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A Physical Archimedean Approach to Affine Geometry and the Remarkable 13 (Mixed) Configuration

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Abstract. We show how to introduce affine geometry via a calculus of balancing weights respecting Archimedes' law of the lever, relying on a fundamental associativity which is simply expressed with multiplicative algebra. Affine subspaces are represented by affine functionals, and vectors are interpreted as null weighted combinations of points. This is then applied to the mixed configuration of thirteen points and lines arising both from the duality between the Menelaus and Ceva theorems and the quadrangle / quadrilateral correspondence.

Key Words: geometry, affine, Menelaus, Ceva, Archimedean, configuration

MSC 2020: 51N10 (primary), 51N05

1 Archimedean affine spaces

We outline an elementary approach to affine geometry that proceeds from physical considerations, and avoids prior theories of linear algebra (as in [1, p.52], [14, p.33]) or synthetic projective geometry (as in [6, p.191], [15, p.3]). This involves a simplified multiplicative algebra of weighted points, and a dual algebra of affine functionals representing lines with respect to an affine basis. To indicate the potential usefulness of this point of view, we look anew at an important planar mixed configuration with thirteen points and lines, which arises both from the classical theorems of Menelaus and Ceva and associated Desargues triangle polarity, as well as the natural affine duality between quadrangles and quadrilaterals. We aim in this introductory paper to lay out some basic notation and results with an emphasis on applications; a more theoretical treatment will be given elsewhere.

The starting point is Archimedes' law of the lever: that the point of balance G of weights $\alpha_1, \ldots, \alpha_n$ on respective points $A_1, \ldots, A_n$ on a (weightless) line is determined by the

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condition that the total moment is zero, that is

$$\sum_{i=1}^{n} \alpha_i \overrightarrow{GA_i} = \vec{0}.$$

If a fulcrum is placed under such a "center of mass" point G then the line will balance according to Archimedes, and the fulcrum will feel an effective weight of $\alpha_1 + \cdots + \alpha_n$ at G.

This situation also includes negative weights, given by upward forces; however in this case there is an important non-null restriction on the weights: they must satisfy $\alpha_1 + \cdots + \alpha_n \neq 0$ as otherwise the balancing point G will be "at infinity", which we exclude for physical reasons. This can be seen already in the impossibility of balancing when we have weights of 1 and -1 at distinct points.

Archimedes' law holds also if the points are in a plane, or indeed in a higher dimensional linear space, although to frame that a prior theory of linear spaces and vectors is needed. We will abstract the main features of this physical situation, which are already present in the one-dimensional case of points on a line, and present the foundations of a theory of affine geometry where the linear algebra is emergent, not prior.

Then the familiar theorems of Menelaus, Ceva and quadrangle/quadrilateral duality can be seen as consequences of the underlying physical assumptions, with computations using a simplified multiplicative algebra of weighted points, and a dual algebra of weighted lines with respect to an affine basis of points.

We will also mention briefly how vector arithmetic arises in a perhaps surprising way by incorporating the null weighted combinations that are excluded when taking centres of mass. And we will show how to practically use these ideas to provide new insights into the important 13 configuration that unifies the above classical theorems, and which opens the door to a more general study of *mixed configurations*. Here we have geometry based on physics — not the more usual other way around, and we conclude with the possibility of extending this calculus of weighted means to spherical and hyperbolic geometries.

Let's begin by assuming that we have some primitive notion of "affine point" or just "point" A which is otherwise unspecified, and define a *weighted point* to be an expression of the form A^α where A is such an affine point and α is a (rational) number.

