presents the corresponding proof.

Key Words: Euclidean geometry, generalization of Pythagorean theorem, equidecomposability, figure dissection, geometric construction

MSC 2020: 51M15 (primary), 97G30

1 Introduction

Thābit ibn Qurra al-Harrānī was a notable mathematician, astronomer, physician, and philosopher who lived in Upper Mesopotamia in 826–921 ([20]). Thābit ibn Qurra was representative of the flourishing Arab-Islamic culture in the 9th century and had different interests in mathematics, such as Algebra, Geometry, Measure Theory and Number theory. A remarkable formula for amicable numbers (see [20]) is attributed to him. In Euclidean geometry, among other investigations, the researcher presented different proofs of the Pythagorean theorem ([23]).

As Pappus (see [8]), Thābit also presented a generalization of the Pythagorean theorem. Specifically, he stated the following result (see [16, p. 213]):

Theorem 1.1 (Thābit)**.** *If from vertex C of a triangle ABC, two lines CD and CE are drawn forming the angles CDA and CEB with the base AB, respectively, both equal to angle ACB, the sum of the squares of sides AC and CB is equal to the rectangle represented as $AD + EB$ times AB (Figure 1).*

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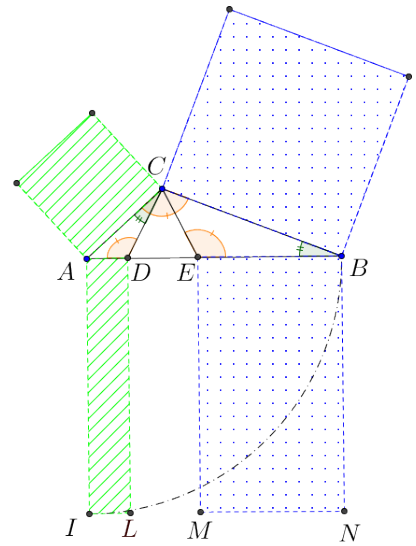

Figure 1: The configuration of Thābit's theorem

Clearly, if $B\hat{C}A$ is a right angle, the Pythagorean theorem is obtained. According to A. Sayili (see [22]), this result is the most important contribution of Thābit ibn Qurra in terms of the generalization of the Pythagorean theorem.

Boyer [16] stated that

> In fact, the theorem provides a beautiful generalization of the pinwheel diagram used by Euclid in the proof of the Pythagorean theorem.

However, the manner in which Thābit proved the result remains unclear, and historical investigations indicate that the researcher probably omitted the proof because it easily follows from properties of similar triangles ([16] p. 214).

To demonstrate this aspect, it is adequate to split Thābit's configuration, as shown in Figure 1 into two configurations. As shown in Figure 2 (left), triangles ABC and ACD are similar and hence the following proportion holds:

$$AB : AC = AC : AD.$$

By construction $AB = AI$, and thus, the square constructed on side AC has the same area as that of rectangle $AIMD$.

Similarly, (Figure 2, right) the square constructed on side BC has the same area as that of rectangle $BEPV$. Then, as Thābit claimed, *the sum of the squares of sides AC and CB is 'equal' to the rectangle represented as $AD + EB$ times AB.*

The understanding of the above-mentioned proof requires knowledge regarding the concept of measurement of planar figures, and therefore, of real numbers. These reasons are likely why this topic rarely appears in scholarly textbooks and is therefore ignored by most students ([23]). Moreover, Euclid's first and second theorems and the Pythagorean theorem and its generalization by Pappus are usually explained without using the concept of similarity, and using only the concept of *equivalence* in terms of equidecomposability, which is more intuitive and contains the concept of equivalence in terms of the area.

This phenomenon likely occurs because images are highly effective in teaching and learning ([2], [5], [7], [11], [19]) and in providing students with a correct representation of the development of mathematical thinking ([9] and [15]).

