Journal for Geometry and Graphics
Volume 25 (2021), No. 2, 171–186

Polyhedral Cylinders Formed by Kokotsakis Meshes

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Dedicated to Prof. Kuno Egle, Karlsruhe Institute of Technology

Abstract. Kokotsakis proved that an infinite planar mesh composed of congruent convex, non-trapezoidal, non-parallelogramic quadrilaterals is deformable with degree of freedom 1 in two modes if the quadrilaterals are rigid and if the edges are revolute joints. Stachel proved that in the deformed state the vertices of all quadrilaterals are located on a circular cylinder the radius of which is a free parameter. In other words: A Kokotsakis mesh forms two polyhedral cylinders which are deformable with degree of freedom one. Later, Stachel also investigated under which conditions a polyhedral cylinder is tiled by quadrilaterals. In the present paper new proofs and new results are obtained by using special parameters for quadrilaterals in combination with cylinder coordinates.

Key Words: Kokotsakis mesh, spherical four-bar, polyhedral cylinder, foldable and self-intersecting tiled polyhedral cylinder, periodic polyhedral cylinder, deltoid, parabola through four points

MSC 2020: 52C25 (primary), 51M20, 70B15

1 Kokotsakis Mesh

A Kokotsakis mesh is an infinite planar tessellation composed of congruent convex, non-trapezoidal, non-parallelogramic quadrilaterals (Figure 1). The exclusion of trapezoids and parallelograms is explained further below. The internal angles in the representative quadrilateral $V_0V_1V_2V_3$ add up to 2π: $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 2\pi$, so that only three angles α_1, α_2, α_3 are independent. As parameters of the quadrilateral the side lengths $V_0 - V_1 = \ell$, $V_0 - V_3 = a$, $V_1 - V_2 = b$ and the lengths $V_0 - V_2 = d_1$ and $V_1 - V_3 = d_2$ of the diagonals 1 and 2 are used. These parameters determine the side length $V_2 - V_3 = c$ and the internal angles.

By reflecting one or both of the triangles $V_0V_1V_2$ and $V_0V_1V_3$ in a side other quadrilaterals with the same lengths ℓ, a, b, d_1, d_2 are obtained. In these quadrilaterals $V_0 - V_2$ and $V_1 - V_3$ are not both diagonals.

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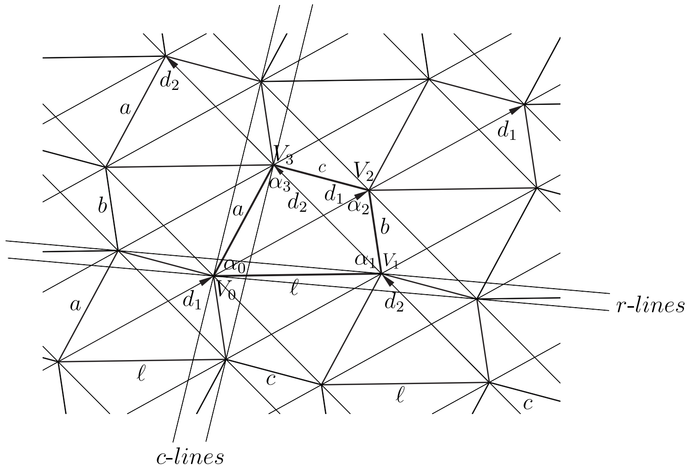

Figure 1: Kokotsakis mesh. Parameters ℓ, a, b, d_1, d_2 and internal angles α_k of the representative quadrilateral $V_0V_1V_2V_3$. Sets 1 and 2 of parallel diagonals. Rows and columns of quadrilaterals. r-lines and c-lines.

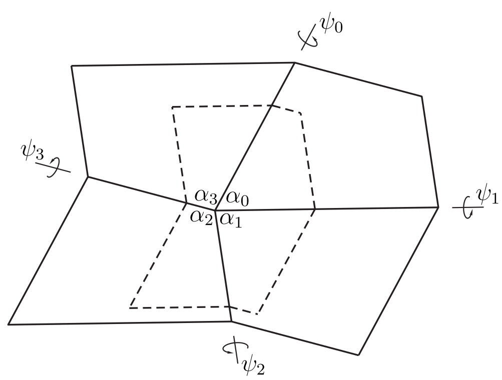

Figure 2: Spherical four-bar, parameters α_k, variables ψ_k $(k = 0, 1, 2, 3)$.

By the diagonals two sets of parallel straight lines with equidistant vertices are defined, the distance being d_1 on one set and d_2 on the other.

In the mesh *rows* and *columns* of quadrilaterals are distinguished. Adjacent rows are separated by a zigzagging line of alternating lengths ℓ, c, ℓ, c etc. Adjacent columns are separated by a zigzagging line of alternating lengths a, b, a, b etc. The quadrilateral in row i and column j is called q_{ij} with q_{00} being an arbitrarily chosen quadrilateral.

By the boundaries of rows and columns two more sets of parallel straight lines with equidistant vertices are defined. In Figure 1 they are denoted *r-lines* and *c-lines*.

The mesh is invariant with respect to

(a) 180°-rotations about the midpoints of sides of quadrilaterals

(b) translations $m\vec{d_1}$ and $n\vec{d_2}$ along the diagonals $(m, n = 0, \pm1, \pm2, \ldots)$.

By the translations $m\vec{d_1}$ and $n\vec{d_2}$ the quadrilateral q_{ij} $(i, j$ arbitrary) is displaced into the position of the quadrilateral q_{uv} with $u = i + m + n$ and $v = j + m - n$.

Every vertex of the mesh is center of a *cluster* of four quadrilaterals. Every quadrilateral is part of four clusters. All clusters are congruent. A single isolated cluster (the solid lines in Figure 2) represents a spherical four-bar mechanism. Parameters are the three independent internal angles α_1, α_2, α_3. Variables are the fold angles ψ_0, ψ_1, ψ_2, ψ_3 indicated in the figure (zero in the flat position). Provided the quadrilaterals are neither trapezoids nor parallelograms a given angle ψ_0 determines two (not necessarily real) sets of angles ψ_1, ψ_2, ψ_3 (Wittenburg [5]). Hence the four-bar is a single-degree-of-freedom mechanism with two modes of deformation. If $(\psi_0, \psi_1, \psi_2, \psi_3)$ is a state of deformation, then also $(-\psi_0, -\psi_1, -\psi_2, -\psi_3)$ is.

Two types of spherical four-bars are distinguished (see [5, pp. 641–643]):

Type I: All fold angles are subject to the constraint $|\psi_i| < \pi$. No two quadrilaterals of the spherical four-bar can intersect each other.

Type II: A single fold angle ψ_k is unconstrained. Up to $\psi_k = \pi$ deformation is possible by folding. Beyond $\psi_k = \pi$ two opposite quadrilaterals of the spherical four-bar intersect each other. Realisation requires to interpret the spherical four-bar as truss made of rods which can be dis-assembled and re-assembled. Since every vertex in Figure 1 is center of a spherical four-bar, intersections result in highly complicated three-dimensional trusses.

The dashed lines in Figure 2 point to the fact that quadrilaterals having the same internal angles result in different meshes with identical relationships between fold angles.

2 Cylinder Coordinates

Kokotsakis [2] proved

Theorem 1. *A mesh is deformable in two different modes.*

Stachel [3, 4] proved

Theorem 2. *In the deformed state the vertices of all quadrilaterals are located on a circular cylinder the radius r of which is a free parameter.*

In the geometrical proofs given in [2] and [3, 4] neither parameters of quadrilaterals nor cylinder coordinates of vertices are used. In what follows new proofs are given.