Now suppose that we have a multiplication of weighted points A^α and B^β in the case $\alpha + \beta \neq 0$; so that there is a unique weighted point $G^{\alpha+\beta} = A^\alpha B^\beta$, called the *weighted mean* of A^α and B^β. This operation should satisfy the following properties, which are abstracted from the physics of balancing weights:

1. *Commutativity*: For points A and B and numbers α and β we have $A^\alpha B^\beta = B^\beta A^\alpha$.
2. *Identity*: For points A and B and a number β we have $A^0 B^\beta = B^\beta A^0 = B^\beta$. This may also be expressed with the equation $A^0 = 1$.
3. *Multiplicativity*: For a point A and numbers α and β we have $A^\alpha A^\beta = A^{\alpha+\beta}$.
4. *Scaling*: If $D^{\alpha+\beta} = A^\alpha B^\beta$ then for any number λ we have $D^{\lambda(\alpha+\beta)} = A^{\lambda\alpha} B^{\lambda\beta}$.
5. *Associativity*: Given points A, B and C and numbers α, β and γ we have

$$A^\alpha (B^\beta C^\gamma) = B^\beta (A^\alpha C^\gamma) = (A^\alpha B^\beta) C^\gamma \tag{1}$$

and henceforth this common expression is denoted by the symmetric expression $A^\alpha B^\beta C^\gamma$.

We here assume the (temporary!) convention that equations become void when sums of weights are 0. If $G^{\alpha+\beta} = A^\alpha B^\beta$, then we say that G is the *base point* of the weighted mean, and write $G = [A^\alpha B^\beta]$.

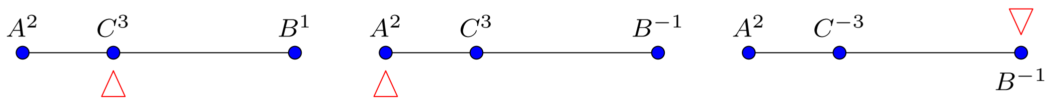

Figure 1: Balancing of weights

2 Some one-dimensional illustrations

Here are some examples involving three points A, B and C of how we may capture the balancing of the three basic types of levers with the fulcrum at various points. On the left of Figure 1 we have a traditional balancing (fulcrum) at C of weights 2 and 1 at A and B respectively, expressed by the equation $C^3 = A^2B^1$.

With the central diagram, we have a balancing at A of a weight 3 at C and a weight of -1 at B, so that the total weight at the fulcrum is 2, and the corresponding equation is $A^2 = C^3B^{-1}$. On the right of Figure 1 we have a balancing at B of a weight 2 at A and a weight of -3 at C; however the fulcrum in this case should be pressing down, as the total weight at B will be -1, and the equation is $B^{-1} = A^2C^{-3}$.

Notice that any one of these equations can be obtained from any other just by elementary algebraic manipulations using the above properties. Using the scaling law, the third equation could also be rewritten as $B^1 = A^{-2}C^3$, which would then represent a more traditional fulcrum position. We see here a curious, perhaps even remarkable, example of a physical law leading directly to algebraic operations.

The crucial associative property is below illustrated with the same three points A, B and C on a line as in Figure 1, but now with weights of $-3, 2$ and 4 respectively. The weighted centre of mass $G^3 = A^{-3}C^4B^2$ can then be expressed in essentially three different ways as

$$G^3 = (A^{-3}C^4)B^2 = D^1B^2 = A^{-3}(C^4B^2) = A^{-3}E^6 = (A^{-3}B^2)C^4 = F^{-1}C^4.$$

Figure 2: Associativity from balancing weights

This law can be extended to more points, so that with four weighted points $A^\alpha, B^\beta, C^\gamma$ and D^δ, the weighted centre of mass $G^{\alpha+\beta+\gamma+\delta} \equiv A^\alpha B^\beta C^\gamma D^\delta$ may be expressed with any of the following equations

$$\begin{aligned} G^{\alpha+\beta+\gamma+\delta} &= (A^\alpha B^\beta C^\gamma)D^\delta = (A^\alpha B^\beta D^\delta)C^\gamma = (A^\alpha C^\gamma D^\delta)B^\beta = (B^\beta C^\gamma D^\delta)A^\alpha \\ &= (A^\alpha B^\beta)(C^\gamma D^\delta) = (A^\alpha D^\delta)(B^\beta C^\gamma) = (A^\alpha C^\gamma)(B^\beta D^\delta) \end{aligned}$$

but some of these expressions might be undefined if the sum of exponents is 0, which makes the usual induction somewhat more complicated.