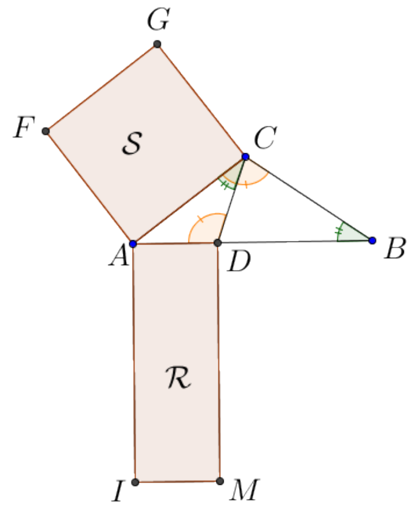

Figure 2: A split Thābit's configuration: in both figures the shaded quadrilaterals have the same area

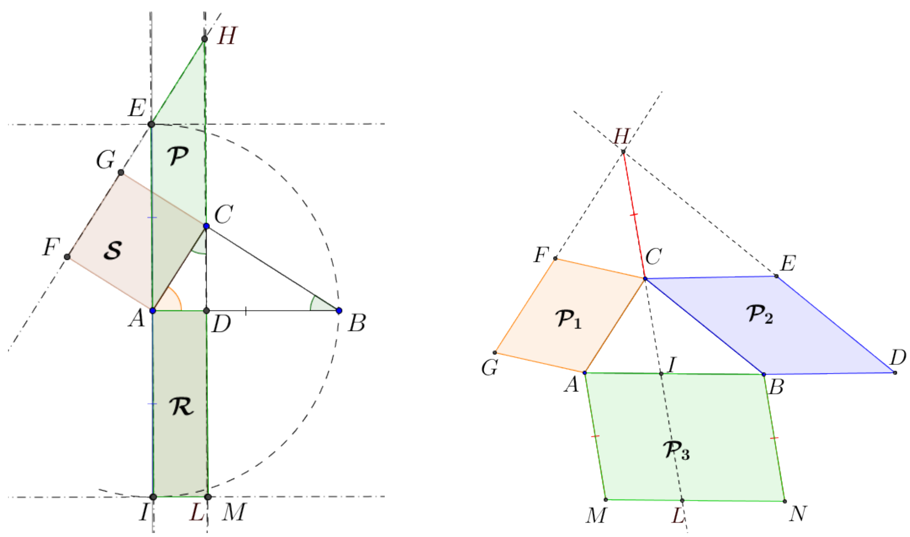

Figure 3: Left, Euclid I theorem: the figures $\mathcal{S}, \mathcal{P}$ and $\mathcal{R}$ are equidecomposable, in particular they are equivalent. Right, Pappus' theorem: the parallelograms $AMLI$ and $\mathcal{P}_1$ are equidecomposable, and similarly parallelograms $ILNB$ and $\mathcal{P}_2$ are equidecomposable, in particular $\mathcal{P}_1 \cup \mathcal{P}_2$ is equivalent to $\mathcal{P}_3$

Therefore, from didactic, historical, epistemological and foundational viewpoints, it is of significance to have a 'visual' and direct proof of Thābit's theorem in the same manner as the classics of Euclid and Pappus. In this article, we present such a proof. The paper is suitable for a wide variety of readers.

2 Dissections of equidecomposable figures

For the convenience of the reader we recall that

Definition 2.1. Two polygons are said to be *equidecomposable* or *equivalent by dissection*

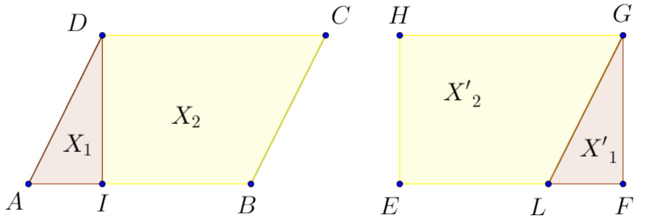

Figure 4: A parallelogram and a rectangle with the same height and congruent basis are equidecomposable

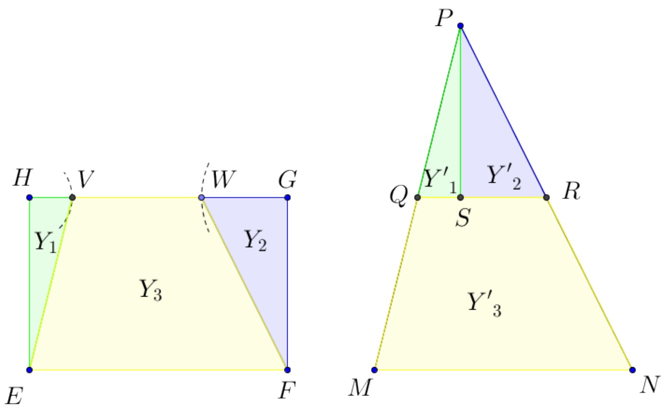

Figure 5: If a rectangle $\mathcal{R}$ has same base of a triangle $\mathcal{T}$ and height of $\mathcal{R}$ is half of $\mathcal{T}$, then they are equidecomposable