First, it is shown that the vertices of a single quadrilateral $V_0V_1V_2V_3$ can be located on a circular cylinder of arbitrary (not arbitrarily small) radius r. The vertices are points on the ellipse in which plane and cylinder intersect. Let r, φ_k, z_k $(k = 0, 1, 2, 3)$ be the cylinder coordinates of V_k. Cartesian coordinates of the position vector $\vec{r}_k$ of V_k are

$$\vec{r}_k = [r\cos\varphi_k,\ r\sin\varphi_k,\ z_k]. \tag{1}$$

Without loss of generality $\varphi_0 = z_0 = 0$, so that there are seven unknowns r, φ_k, z_k $(k = 1, 2, 3)$. These seven unknowns are subject to only six constraint equations expressing the prescribed lengths ℓ, a, b, d_1, d_2 and the planarity condition $(\vec{r}_1 - \vec{r}_0) \cdot [(\vec{r}_2 - \vec{r}_0) \times (\vec{r}_3 - \vec{r}_0)] = 0$. The square of the distance of two vertices V_i and V_j on the cylinder is $(\vec{r}_i - \vec{r}_j)^2 = 2r^2[1 - \cos(\varphi_i - \varphi_j)] + (z_i - z_j)^2$. The six constraint equations are

$$2r^2(1 - \cos\varphi_1) + z_1^2 = \ell^2, \tag{2}$$
$$2r^2(1 - \cos\varphi_2) + z_2^2 = d_1^2, \tag{3}$$
$$2r^2(1 - \cos\varphi_3) + z_3^2 = a^2, \tag{4}$$
$$2r^2[1 - \cos(\varphi_1 - \varphi_2)] + (z_1 - z_2)^2 = b^2, \tag{5}$$
$$2r^2[1 - \cos(\varphi_1 - \varphi_3)] + (z_1 - z_3)^2 = d_2^2, \tag{6}$$
$$\sum_{i=1}^{3} z_i[\sin\varphi_j - \sin\varphi_k - \sin(\varphi_j - \varphi_k)] = 0 \quad (i, j, k = 1, 2, 3 \quad \text{cyclic}). \tag{7}$$

As predicted, the radius r is a free parameter.

Two vertices on the cylinder define the helix passing through these vertices. The equidistant vertices on the two sets of parallel lines along the diagonals 1 and 2 in Figure 1 are mapped into equidistant vertices on two sets of parallel helices. Because of the invariance property (a) of the planar mesh the deformed mesh is invariant with respect to turning the cylinder upside down. Translatory displacements $m\vec{d_1}$ and $n\vec{d_2}$ of quadrilaterals along the diagonals in the planar mesh are mapped into screw displacements $(m\varphi_2, mz_2)$ and $[n(\varphi_3 - \varphi_1), n(z_3 - z_1)]$, respectively, the screw axis being the cylinder axis. The resultant screw displacement is $[m\varphi_2 + n(\varphi_3 - \varphi_1), mz_2 + n(z_3 - z_1)]$.

The resultant screw displacement has no effect on the state of deformation of the cluster of four quadrilaterals shown in Figure 2. From this it follows that in the deformed state of the mesh all clusters are congruent. Since a single cluster has two modes of deformation, the mesh as a whole has two modes of deformation. This proves the existence of two polyhedral cylinders (PCs) mobile with degree of freedom one and specified by roots $\varphi_{k_{1,2}}(r)$, $z_{k_{1,2}}(r)$ $(k = 1, 2, 3)$ of the equations. This concludes the proof.

The minimal radius $r_{\min}$ allowing real solutions is determined in Section 3. If Equations (2)–(7) are solved by (z_1, z_2, z_3), then also by $(-z_1, -z_2, -z_3)$, and if they are solved by $(\varphi_1, \varphi_2, \varphi_3)$, then also by $(-\varphi_1, -\varphi_2, -\varphi_3)$. Relevant solutions are those for which $V_0 - V_2$ and $V_1 - V_3$ are diagonals of the quadrilateral.

Equations (2)–(7) cannot be decoupled. This is true also for equations resulting from the transformation $u_k = \tan\varphi_k/2$, $1 - \cos\varphi_k = 2u_k^2/(1 + u_k^2)$, $\sin\varphi_k = 2u_k/(1 + u_k^2)$.

Special case: A quadrilateral inscribed in a circle of radius ϱ is specified by the polar coordinates β_1, β_2, β_3 of V_1, V_2, V_3, respectively. In these terms $\ell^2 = 2\varrho^2(1 - \cos\beta_1)$, $d_1^2 = 2\varrho^2(1 - \cos\beta_2)$, $a^2 = 2\varrho^2(1 - \cos\beta_3)$, $b^2 = 2\varrho^2[1 - \cos(\beta_1 - \beta_2)]$, $d_2^2 = 2\varrho^2[1 - \cos(\beta_1 - \beta_3)]$. With these parameters Equations (2)–(7) have with $r = \varrho$ in both modes of deformation the solution $z_k = 0$, $\varphi_k = \beta_k$ $(k = 1, 2, 3)$. This state of deformation cannot be produced by folding the mesh.

The outward normal vector of the quadrilateral $V_0V_1V_2V_3$ is $(\vec{r}_1 - \vec{r}_0) \times (\vec{r}_2 - \vec{r}_0)$. The scalar product of the unit normal vectors of two quadrilaterals sharing an edge is the cosine of the fold angle in this edge (one of the angles ψ_k $(k = 0, 1, 2, 3)$). For all quadrilaterals the cosine of the angle between the cylinder axis and the normal to the quadrilateral is equal in magnitude and alternating in sign along rows and along columns.

In every state of deformation of a mesh the projections of all quadrilaterals along the cylinder axis are congruent quadrilaterals inscribed in a circle. The projected quadrilaterals are divided into two groups of equidistant quadrilaterals, the angular distance being φ_2 in one group and $\varphi_3 - \varphi_1$ in the other. For being convex it is necessary and sufficient that

$$\text{either} \qquad 0 < \varphi_1 < \varphi_2 < \varphi_3 \qquad \text{or} \qquad 0 < \varphi_3 < \varphi_2 < \varphi_1. \tag{8}$$

By the inequality signs the case is excluded that the projected quadrilaterals are secants of the circle. This case is the subject of Section 6.1 on trapezoidal quadrilaterals.

Let angles φ_1, φ_2, φ_3 satisfying the convexity condition (8) and parameters z_1, z_2 (or z_1, z_3) and $\ell > |z_1|$ be arbitrarily prescribed. With these data Equations (2)–(7) determine the parameters z_3 (or z_2), r, d_1, a, b and d_2 specifying a quadrilateral $V_0V_1V_2V_3$ and a state of deformation of the mesh. With the parameters ℓ, d_1, a, b, d_2 of the quadrilateral the variables $\varphi_{k_{1,2}}$, $z_{k_{1,2}}$ $(k = 1, 2, 3)$ can then be calculated numerically as functions of r.

Example 1. The prescribed data $\varphi_1 = 180°$, $\varphi_2 = 90°$, $\varphi_3 = 60°$, $z_1 = 1/2$, $z_3 = 1$, $\ell = 1$ determine $z_2 = (3 + 7\sqrt{3})/12$, $r = \sqrt{3}/4$, $d_1^2 = (7/24)(5 + \sqrt{3})$, $b^2 = (7/24)(5 - \sqrt{3})$,

$a = \sqrt{19}/4$, $d_2 = \sqrt{13}/4$. The unit outward normal vector has the cartesian coordinates $\sqrt{1/61}[-\sqrt{3}, 7, -3]$. From five unit normal vectors the fold angles are calculated: $\cos\psi_0 = 53/61$, $\cos\psi_1 = 37/61$, $\cos\psi_2 = (14\sqrt{3}-9)/61$, $\cos\psi_3 = (16\sqrt{3}-9)/61$. ◇

3 Geometrical Solution

In every state of deformation of a mesh all quadrilaterals are circumscribed by congruent ellipses. Figure 3 shows a quadrilateral $V_0V_1V_2V_3$ and a circumscribing ellipse with semi axes a and $b < a$ (not to be confused with the parameters a, b in Figure 1). The radius r of the cylinder is b, and its axis is tilted against the plane of the ellipse by the angle $\alpha = \arctan(b/a)$ shown in the figure. The cylinder coordinates φ_k, z_k $(k = 0,1,2,3)$ are found by projections as is demonstrated by φ_0, z_0. The vertex V_k $(k = 0,1,2,3)$ on the ellipse and its axial projection on the circle are located on one and the same side of the minor principal axis of the ellipse. By the angles φ_0 ,φ_1, φ_2, φ_3 the projection of the quadrilateral onto the circle is determined. Every ellipse circumscribing the quadrilateral determines cylinder coordinates r, φ_k, z_k $(k = 0,1,2,3)$ defining a state of deformation of the mesh.