Proposition 1. *If $\alpha_1 + \cdots + \alpha_k \neq 0$ then we can use the (three-fold) associative law to reduce $A_1^{\alpha_1} \cdots A_k^{\alpha_k}$ to a single weighted point.*

Proof. Suppose $\alpha_1 + \cdots + \alpha_k = M \neq 0$. If $\sum_{i=1}^{k} \alpha_i - \alpha_j = 0$ for all $j = 1, \ldots, k$ (i.e. all partial sums with $k-1$ terms are zero), then $\alpha_1 = \alpha_2 = \cdots = \alpha_k = M$ or $kM = M$. We note that this is possible when the underlying field is a finite field. In such case, we can write

$$A_1^{\alpha_1} \cdots A_k^{\alpha_k} = (A_1^M A_2^M)(A_3^M \cdots A_k^M)$$

since both $2M$ and $(k-2)M$ are non-zero.

Otherwise, $\sum_{i=1}^{k} \alpha_i - \alpha_j \neq 0$ for at least one j in the range $1, \dots, k$, and so we can write

$$A_1^{\alpha_1} \cdots A_k^{\alpha_k} = A_j^{\alpha_j} (A_1^{\alpha_1} \cdots A_{j-1}^{\alpha_{j-1}} A_{j+1}^{\alpha_{j+1}} \cdots A_k^{\alpha_k})$$

and then use induction. □

2.1 Three points in the plane

It is a somewhat remarkable physical, empirical, fact that the laws for an Archimedean affine space, which can be discovered by experimenting with points on a line, extend physically to two and three dimensions. Consider then the case of a horizontal plane, assuming a uniform gravitational force pointing down, with three weighted points A^1, B^2 and C^3 as in Figure 3.

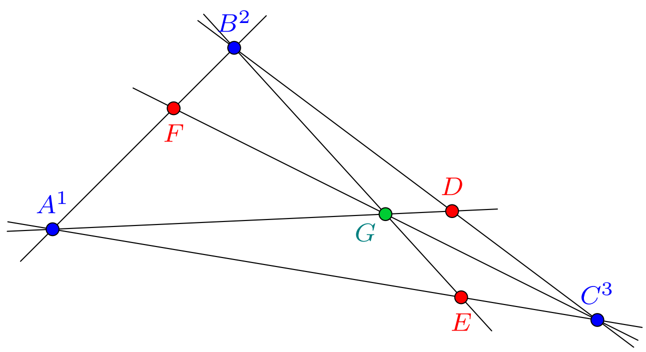

Figure 3: Balancing weights in the plane

The weighted centre of mass is $G^6 = A^1B^2C^3$ in green, and we may physically verify that we can obtain it in three different ways corresponding to the expressions

$$G^6 = A^1(B^2C^3) = A^1D^5 = (A^1C^3)B^2 = E^4B^2 = (A^1B^2)C^3 = F^3C^3.$$

From our physical intuition, this makes sense since the centre of mass G ought to lie on each of the lines AD, BE and CF, since each of these will be a line of balance of the triangle, hence of the plane. It follows that the three lines AD, BE and CF do meet at a common point, which must then be G. Note that here a geometrical result is effectively a consequence of a physical argument. As an aside, the final equation establishes that G is the *midpoint* of the *side* $\overline{CF} \equiv \{C\ F\}$ where we use the convention that braces without commas denotes a set, while if commas are present this denotes an ordered set.

2.2 Examples of Archimedean affine spaces

Example 2. *Consider* $\mathbb{A} = \mathbb{A}^1$ *consisting of* affine points $A = [a]$ *where* a *is a rational number, so that a weighted point has the form* $A^\alpha = [a]^\alpha$. *For* $A = [a]$ *and* $B = [b]$ *affine points, and rational numbers* α *and* β *subject to* $\alpha + \beta \neq 0$*, we define the weighted mean*

$$A^\alpha B^\beta \equiv \left[\frac{\alpha}{\alpha+\beta}a + \frac{\beta}{\alpha+\beta}b\right]^{(\alpha+\beta)}.$$