([12, Chapter 12]), if it is possible to dissect, or decompose, one of them into a finite number of pieces that can then be rearranged to form the second polygon, c.f. Figures 4 and 5, for example.

We highlight that this concept leads to an equivalence relation (see for example [24]) among planar polygonal figures (see Figure 6).

The concept of equidecomposability is often used in elementary geometry. Notably, it is not always so trivial to identify the equivalent dissections of equidecomposable figures. Consequently, relevant theorems, such as that of Pappus are often ignored in high school and undergraduate courses, which is a lost opportunity for an interesting didactic experience. As indicated by Eves [8],

> The student of high school geometry can hardly fail to be interested in the Pappus extension of the Pythagorean theorem, and the proof of the extension can serve as a nice exercise for the students.

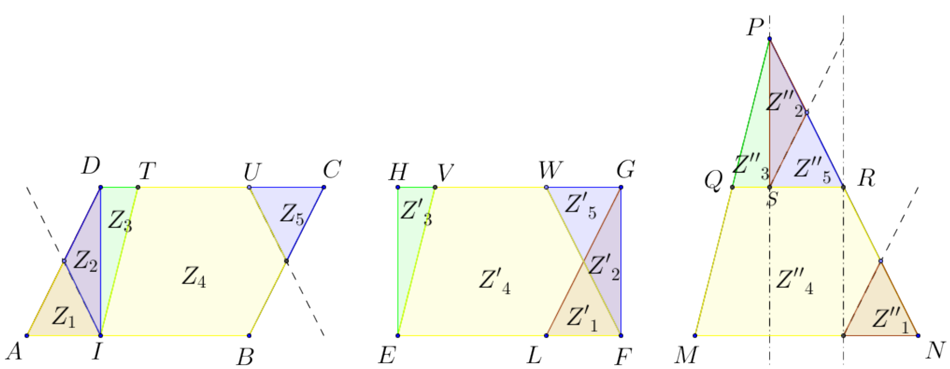

Figure 6: The parallelogram shown in Figure 4 and the triangle shown in Figure 5 are equidecomposable

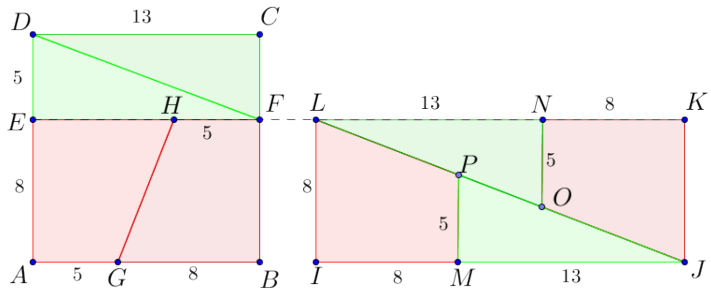

Figure 7: The pieces of the same color are pairwise congruent, but the area of the square is 169 and the area of the rectangle on the left is 168. In fact, the second figure is apparently dissected: L, P, O and J are not aligned

We conclude this section by noting that the dissection into same pieces of two planar figure must be checked carefully, as indicated by the following classic example shown in Figure 7 (see for example [14]):

This example can be attributed to Sam Loyd. This Example was first published in 1774 (for more details, see [25], and references therein).

3 The proof

In this section, we demonstrate Thābit's theorem (1.1) via equidecomposability. As shown in Figure 2 it is adequate to demonstrate the following aspect:

Theorem 3.1. *If from vertex C of a triangle ABC, a line CD is drawn forming the angle $D\hat{C}A$ with the base AC, which is equal to angle $A\hat{B}C$, the square of side AC is equal to the rectangle whose sides are congruent to AD and AB (Figure 2, left).*

Proof. We implement six steps.