In what follows, all ellipses circumscribing a given quadrilateral are determined in analytical form. For better understanding it is helpful to make a sketch of two parabolas 1 and 2 intersecting each other in four points and of the quadrilateral $V_0V_1V_2V_3$ defined by these points. Everything which follows (x, y-axes, domains, ellipses in these domains, lines $y = \text{const}$) should be marked in this sketch.

By the x,y-coordinates of V_0, V_1, V_2, V_3 in some arbitrarily chosen x,y-system and by the coordinates x_4 y_4 of an additional auxiliary point P a second-order curve circumscribing the quadrilateral is defined. Its equation is

$$\det\begin{bmatrix} x^2 & y^2 & xy & x & y & 1\\ x_0^2 & y_0^2 & x_0y_0 & x_0 & y_0 & 1\\ x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1\\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 & 1\\ x_3^2 & y_3^2 & x_3y_3 & x_3 & y_3 & 1\\ x_4^2 & y_4^2 & x_4y_4 & x_4 & y_4 & 1 \end{bmatrix} = 0. \tag{9}$$

In the x,y-system with V_0 as origin and with V_2 on the x-axis this equation reads

$$x_2\cdot\det\begin{bmatrix} y^2 & xy & x & y\\ y_1^2 & x_1y_1 & x_1 & y_1\\ y_3^2 & x_3y_3 & x_3 & y_3\\ y_4^2 & x_4y_4 & x_4 & y_4 \end{bmatrix} - \det\begin{bmatrix} x^2 & y^2 & xy & y\\ x_1^2 & y_1^2 & x_1y_1 & y_1\\ x_3^2 & y_3^2 & x_3y_3 & y_3\\ x_4^2 & y_4^2 & x_4y_4 & y_4 \end{bmatrix} = 0.$$

This is the equation $A(x^2 - x_2x) + 2Bxy + Cy^2 + Ey = 0$ with coefficients A, B, C, E which are second-order functions of x_4, y_4. P determines an ellipse if $AC - B^2 > 0$.

The fourth-order equation $AC-B^2 = 0$ separating ellipses from hyperbolas is the equation of the two parabolas passing through the vertices of the quadrilateral. Let (ϱ_1, γ) and (ϱ_2, γ) be the polar coordinates of these parabolas. Substitution of $x_4 = \varrho\cos\gamma$, $y_4 = \varrho\sin\gamma$ results in a quadratic equation with real roots $\varrho_{1,2}(\gamma)$. For geometrical constructions of the parabolas see [1, Example 7.4.1].

Let Δ_1 be the domain inside parabola 1, Δ_2 be the domain inside parabola 2 and Δ_{12} be the intersection of Δ_1 and Δ_2. The curve is an ellipse if P is located either in Δ_1 or in

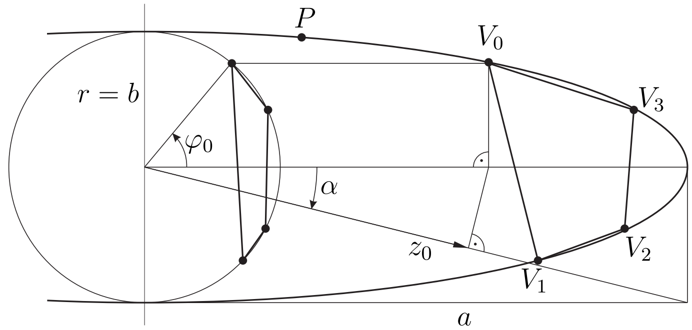

Figure 3: Quadrilateral $V_0V_1V_2V_3$ and auxiliary point P defining a circumscribing ellipse. Cylinder coordinates r and φ_0, z_0 of V_0. Projected quadrilateral.

Δ_2, but not in Δ_{12}. The sketch shows that every ellipse is located in Δ_1 as well as in Δ_2. Slender ellipses have the most part of their circumference either in Δ_1 or in Δ_2. Slender ellipses occur when the planar mesh is only slightly deformed. Hence the conclusion: Each mode of deformation is associated with one parabola and with a family of ellipses. When the deformed mesh becomes planar, the ellipses tend, in the limit, toward the parabolas. With increasing deformation of the mesh the tilt angle α and with it the ratio $b/a = \tan\alpha$ is increasing, while the radius b of the cylinder is decreasing. b is calculated from A, B, C, E as function of x_4 and y_4. It suffices to calculate b for all points (x_4, y_4) of a straight line $y =$ const which intersects all ellipses (in points other than V_1, V_2, V_3). Then, b is a function of the single variable x_4. The minimum $b_{\min}$ of this function is the minimal radius $r_{\min}$ for which Equations (2)–(7) have a real solution φ_k, z_k $(k = 1, 2, 3)$.

The principal-axes system of an ellipse is inclined against the x, y-system by the angle $\beta = (1/2)\arctan[2B/(A - C)]$. In general, β depends on x_4, y_4. In the special case of quadrilaterals inscribed in a circle, 2β is, independent of x_4, y_4, the angle enclosed by the diagonals of the quadrilateral: $\tan 2\beta = (y_3 - y_1)/(x_3 - x_1)$. The equation of the circumcircle is $2B = A - C = 0$. As was said already, the flat position of the mesh inside this circle cannot be produced by folding the mesh.

4 Tiled Polyhedral Cylinders

An infinite mesh is wrapped around a cylinder of arbitrary radius r infinitly many times. A polyhedral cylinder is said to be tiled if integers m and n exist such that the resultant screw displacement $[m\varphi_2 + n(\varphi_3 - \varphi_1), mz_2 + n(z_3 - z_1)]$ equals $(2\pi, 0)$. This is the set of *closure conditions* (compare Stachel [4])

$$m\varphi_2 + n(\varphi_3 - \varphi_1) = 2\pi, \qquad mz_2 + n(z_3 - z_1) = 0 \quad (m,\, n \text{ integer}). \tag{10}$$

On a tiled PC with integers (m, n) the quadrilateral q_{ij} $(i, j$ arbitrary) coalesces with all quadrilaterals q_{uv} with $u = i + k(m + n), v = j + k(m - n)$ $(k = \pm1, \pm2, \ldots)$. m and n may be positive, zero or negative. The equations show:

1. $(z_3 - z_1)/z_2$ is rational.
2. Reversing the signs of m, n, φ_k, z_k $(k = 1, 2, 3)$ has no effect on the tiled PC.
3. Replacing n by $-n$, V_1 by V_3 and V_1 by V_3 has no effect on the tiled PC.
4. Tiled PCs are flatfolded if $|m| + |n| = 2$. They are three-dimensional if $|m| + |n| \geq 3$.
5. On a tiled PC with integers $(m \geq 3,\, n = 0)$ every string of m diagonals d_1 connecting two vertices is mapped into a regular polygon with m sides of length d_1 in a plane

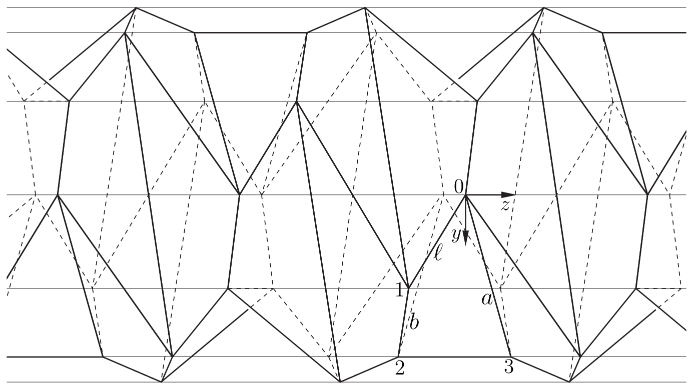

Figure 4: The foldable tiled PC of Example 2.

$z = \text{const}$. Likewise, on a tiled PC with integers ($m = 0$, $n \geq 3$) every string of n diagonals d_2 connecting two vertices is mapped into a regular polygon with n sides of length d_2 in a plane $z = \text{const}$.

6. A tiled PC is formed by $|m + n|$ rows of quadrilaterals if $m \neq 0$ and $mn \geq 0$. It is formed by $|m - n|$ columns of quadrilaterals if $n \neq 0$ and $mn \leq 0$.

Equations (2)–(7) and (10) are eight equations for the nine unknowns m, n, r, φ_k, z_k ($k = 1, 2, 3$). Necessary conditions on the five parameters ℓ, a, b, d_1, d_2 for the existence of a solution are not available.