Step 1) Rotate triangle ABC around vertex A by 90° counterclockwise. In this case, $B\hat{A}E$ is a right angle, and $AB = AE$ (Figure 8, left).

Step 2) Consider the extension of the EA side. Segment AI on this side is congruent to AB. Consider $\mathcal{S} = ACGF$ as the square of AC side. Insert H, which the intersection of the extension of FG and parallel line to AB, through E.

Step 3) The straight line parallel to AB passing through I intersects the line passing through HA at a point, L. In this case, triangle HEA is congruent to triangle LIA (Figure 8, right); in particular $HA = AL$.

Step 4) The right triangles HEA and HFA have the same hypotenuse. Therefore the quadrilateral $AEHF$ is cyclic, and angles $F\hat{H}A$ and $F\hat{E}A$ are congruent. It follows that $F\hat{H}A = C\hat{B}A = A\hat{C}D$. Moreover, when the line through FG is parallel to the line through CA, the angle $F\hat{H}A$ is equal to $C\hat{A}H$, and hence $C\hat{A}H = A\hat{C}D$.
The straight line CA intersects both CD and the HL, generating a pair of equal alternating internal angles: $C\hat{A}H = A\hat{C}D$. This aspects proves that CD and HL are parallel (Figure 9).

Step 5) Let now $\mathcal{P}_1 = DALN$ be the parallelogram constructed on the consecutive sides DA and AL, and let $\mathcal{P}_2 = HACO$ the parallelogram constructed on the consecutive sides HA and AC. Clearly $\mathcal{P}_1$ and $\mathcal{P}_2$ have the same height (see Step 4); moreover, these

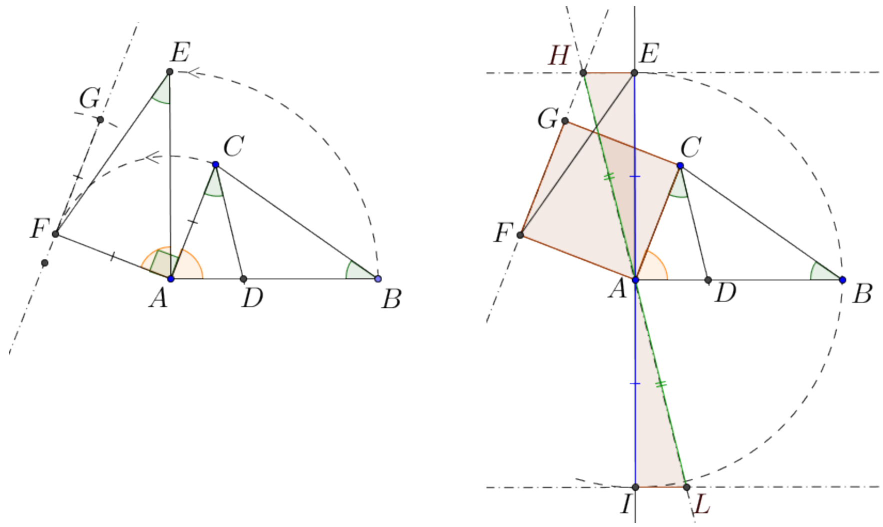

Figure 8: Starting from a triangle ABC, we construct two congruent right triangles, HEA and AIL. In particular, $HA = AL$

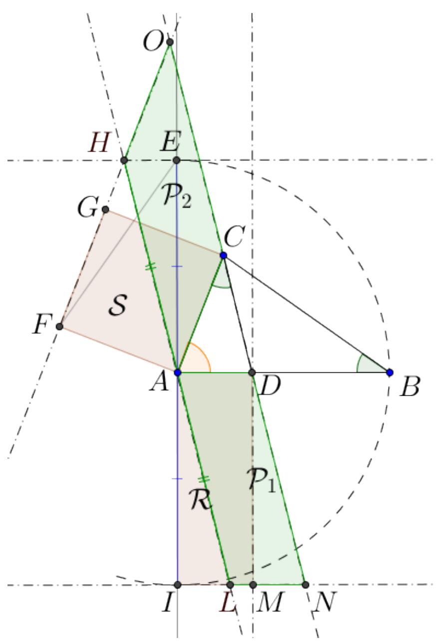

Figure 9: Visual proof of Thābit's theorem: the four coloured quadrilaterals are pairwise equidecomposable

entities have congruent bases HA and AL (see Step 2). It follows that $\mathcal{P}_1$ and $\mathcal{P}_2$ are equidecomposable.

Step 6) The square $\mathcal{S} = ACGF$ and parallelogram $\mathcal{P}_2$ have the same base CA and same height and are thus equidecomposable. Similarly, the rectangle $\mathcal{R} = DAIM$ and parallelogram $\mathcal{P}_1$ have the same base AD and same height AI and are thus equidecomposable.

We have proven that

- the square $\mathcal{S}$ and parallelogram $\mathcal{P}_2$ are equidecomposable (Step 6);
- the parallelogram $\mathcal{P}_2$ and parallelogram $\mathcal{P}_1$ are equidecomposable (Step 5);
- the parallelogram $\mathcal{P}_1$ and rectangle $\mathcal{R}$ are equidecomposable (Step 6).

Therefore, the square $\mathcal{S}$ and rectangle $\mathcal{R}$ are equidecomposable. □

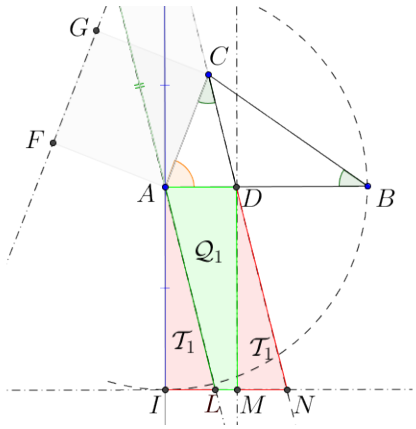

Figure 10: The rectangle $\mathcal{R} = DAIM$ and parallelogram $\mathcal{P}_1 = ALND$ can be dissected into the triangle $\mathcal{T}_1$ and quadrilateral $\mathcal{Q}_1$

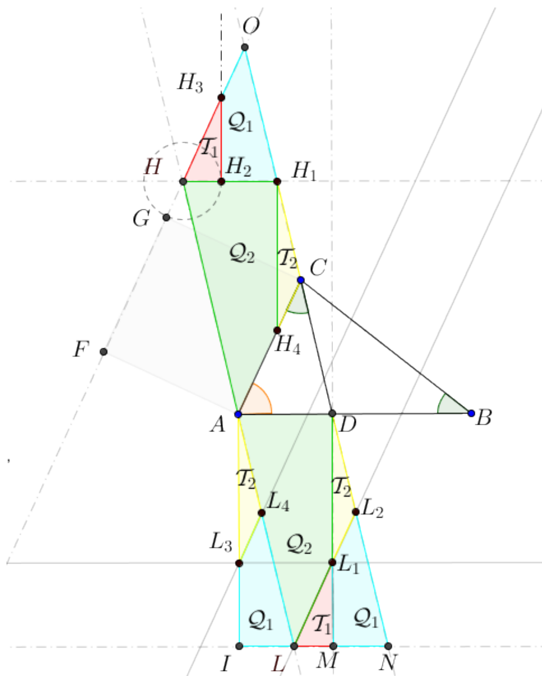

Figure 11: The rectangle $\mathcal{R} = AIMD$ and Parallelogram $\mathcal{P}_2 = HACO$ can be dissected into the red triangle $\mathcal{T}_1$, the yellow triangle $\mathcal{T}_2$ the sky blue quadrilateral $\mathcal{Q}_1$ and the green quadrilateral $\mathcal{Q}_2$

4 Determination of a dissection of $\mathcal{S}$ and $\mathcal{R}$ in the same pieces

Theorem 1.1 shows that there exists a dissection of the square $\mathcal{S}$ and rectangle $\mathcal{R}$ (shown in Figure 9) in the same pieces. It is natural to ask:

Which are these pieces? How can $\mathcal{S}$ and $\mathcal{R}$ be dissected in the same pieces?

In this section, we outline the guiding principles to determine the dissection of $\mathcal{S}$ and $\mathcal{R}$ in seven pairwise congruent pieces (Figure 13).

Figure 9, illustrates three steps to determine a dissection of the rectangle $\mathcal{R}$ and the square $\mathcal{S}$, in the same pieces.