Tiled PCs can be constructed as follows. Integers m, n and angles φ_1, φ_2, φ_3 satisfying the convexity condition (8) and the first Equation (10) are arbitrarily prescribed. In addition, one of the coordinates z_1, z_2, z_3 and either the radius r or one of the lengths ℓ, d_1, a, b, d_2 are prescribed. Equation (7) and the second Equation (10) determine the unknowns among z_1, z_2, z_3. Finally, Equations (2)–(6) determine the squares of the remaining unknowns. If one of the lengths ℓ, d_1, a, b, d_2 is prescribed, then it must be sufficiently large so as to determine a value $r^2 > 0$. Whether a tiled PC constructed this way can be produced by folding the mesh or whether it is self-intersecting must be investigated separately.

A tiled PC is said to be M-periodic if a (smallest) integer M exists such that the screw displacements $M(\varphi_2, z_2)$ and $M(\varphi_3 - \varphi_1, z_3 - z_1)$ are both pure translations. Hence $M\varphi_2$ and $M(\varphi_3 - \varphi_1)$ must both be integer multiples of 2π. Rationality of φ_2/π is a necessary and sufficient condition for periodicity. Periodicity means that every quadrilateral is periodically repeated in translation along the cylinder axis.

Example 2. Given are $m = 3$, $n = 2$, $\varphi_1 = 30°$, $\varphi_2 = 60°$, $\varphi_3 = 120°$, $z_1 = -1/2$, $\ell = 1$.

The tiled PC determined by these data is M-periodic with $M = 12$. The data yield $z_2 = -2(12 + 5\sqrt{3})/69 \approx -.5988$, $z_3 = (1 + 10\sqrt{3})/46 \approx .3983$, $r = (3\sqrt{2} + \sqrt{6})/4 \approx 1.6730$, $d_1^2 = (10690 + 5401\sqrt{3})/(12 \cdot 23^2)$, $d_1 \approx 1.7770$, $a^2 = (9823 + 4781\sqrt{3})/(4 \cdot 23^2)$, $a \approx 2.9250$, $b^2 = (1327 - 70\sqrt{3})/(3 \cdot 23^2)$, $b \approx .8716$, $d_2^2 = 3(1204 + 609\sqrt{3})/(2 \cdot 23^2)$, $d_2 \approx 2.5308$.

In Figure 4 the tiled PC is shown in projection onto the y, z-plane. It is formed by folding $m + n = 5$ rows of quadrilaterals. The periodic repetition of quadrilaterals in translation along the cylinder axis is clearly visible. ◇

Example 3. The data $m = 8$, $n = -4$, $\varphi_1 = 60°$, $\varphi_2 = 120°$, $\varphi_3 = 210°$, $z_3 = 1$, $r = 1$ determine the tiled PC with the parameters $M = 12$, $z_1 = (25 - 14\sqrt{3})/37 \approx .0203$, $z_2 = (6 + 7\sqrt{3})/37 \approx .4898$, $\ell^2 = 2(1291 - 350\sqrt{3})/37^2$, $\ell \approx 1.0002$, $d_1^2 = 6(715 + 14\sqrt{3})/37^2$, $d_1 \approx 1.8000$, $a^2 = 3 + \sqrt{3}$, $a \approx 2.1753$, $b^2 = (3053 - 798\sqrt{3})/37^2$, $b \approx 1.1047$, $d_2^2 = 5(694 + 341\sqrt{3})/37^2$, $d_2 \approx 2.1661$.

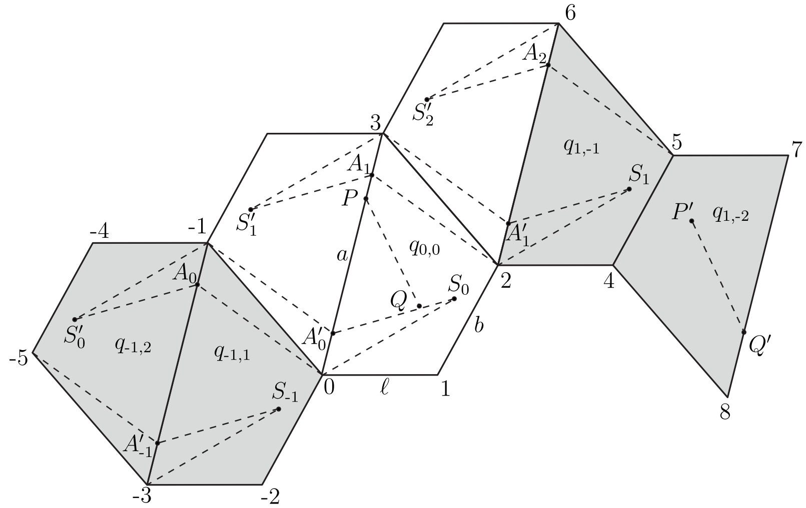

Figure 5: The planar mesh of Example 3. On the tiled PC the quadrilateral $V_0V_1V_2V_3$ is intersected by the four shaded quadrilaterals in the dashed line segments.

This tiled PC which is formed by $m-n=12$ columns of quadrilaterals cannot be produced by folding the mesh. In Figure 5 the mesh is shown. On the tiled PC every quadrilateral is intersected in like manner by four other quadrilaterals The quadrilateral $V_0V_1V_2V_3$ labeled $q_{0,0}$ is intersected by the four shaded quadrilaterals $q_{-1,1}$, $q_{1,-1}$, $q_{-1,2}$ and $q_{1,-2}$. A point S_0 inside $q_{0,0}$ is intersected by the edge $V_{-1}-V_{-3}$ at a point A_0 on this edge. The equation $\vec{r}=\vec{r}_{-1}+\mu(\vec{r}_{-3}-\vec{r}_{-1})$ of the edge with parameter μ and the equation $\vec{m}\cdot\vec{r}=1$ of the plane with $\vec{m}$ given by $\vec{r}_0$, $\vec{r}_1$, $\vec{r}_2$ determine μ and the positions of A_0 and S_0: $\mu\approx .1699$, distances $V_0-A_0=V_0-S_0\approx 1.3264$, $V_3-S_0\approx 1.5636$. The quadrilaterals $q_{0,0}$ and $q_{-1,1}$ intersect in the dashed line segments V_0-A_0 and V_0-S_0.

All pairs of quadrilaterals sharing a single vertex intersect in like manner. This explains the dashed line segments $A_1-V_2-S_1$, $A'_1-V_3-S'_1$, $A'_0-V_{-1}-S'_0$ etc. S'_0 coalesces with A'_0 and A_0 with S_0. Hence $q_{0,0}$ is intersected by $q_{-1,2}$ in the dashed line segment $S_0-A'_0$.

In the same way it is shown that $q_{0,0}$ and $q_{1,-2}$ intersect in the dashed line segments $P-Q$ and $P'-Q'$. Distances: $V_0-P=V_7-Q'\approx 2.1753$, $V_0-Q=V_7-P'\approx 1.0155$, $V_3-Q=V_8-P'\approx 1.5675$. ◇

5 Deltoids

In this section a two-parametric family of deltoids forming tiled PCs with prescribed integers (m,n) is determined. A deltoid is a quadrilateral with mutually orthogonal diagonals one of which, say V_1-V_3, is an axis of symmetry. Then, $\ell=b$ and $(\vec{r}_2-\vec{r}_0)\cdot(\vec{r}_3-\vec{r}_1)=0$. With (1), (2) and (5) these equations are

$$2r^2[\cos\varphi_1-\cos(\varphi_1-\varphi_2)]+z_2(z_2-2z_1)=0, \tag{11}$$

$$r^2[\cos\varphi_1-\cos(\varphi_1-\varphi_2)-\cos\varphi_3+\cos(\varphi_2-\varphi_3)]+z_2(z_3-z_1)=0. \tag{12}$$

The second equation is replaced by a linear combination of both equations:

$$2r^2[\cos\varphi_3-\cos(\varphi_2-\varphi_3)]+z_2(z_2-2z_3)=0. \tag{13}$$

From the fact that a diagonal cannot be on a generator of the cylinder and from the orthogonality of the diagonals it follows that $z_2 \neq 0$. Therefore, elimination of r^2 from (11) and (13) results in

$$z_2[\cos\varphi_3 - \cos(\varphi_2 - \varphi_3) - \cos\varphi_1 + \cos(\varphi_1 - \varphi_2)] + \\ 2\{z_3[\cos\varphi_1 - \cos(\varphi_1 - \varphi_2)] - z_1[\cos\varphi_3 - \cos(\varphi_2 - \varphi_3)]\} = 0. \quad (14)$$

This equation, Equation (7) and the second closure condition (10) are a set of three homogeneous linear equations for z_1, z_2, z_3. Setting the coefficient determinant equal to zero results in

$$\left.\begin{aligned} &A[\cos\varphi_1 - \cos(\varphi_1 - \varphi_2)] + B[\cos\varphi_3 - \cos(\varphi_2 - \varphi_3)] = 0, \\ &A = 2m[\sin\varphi_2 - \sin\varphi_3 - \sin(\varphi_2 - \varphi_3)] - nC, \\ &B = 2m[\sin\varphi_1 - \sin\varphi_2 - \sin(\varphi_1 - \varphi_2)] + nC, \\ &C = \sin\varphi_1 - \sin\varphi_3 + \sin(\varphi_1 - \varphi_2) + \sin(\varphi_2 - \varphi_3) + 2\sin(\varphi_3 - \varphi_1). \end{aligned}\right\} \quad (15)$$

This equation and the first closure condition (10) determine φ_1 and φ_3 as functions of φ_2. The angle φ_2 can be chosen freely subject to the convexity condition (8). With angles φ_1, φ_2, φ_3 thus determined Equation (7) and the second closure condition (10) determine z_1 and z_3 as functions of z_2. The coordinate $z_2 \neq 0$ can be chosen freely subject to the condition that (11) yields a value $r^2 > 0$. Subsequently, Equations (3)–(6) determine d_1^2, a^2, $b^2 = \ell^2$ and d_2^2.

The symmetry of deltoids has the effect that in both modes of deformation the same tiled PC is formed.

Example 4. A model made of cardboard conveyed the impression that the deltoid with parameters $\ell = b = d_1 = d_2 = 1$, $a^2 = 2 - \sqrt{3}$ forms a tiled PC with $m = 2$, $n = 3$. This impression is shown to be wrong by solving Equations (2)–(7) and the first closure condition (10). The solutions $\varphi_1 \approx -48.19306°$, $\varphi_2 \approx 54.17303°$, $\varphi_3 \approx 35.69159°$, $z_1 \approx -.85198$, $z_2 \approx -.81181$, $z_3 \approx -.33690$, $r \approx .64121$ do not satisfy the second Equation (10).

In order to find an almost identical deltoid forming a tiled PC with $m = 2$, $n = 3$ the almost identical values $\varphi_2 = 54.18°$, $z_2 = -.81$ are chosen. Equations(10) determine $\varphi_3 - \varphi_1 = 83.88°$, $M = 2000$ and $z_3 - z_1 = .54$. The remaining equations determine $\varphi_1 \approx -48.18143°$, $\varphi_3 \approx 35.69857°$, $z_1 \approx -.872624$, $z_3 \approx -.332624$, $r \approx .655753$ and the desired parameters $\ell = b \approx 1.023747$, $d_1 \approx 1.006381$, $d_2 \approx 1.029532$, $a \approx .521764$. This tiled PC is formed by folding $m + n = 5$ rows of deltoids. In Figure 6a it is shown in projection onto the y, z-plane. ◇

Example 5. The data $m = 2$, $n = 1$, $\varphi_2 = 100°$, $z_2 = 1$ determine a deltoid and the tiled PC formed by this deltoid. The closure conditions (10) determine $\varphi_3 - \varphi_1 = 160°$, $M = 18$ and $z_3 - z_1 = -2$. The remaining equations determine the parameters $\varphi_1 \approx 44.3269°$, $\varphi_3 \approx 204.3269°$, $z_1 \approx .87156$, $z_3 \approx -1.12844$, $r \approx 1.56632$, $\ell = b \approx 1.46842$, $d_1 \approx 2.59976$, $a \approx 3.26361$, $d_2 \approx 3.67662$. The tiled PC is formed by folding $m + n = 3$ rows of deltoids. In Figure 6b it is shown in projection onto the y, z-plane. ◇

Remark. There are infinitly many more nonperiodic than periodic tiled PCs. However, there is no angle φ_2 which is an irrational multiple of π, and for which Equation (15) and the first Equation (10) can be solved in nonnumerical form. Computers can handle only rational numbers. Hence the conclusion: Nonperiodic tiled PCs ($M \to \infty$) exist, but examples cannot be given.

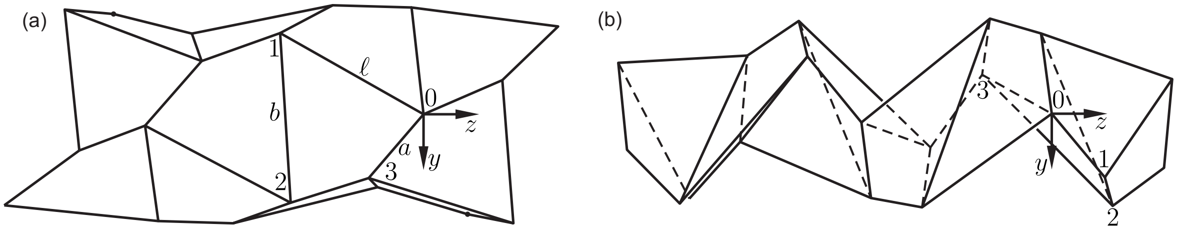

Figure 6: The foldable tiled PCs of Example 4 (Figure a) and Example 5 (Figure b).

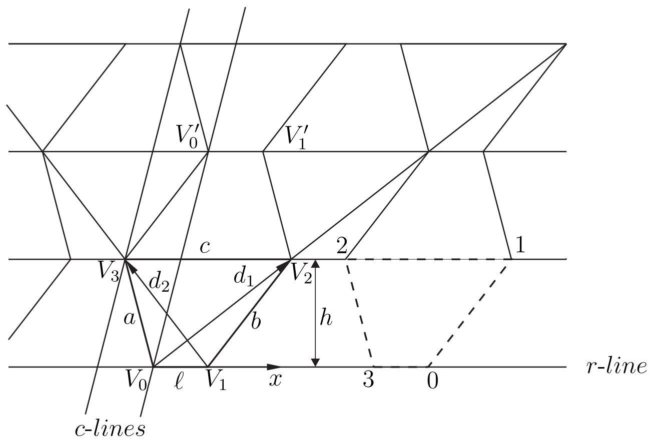

Figure 7: Mesh composed of trapezoids. Parameters x_1, x_2, x_3, h.

6 Trapezoids

Figure 7 depicts a mesh composed of trapezoids with parallel sides $V_0 - V_1$ and $V_2 - V_3$. As parameters of a single trapezoid the coordinates x_1, x_2, x_3 of V_1, V_2, V_3 and the height h are used (without loss of generality $x_0 = 0$, $x_1 > 0$). In these terms the previously used parameters are

$$\ell = x_1,\ d_1^2 = h^2 + x_2^2,\ b^2 = h^2 + (x_2 - x_1)^2,\ a^2 = h^2 + x_3^2,\ d_2^2 = h^2 + (x_3 - x_1)^2. \tag{16}$$

6.1 Trivial Tiled Polyhedral Cylinders

Rows of trapezoids are separated by equidistant parallel straight lines which are the r-lines defined in Figure 1. One of the two modes of deformation is trivial. The mesh can be deformed with arbitrary fold angles along r-lines leaving rows planar. If identical fold angles are chosen, then all vertices of the mesh are located on a circular cylinder.

The PC is tiled if integers m and n exist such that the component of the vector $m\vec{d_1} + n\vec{d_2}$ along the r-lines equals zero, i.e., if $mx_2 + n(x_3 - x_1) = 0$ (this is the second closure condition (10); the first closure condition does not apply). The condition is satisfied if the ratio $q = (x_3 - x_1)/x_2$ is rational. Let (m, n) be a pair of integers with no common divisor satisfying the condition. Then the condition is satisfied by all (km, kn) with $k = 1, 2, \ldots$ The cross section of a tiled PC with integers (km, kn) is a polygon with $k(m+n)$ sides of length h inscribed in a circle. If this polygon is regular (irregular), then the tiled PC is foldable (self-intersecting).