Step I) Determination of the dissection of the rectangle $\mathcal{R} = DAIM$ and parallelogram $\mathcal{P}_1 = ALND$ in the same pieces (see Figure 10). It is adequate to highlight that triangles AIL and DMN are congruent.

Step II) Determination of the dissection of the rectangle $\mathcal{R}$ and parallelogram $\mathcal{P}_2$ in the same pieces (Figure 11). This part requires additional considerations.

(1) Consider the parallel line to CA through L: we have intersections L_1 and L_2.

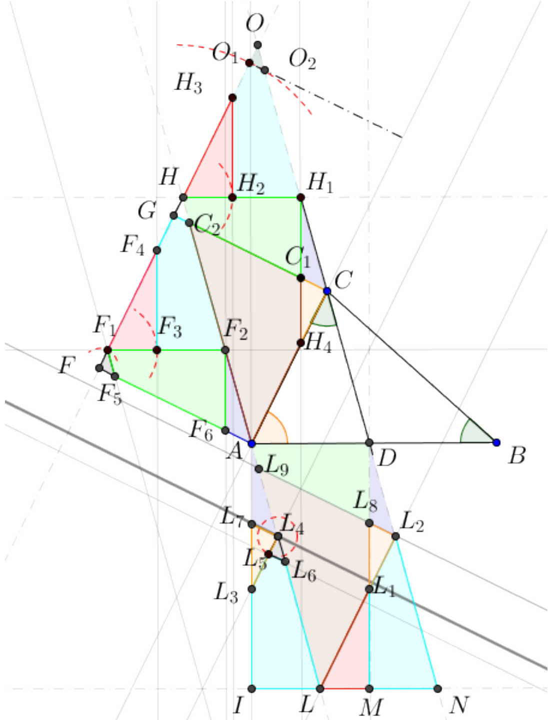

Figure 12: A visualization of the final step to find the dissection of $\mathcal{S}$ and $\mathcal{R}$ in seven pieces pairwise congruent

(2) Consider the parallel line to AB through L_1, and let L_3 be its intersection with AI. Consider the parallel line to AC thorough L_3 and let L_4 be its intersection with LH.
(3) Consider the parallel line to AB through H, and let H_1 be its intersection with ON.
(4) Consider on HH_1 point H_2 such that $HH_2 = LM$.
(5) Consider the perpendicular to AB through H_2, and let H_3 be its intersection with FO.
(6) Consider the perpendicular to AB through H_1, and let H_4 be its intersection with CA.
(7) Using congruence theorems, the reader can verify that:
   * the red triangle HH_2H_3 is congruent to $\mathcal{T}_1$;
   * the blue quadrilaterals $H_1H_2H_3O$ and LIL_3L_4 are congruent;
   * the yellow triangles L_3L_4A, L_1L_2D and H_4CH_1 are congruent;
   * the green quadrilaterals LL_1DA and AH_4H_1H are congruent.

In this manner we have determined four pieces in which $\mathcal{R}$ and $\mathcal{P}_2$ can be dissected.

Step III) Determination of the dissection of the rectangle $\mathcal{R}$ and square $\mathcal{S}$ in the same pieces (Figure 12).

(8) Let C_1 be the intersection between CG and the perpendicular line to AB through H_1
(9) Let $O_1 \in GO$ such that $GO_1 = FG$. As $HO = FG$, $GH = O_1O$.
(10) Let $F_1 \in FG$ such that $FF_1 = GH$. In this case, $F_1H = HO$.
(11) Let F_2 be the intersection between HA and the parallel line to AB through F_1. Thus, according to (10) and the second criterion, F_1F_2H and HH_1O are congruent triangles.
(12) Let $F_3 \in F_1F_2$ such that $F_1F_3 = HH_2$. Let F_4 be the intersection between FG and the perpendicular line to AB through F_3. In this case, $F_1F_3F_4$ is congruent to the red triangle HH_2H_3. According to (9), $F_4G = H_3O_1$.
(13) Let F_5 be the intersection between AF and the parallel line to HL through F_1.