Examples:

1. $k(m + n) = 9$: The cross section is either the regular polygon 1, 2, 3, 4, 5, 6, 7, 8, 9, 1 or the star 1, 3, 5, 7, 9, 2, 4, 6, 8, 1 or the star 1, 5, 9, 4, 8, 3, 7, 2, 6, 1.

2. $k(m+n) = 14$: The cross section is either the regular polygon 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 1 or the star 1, 4, 7, 10, 13, 2, 5, 8, 11, 14, 3, 6, 9, 12, 1 or the star 1, 6, 11, 2, 7, 12, 3, 8, 13, 4, 9, 14, 5, 10, 1.

6.2 Nontrivial Deformations

The second mode of deformation is nontrivial. Trapezoids inscribed in a circle are symmetric. For this reason the axial projections of the trapezoids $V_0V_1V_2V_3$ and $V_2V_3V_0'V_1'$ shown in Figure 7 coalesce. From this it follows that vertices which in the planar mesh are located on a c-line are located on a generator of the circular cylinder. This is expressed by the equation

$$\varphi_3 = \varphi_1 - \varphi_2. \tag{17}$$

The trapezoid drawn with dashed lines shows that in the trivial mode of deformation this equation is valid in the special form $\varphi_3 = \varphi_1 - \varphi_2 = 0$. With (16) and (17) Equations (2)–(7) are

$$2r^2(1-\cos\varphi_1) + z_1^2 = x_1^2, \tag{18}$$
$$2r^2(1-\cos\varphi_2) + z_2^2 = h^2 + x_2^2, \tag{19}$$
$$2r^2[1-\cos(\varphi_1-\varphi_2)] + z_3^2 = h^2 + x_3^2, \tag{20}$$
$$2r^2[1-\cos(\varphi_1-\varphi_2)] + (z_1-z_2)^2 = h^2 + (x_2-x_1)^2, \tag{21}$$
$$2r^2(1-\cos\varphi_2) + (z_3-z_1)^2 = h^2 + (x_3-x_1)^2, \tag{22}$$
$$z_1[\sin\varphi_2 - \sin(\varphi_1-\varphi_2) - \sin(2\varphi_2-\varphi_1)] + (z_2-z_3)[\sin\varphi_2 + \sin(\varphi_1-\varphi_2) - \sin\varphi_1] = 0. \tag{23}$$

These equations are decoupled as follows. The difference of (19) and (22) and the difference of (20) and (21) read

$$(z_2+z_3-z_1)(z_2-z_3+z_1) = (x_2+x_3-x_1)(x_2-x_3+x_1), \tag{24}$$
$$(z_2+z_3-z_1)(z_3-z_2+z_1) = (x_2+x_3-x_1)(x_3-x_2+x_1). \tag{25}$$

The sum and the difference of these two equations are

$$z_1(z_2+z_3-z_1) = x_1(x_2+x_3-x_1), \tag{26}$$
$$(z_2-z_3)(z_2+z_3-z_1) = (x_2-x_3)(x_2+x_3-x_1), \tag{27}$$

whence it follows that with the given ratio of lengths of the parallel sides, $\lambda = (x_2-x_3)/x_1 > 0$,

$$(z_2-z_3)/z_1 \equiv \lambda \text{ independent of } r. \tag{28}$$

λ is independent of h and independent of whether the trapezoid is symmetric or not.

Equation (23) is a relationship between φ_1 and φ_2:

$$\sin\varphi_2 - \sin(\varphi_1-\varphi_2) - \sin(2\varphi_2-\varphi_1) + \lambda[\sin\varphi_2 + \sin(\varphi_1-\varphi_2) - \sin\varphi_1] = 0. \tag{29}$$

It has the solution $\varphi_1 = \varphi_2$ associated with the trivial mode of deformation and a solution $\varphi_1 \neq \varphi_2$ associated with the nontrivial mode of deformation. Explicit expressions are obtained by writing the equation in the form $A\sin\varphi_1 + B\cos\varphi_1 = C$ with

$$A = [2\cos\varphi_2 + (\lambda-1)]\cos\varphi_2 - (\lambda+1), \quad B = -[2\cos\varphi_2 + (\lambda-1)]\sin\varphi_2,$$
$$C = -(\lambda+1)\sin\varphi_2, \quad A^2+B^2 = 2(1-\cos\varphi_2)(\lambda^2+1+2\lambda\cos\varphi_2),$$
$$R = \sqrt{A^2+B^2-C^2} = (\lambda-1)(1-\cos\varphi_2).$$

The solutions are $\sin\varphi_1 = (AC \mp BR)/(A^2+B^2)$, $\cos\varphi_1 = (BC \pm AR)/(A^2+B^2)$. This yields $\varphi_1 = \varphi_2$ and

$$\sin\varphi_1 = \frac{2(\lambda+\cos\varphi_2)\sin\varphi_2}{\lambda^2+1+2\lambda\cos\varphi_2}, \qquad 1-\cos\varphi_1 = \frac{2(1-\cos^2\varphi_2)}{\lambda^2+1+2\lambda\cos\varphi_2}, \tag{30}$$

$$\sin(\varphi_1-\varphi_2) = \frac{(1-\lambda^2)\sin\varphi_2}{\lambda^2+1+2\lambda\cos\varphi_2}, \qquad 1-\cos(\varphi_1-\varphi_2) = \frac{(\lambda-1)^2(1-\cos\varphi_2)}{\lambda^2+1+2\lambda\cos\varphi_2}. \tag{31}$$

With these expressions and with $x_3 = x_2 - \lambda x_1$ and $z_3 = z_2 - \lambda z_1$ Equations (18)–(20) are

$$2r^2\frac{2(1-\cos^2\varphi_2)}{\lambda^2+1+2\lambda\cos\varphi_2} + z_1^2 = x_1^2, \tag{32}$$

$$2r^2(1-\cos\varphi_2) + z_2^2 = h^2 + x_2^2, \tag{33}$$

$$2r^2\frac{(\lambda-1)^2(1-\cos\varphi_2)}{\lambda^2+1+2\lambda\cos\varphi_2} + (z_2-\lambda z_1)^2 = h^2 + (x_2-\lambda x_1)^2. \tag{34}$$

Equation (32) multiplied by λ^2 plus Equation (33) minus Equation (34) is, following division by 2λ,

$$2r^2\frac{(\lambda+1)(1-\cos^2\varphi_2)}{\lambda^2+1+2\lambda\cos\varphi_2} - x_1x_2 = -z_1z_2\,. \tag{35}$$

Squaring and eliminating $z_1^2z_2^2$ by means of (32) and (33) results in

$$\left[2r^2\frac{(\lambda+1)(1-\cos^2\varphi_2)}{\lambda^2+1+2\lambda\cos\varphi_2} - x_1x_2\right]^2 = \left[x_1^2 - 2r^2\frac{2(1-\cos^2\varphi_2)}{\lambda^2+1+2\lambda\cos\varphi_2}\right]\left[h^2+x_2^2-2r^2(1-\cos\varphi_2)\right]. \tag{36}$$

This is a fourth-order equation for $\cos\varphi_2$ with parameter r. For $y = 2r^2$ it is the quadratic equation $Py^2 - Qy = -h^2x_1^2$ with

$$P = \left[\frac{(\lambda-1)(1-\cos\varphi_2)\sin\varphi_2}{\lambda^2+1+2\lambda\cos\varphi_2}\right]^2,$$

$$Q = \frac{1-\cos\varphi_2}{\lambda^2+1+2\lambda\cos\varphi_2}\left\{2(1+\cos\varphi_2)\left[h^2+\left(x_2-x_1\frac{\lambda+1}{2}\right)^2\right] + \frac{1}{2}x_1^2(\lambda-1)^2(1-\cos\varphi_2)\right\} > 0.$$

Both roots $y = (Q \pm \sqrt{Q^2 - 4h^2x_1^2P}\,)/(2P)$ are real and positive since

$$Q^2 - 4h^2x_1^2P = \frac{4\sin^4\varphi_2}{(\lambda^2+1+2\lambda\cos\varphi_2)^2}F_1F_2,$$

$$F_{1,2} = \left[h \pm x_1\frac{(\lambda-1)\sin\varphi_2}{2(1+\cos\varphi_2)}\right]^2 + \left[x_2 - x_1\frac{\lambda+1}{2}\right]^2 > 0.$$

Only with the smaller of the roots $2r^2$ Equations (32) and (33) yield quantities $z_1^2 > 0$ and $z_2^2 > 0$.