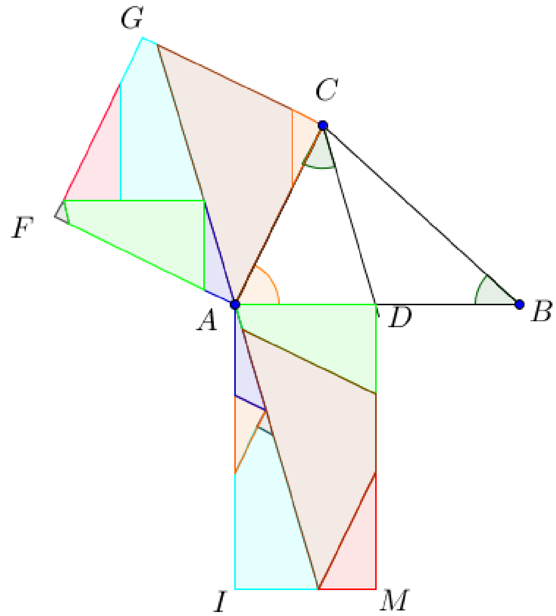

Figure 13: The rectangle $\mathcal{R} = AIMD$ and the square $\mathcal{S} = ACGF$ can be dissected into seven pairwise congruent pieces

Note that the small triangles FF_1F_5, GHC_2 and OO_1O_2 are congruent.

(14) According to (12), $F_3F_2 = H_2H_1$ and $F_3F_4 = H_2H_3$. According to (11–13) $F_4G = H_3O_1$ and $GC_2 = O_1O_2$. Therefore, the sky-blue pentagons $F_3F_2C_2GF_4$ and $H_2H_1O_2O_1H_3$ are congruent.

(15) Clearly $HA = OC$, and thus, by (13) and (14), we obtain $F_2A = H_1C$. It follows that F_2AF_6 and H_1CC_1 are congruent.

(16) Let $L_5 \in L_4L_3$ such that $L_4L_5 = FF_1$. Let L_6 be the intersection between AL and the parallel line to FA through L_5. In this case, the black triangles $L_4L_5L_6$ and F_1FF_5 are congruent.

(17) Let L_7 be the intersection between AI and the parallel line to FA through L_4. Let L_8 be the intersection between DM and the parallel line to FA through L_2. According to Step II (see (7)), $AL_4 = DL_2 = H_1C$. Therefore, according to (15-16), the violet triangles AL_7L_4 DL_8L_2, H_1C_1C and F_2F_6A are congruent. It follows that $L_7L_3 = L_8L_1 = C_1H_4$, and thus orange triangles $L_3L_4L_7$ and H_4CC_1 are congruent.

(18) According to Step II, the quadrilaterals ILL_4L_3 and $H_1H_2H_3O$ are congruent, and thus, according to (16-17), it follows that the sky-blue pentagons $ILL_6L_5L_3$, and $H_3H_2H_1O_2O_1$ are congruent. Consequently, by (14) $ILL_6L_5L_3$ and $F_3F_2C_2GF_4$ are congruent.

(19) Clearly, triangles LL_2L_9 and ACC_2 are congruent. Therefore, according to (17), $L_9L_8 = C_2C_1 = F_5F_6$, and by Step II (see (7)) $AL_9 = HC_2 = F_1F_5$. It follows that the green quadrilaterals AL_9L_8D and $F_1F_5F_6F_2$ are congruent, and the brown quadrilaterals $LL_1L_8L_9$ and $AH_4C_1C_2$ are congruent.

In this manner, we have determined seven pieces in which $\mathcal{R}$ and $\mathcal{S}$ can be dissected (Figure 13).

5 Conclusions

Reasoning based on figures is becoming a growing interdisciplinary field in logic, philosophy and cognitive sciences, and is also of considerable interest in the field of education (for a summary, see [21]). The hypothesis according to which geometric figures are constituent parts of the logical structure of the geometric theory (cf. [4], [3], [6], [13], [17], [18], for example) that offsets the limitations of the algebraic /logical language (see E. Agazzi in [1] or [10], L. Kvasz in [13], for example) is increasingly being accepted. The validity of the geometry of the Elements of Euclid is being re-evaluated and its recovery considering the developments in contemporary mathematics (see [17], [11], for example) is considered reasonable. In this context, a visual proof of Thābit's theorem based exclusively on equidecomposibility could be interesting from a didactic, historical, epistemological and foundational viewpoints.

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