For angles φ_1, $\varphi_2 \ll 1$ Taylor expansion of (30) and (36) yields the approximations

$$\varphi_2 \approx \frac{\lambda+1}{2}\varphi_1, \qquad r\varphi_1 \approx x_1\frac{h}{\sqrt{h^2+[x_2-x_1(\lambda+1)/2]^2}} = \text{const.} \tag{37}$$

The square root is the distance between the midpoints of the parallel sides of the trapezoid. $r\varphi_1 \approx x_1$ if the trapezoid is symmetric.

6.3 Geometrical Solution

Nontrivial states of deformation are constructed as shown in Figure 3. Equation (9) reads

$$\det\begin{bmatrix} x^2 & y^2 & xy & y \\ x_2^2 & h^2 & x_2h & h \\ x_3^2 & h^2 & x_3h & h \\ x_4^2 & y_4^2 & x_4y_4 & y_4 \end{bmatrix} - x_1\cdot\det\begin{bmatrix} y^2 & xy & x & y \\ h^2 & x_2h & x_2 & h \\ h^2 & x_3h & x_3 & h \\ y_4^2 & x_4y_4 & x_4 & y_4 \end{bmatrix} = 0.$$

This is the equation

$$\left.\begin{aligned} &A(x^2-x_1x)+2Bxy+Cy^2+Ey=0,\\ &A=h^2(x_2-x_3)(y_4^2-hy_4),\quad 2B=-h(x_2-x_3)(x_2+x_3-x_1)(y_4^2-hy_4),\\ &C=-h(x_2-x_3)[h(x_4^2-x_1x_4)-(x_2+x_3-x_1)x_4y_4+x_2x_3y_4],\\ &E=h(x_2-x_3)[h^2(x_4^2-x_1x_4)-h(x_2+x_3-x_1)x_4y_4+x_2x_3y_4^2]. \end{aligned}\right\}\tag{38}$$

The fourth-order equation $AC-B^2=0$ separating ellipses and hyperbolas is

$$(y_4^2-hy_4)(A^*x_4^2+2B^*x_4y_4+C^*y_4^2+D^*x_4+E^*y_4)=0\tag{39}$$

with coefficients $A^*=h^2$, $2B^*=-h(x_2+x_3-x_1)$, $C^*=(x_2+x_3-x_1)^2/4$ satisfying the equation $A^*C^*-B^{*2}=0$. Equation (39) is the equation of the parallel lines $y_4=0$ and $y_4=h$ and of the single parabola passing through the vertices of the trapezoid (see [1, Example 7.4.1]).

Let Δ_1 be the domain inside the parabola, Δ_2 be the domain between the parallel lines and Δ_{12} be the intersection of Δ_1 and Δ_2. Equation (38) determines an ellipse if the point (x_4,y_4) is located either in Δ_1 or in Δ_2, but not in Δ_{12}. Every ellipse is located in Δ_1 as well as in Δ_2. The semi axes b of these ellipses are in the range $h/2<b<\infty$. Since there are no ellipses in the trivial mode of deformation, every ellipse determines a nontrivial deformation of the mesh. The lines $y_4=0$ and $y_4=h$ determine a trivial deformation.

6.4 Nontrivial Tiled Polyhedral Cylinders

Because of (17) the closure conditions (10) are

$$\varphi_1-\varphi_3=\varphi_2=2\pi/(m-n),\qquad mz_2+n(z_3-z_1)=0.\tag{40}$$

The first condition tells that (i) integers (m,n) with $m-n=0$ or ±1 can occur only in tiled PCs in the trivial mode of deformation and that (ii) nontrivial tiled PCs are M-periodic with $M=|m-n|$.

Nontrivial tiled PCs can be constructed as follows. Given are

1. Arbitrary integers m,n $(m-n\neq 0,\pm1)$ and arbitrary angles φ_1, φ_2, φ_3 satisfying Equations (40),
2. one of the coordinates z_1, z_2, z_3 ($\neq 0$ arbitrary),
3. either one of the parameters r, ℓ, d_1, a, b, d_2 or one of the parameters r, x_1, x_2, x_3, h.

The unknowns among z_1,z_2,z_3 are determined by Equation (7) and by the second Equation (40). The unknowns among the parameters r, ℓ, d_1, a, b, d_2 are determined by Equations (2)–(6). The unknowns among the parameters r, x_1, x_2, x_3, h are determined by Equations (35), (28) and (19). If r is not prescribed, then the prescribed parameter must be sufficiently large so as to determine a value $r^2>0$.

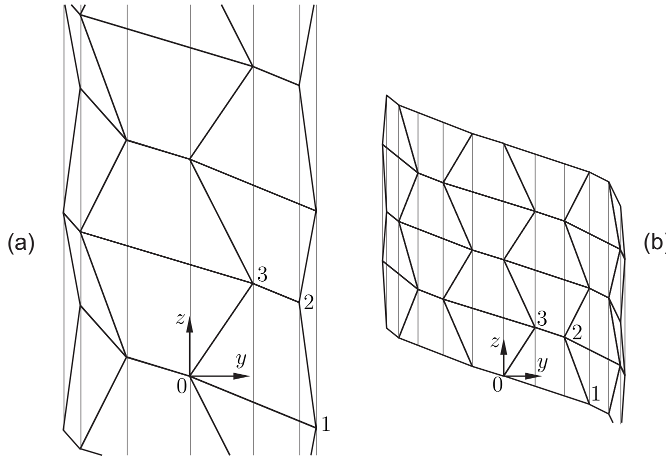

Figure 8: Foldable tiled PCs of Example 6 with the parameters (a) and (b).

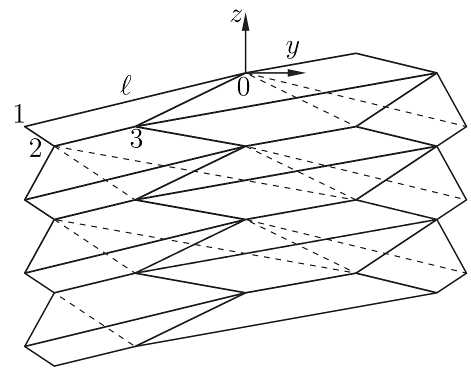

Figure 9: Foldable tiled PC of Example 7.

Example 6. Given are the two sets of parameters

(a) $m = 4$, $n = -2$, $\varphi_1 = 90°$, $x_1 = 1$, $z_3 = 1/2$,

(b) $m = 8$, $n = -4$, $\varphi_1 = 45°$, $x_1 = 1/2$, $z_3 = 1/4$.

Note that m and n in (b) are twice as large as in (a) while φ_1, x_1, z_3 in (a) are twice as large as in (b).

Solution to Problem (a): $M = 6$, $\varphi_2 = 60°$, $\varphi_3 = 30°$, $z_1 = -\sqrt{3}/6 \approx -.2887$, $z_2 = (3+\sqrt{3})/12 \approx .3943$, $\lambda = (\sqrt{3}-1)/2 \approx .3660$, $r^2 = 11/24$, $r \approx .6770$, $x_2 = 3(1+\sqrt{3})/16 \approx .5123$, $x_3 = (11-5\sqrt{3})/16 \approx .1462$, $h^2 = 11(14-\sqrt{3})/384$, $h \approx .5928$, $q = (x_3 - x_1)/x_2 = -5/3$.

The tiled PC is formed by folding $m - n = 6$ columns of trapezoids. In Figure 8a it is shown in projection onto the y, z-plane. The result for q shows that with the same mesh tiled PCs in the trivial mode of deformation can be formed with integers $(m^*, n^*) = (5k, 3k)$ $(k = 1, 2, \ldots)$.

Solution to Problem (b): $M = 12$, $\varphi_2 = 30°$, $\varphi_3 = 15°$, $z_1 = -(2+5\sqrt{2}-3\sqrt{3}+4\sqrt{6})/92 \approx -.1486$, $z_2 = (25+5\sqrt{2}-3\sqrt{3}+4\sqrt{6})/184 \approx .1993$, $\lambda = (1+\sqrt{2}-\sqrt{3})/2 \approx .3411$, $2r^2 = (3982+2043\sqrt{2}-108\sqrt{3}-40\sqrt{6})/92^2$, $r \approx .6238$, $x_2 = (5679+1927\sqrt{2}-2113\sqrt{3}-234\sqrt{6})/(2\cdot 92^2) \approx .2464$, $x_3 = (1447-2305\sqrt{2}+2119\sqrt{3}-234\sqrt{6})/(2\cdot 92^2) \approx .0759$, $h^2 = 2r^2(1-\sqrt{3}/2)+z_2^2-x_2^2$, $h \approx .2885$.

The tiled PC is formed by folding $m - n = 12$ columns of trapezoids. In Figure 8b it is shown in projection onto the y, z-plane. The scale is the same as in Figure 8a. Since $q = (x_3 - x_1)/x_2$ is irrational no tiled PC can be formed in the trivial mode of deformation. ◇

Example 7. Given are $m = 0$, $n = 6$, $\varphi_1 = -90°$, $z_1 \neq 0$ arbitrary, d_2 arbitrary. Definition: $\mu = z_1^2/d_2^2$. The parameters of the tiled PCs determined by these data are $M = 6$, $\varphi_2 = -60°$, $\varphi_3 = -30°$, $z_2 = z_1(\sqrt{3}+1)/2$, $z_3 = z_1$, $\lambda = (\sqrt{3}-1)/2$, $r = d_2$, $\ell^2 = d_2^2(2+\mu)$, $d_1^2 = d_2^2[1+\mu(1+\sqrt{3}/2)]$, $a^2 = d_2^2(2-\sqrt{3}+\mu)$, $b^2 = d_2^2(2-\sqrt{3})(1+\mu/2)$, $q = -1/(1+\mu)$.

In Figure 9 the foldable tiled PC with the parameters $z_1 = -1/2$, $d_2 = 2$ is shown in projection onto the y, z-plane. From $z_3 = z_1$ and $(m, n) = (0, 6)$ it follows that every string of six diagonals d_2 connecting two vertices in the planar mesh is mapped into a regular hexagon in a plane $z = \text{const}$. One out of six trapezoids is seen edge-on. From $q = -16/17$ it follows that with the same mesh tiled PCs in the trivial mode of deformation can be formed with integers $(m^*, n^*) = (16k, 17k)$ $(k = 1, 2, \ldots)$. ◇

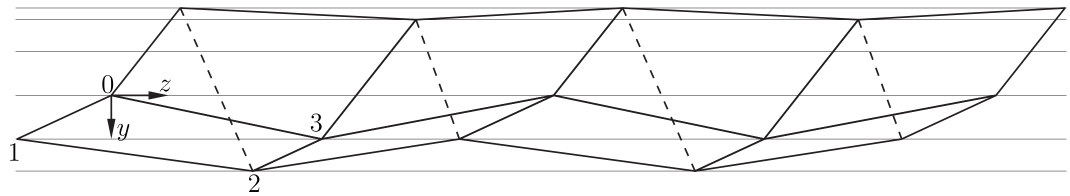

Figure 10: Foldable tiled PC of Example 8.

Example 8. The data $m = 2$, $n = -1$, $\varphi_1 = 150°$, $z_1 = -1$, $r = 1$ determine the tiled PC with the parameters $M = 3$, $\varphi_2 = 120°$, $\varphi_3 = 30°$, $z_2 = \sqrt{3}$, $z_3 = 2\sqrt{3} - 1$, $\lambda = \sqrt{3} - 1$, $\ell^2 = 3 + \sqrt{3}$, $d_1^2 = 6$, $a^2 = 5(3 - \sqrt{3})$, $b^2 = 6 + \sqrt{3}$, $d_2^2 = 15$, $h^2 = 3(13 + \sqrt{3})/8$, $q = 1 - 2\sqrt{3}$.

In Figure 10 the tiled PC is shown in projection onto the y, z-plane. It is formed by folding $m - n = 3$ columns of quadrilaterals. Since q is irrational no tiled PC can be formed in the trivial mode of deformation. ◇

6.5 Symmetric Trapezoids

A symmetric trapezoid is specified by the parameters $x_1 = \ell$, $x_2 - x_3 = c$ and h. The symmetry has the effect that c-lines are orthogonal to r-lines. The closure condition $mx_2 + n(x_3 - x_1) = 0$ is satisfied by all integers $(m, n = m)$ with $m = 2, 3, \ldots$ Hence infinitly many trivial tiled PCs can be formed.

In what follows, it is shown that also infinitly many nontrivial tiled PCs can be formed. The orthogonality of r-lines and c-lines is preserved when the mesh is deformed. Edges of alternating length ℓ, c, ℓ, c etc. on an r-line are mapped into secants of the circle of radius r (arbitrary) in a plane $z =$ const. In particular, $z_1 = 0$ and $z_2 = z_3$ independent of r. With this, the second condition (40) is satisfied. The PC is tiled if $m \geq 2$ pairs of secants (ℓ, c) form a regular polygon. This requires $n = -m$. With this, the first condition (40) reads $\varphi_1 - \varphi_3 = \varphi_2 = \pi/m$. The angle φ_1 and the radius r as functions of m are determined by Equation (30) and by Equation (35) with $z_1 = 0$, $x_1 x_2 = \ell(\lambda + 1)/2$. The results are, independent of h,

$$\varphi_2 = \frac{\pi}{m}, \quad \sin\varphi_1 = \frac{2(\lambda + \cos\varphi_2)\sin\varphi_2}{\lambda^2 + 1 + 2\lambda\cos\varphi_2}, \quad \varphi_3 = \varphi_1 - \varphi_2, \quad r = \frac{\ell}{2}\,\frac{\sqrt{\lambda^2 + 1 + 2\lambda\cos\varphi_2}}{\sin\varphi_2} \tag{41}$$

$(m = 2, 3, \ldots)$ Only fold angles along edges depend on h.

Summary

A Kokotsakis mesh formed by congruent convex, non-trapezoidal, non-parallelogramic quadrilaterals is a single-degree-of-freedom mechanism with two modes of deformation. In both modes the mechanism is a polyhedral cylinder (PC) since the vertices of all quadrilaterals are located on a circular cylinder the radius r of which is a free parameter. In Section 2 six equations are formulated for six cylinder coordinates φ_k, z_k $(k = 1, 2, 3)$ as functions of r. These equations depend on five parameters specifying the quadrilaterals. The equations can only be solved numerically.

In Section 3 a simple geometrical method of solution is shown. It leads to equations of the two parabolas passing through the vertices of a convex quadrilateral and to an algorithm determining the smallest radius r for which the six equations for φ_k, z_k ($k = 1, 2, 3$) have real solutions.

In Section 4 it is shown that tiled PCs with coordinates r, φ_k, z_k ($k = 1, 2, 3$) exist if two additional equations with two additional unknown integers m, n are satisfied. Necessary conditions on the five parameters for the existence of a solution are not available. The construction of tiled PCs is explained. Tiled PCs are either foldable or self-intersecting. They are periodic if φ_2/π is rational.

In Section 5 a two-parametric family of deltoids forming tiled PCs with prescribed integers (m, n) is determined. In both modes of deformation the same tiled PC is formed.

Section 6 is devoted to trapezoidal quadrilaterals. In this case, the six equations for φ_k, z_k ($k = 1, 2, 3$) are decoupled. The two modes of deformation are distinguished as trivial and nontrivial. For both modes the construction of tiled PCs is explained. Nontrivial tiled PCs are periodic. Symmetric trapezoids form in both modes infinitly many tiled PCs.

An open problem: Are there non-deltoidal, non-trapezoidal quadrilaterals forming tiled PCs in both modes of deformation?

Acknowledgements

The author is grateful to Dr. Krause, Prof. Dörfler, Mrs. M. Wu and Mr. J. Marquard (Inst. for Appl. Num. Math., KIT) for numerical calculations verifying predicted results and disproving false conjectures. Mr. M. Bücker converted my hand drawings into digital form.

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Received June 7, 2021; final form September 7, 2021.