If $\alpha+\beta+\gamma \neq 0$ and $A=[a]$, $B=[b]$ and $C=[c]$, then the associative property (the only non-trivial one here) can be checked by observing that

$$\begin{aligned} A^\alpha(B^\beta C^\gamma) &= [a]^\alpha \left[\frac{\beta}{\beta+\gamma}b + \frac{\gamma}{\beta+\gamma}c\right]^{(\beta+\gamma)} \\ &= \left[\frac{\alpha}{\alpha+(\beta+\gamma)}a + \frac{(\beta+\gamma)}{\alpha+(\beta+\gamma)}\left(\frac{\beta}{\beta+\gamma}b + \frac{\gamma}{\beta+\gamma}c\right)\right]^{(\alpha+\beta+\gamma)} \\ &= \left[\frac{\alpha}{\alpha+\beta+\gamma}a + \frac{\beta}{\alpha+\beta+\gamma}b + \frac{\gamma}{\alpha+\beta+\gamma}c\right]^{(\alpha+\beta+\gamma)} \end{aligned}$$

is indeed a symmetric expression, and so it equals $(A^\alpha B^\beta)C^\gamma$ and $B^\beta(C^\gamma A^\alpha)$.

We may call $\mathbb{A}^1$ the *Archimedean affine line*. Note that it has otherwise no additive or linear structure, and that the point $O \equiv [0]$ plays no distinguished role.

Example 3. *The previous example extends straightforwardly to $\mathbb{A} = \mathbb{A}^2$ consisting of affine points $A=[a_1,a_2]$ so that if $A=[a_1,a_2]$ and $B=[b_1,b_2]$ and α and β are rational numbers subject to $\alpha+\beta \neq 0$, then we have*

$$A^\alpha B^\beta \equiv \left[\frac{\alpha}{\alpha+\beta}a_1 + \frac{\beta}{\alpha+\beta}b_1, \frac{\alpha}{\alpha+\beta}a_2 + \frac{\beta}{\alpha+\beta}b_2\right]^{(\alpha+\beta)}.$$

If $C=[c_1,c_2]$ and $\alpha+\beta+\gamma \neq 0$ then one can check that we have a well-defined weighted mean of three points, given by

$$A^\alpha B^\beta C^\gamma = \left[\frac{\alpha a_1}{\alpha+\beta+\gamma} + \frac{\beta b_1}{\alpha+\beta+\gamma} + \frac{\gamma c_1}{\alpha+\beta+\gamma}, \frac{\alpha a_2}{\alpha+\beta+\gamma} + \frac{\beta b_2}{\alpha+\beta+\gamma} + \frac{\gamma c_2}{\alpha+\beta+\gamma}\right]^{(\alpha+\beta+\gamma)}.$$

This example extends in the obvious way to $\mathbb{A} = \mathbb{A}^n$, a space consisting of affine points $A = [a_1, a_2, \ldots, a_n]$.

Example 4. *Suppose that $\mathbb{A} = \mathbb{A}^n$ of the previous example, and consider the space $\mathbb{W}$ of those affine points $A=[a_1,a_2,\ldots,a_n]$ that satisfy a fixed equation of the form*

$$c_1a_1 + c_2a_2 + \cdots + c_na_n = d$$

for (fixed) rational numbers $c_1, c_2, \ldots, c_n$ and d. Then it makes sense to restrict the affine structure of $\mathbb{A}$ to $\mathbb{W}$, since if $A=[a_1,a_2,\ldots,a_n]$ and $B=[b_1,b_2,\ldots,b_n]$ are in $\mathbb{W}$, then so is the affine point $[A^\alpha B^\beta]$ for any rational numbers α and β subject to $\alpha+\beta \neq 0$. The reason is shown here:

$$\begin{aligned} & c_1\left(\frac{\alpha}{\alpha+\beta}a_1 + \frac{\beta}{\alpha+\beta}b_1\right) + c_2\left(\frac{\alpha}{\alpha+\beta}a_2 + \frac{\beta}{\alpha+\beta}b_2\right) + \cdots + c_n\left(\frac{\alpha}{\alpha+\beta}a_2 + \frac{\beta}{\alpha+\beta}b_2\right) \\ &= \frac{\alpha}{\alpha+\beta}(c_1a_1 + c_2a_2 + \cdots + c_na_n) + \frac{\beta}{\alpha+\beta}(c_1b_1 + c_2b_2 + \cdots + c_nb_n) \\ &= \frac{\alpha}{\alpha+\beta}d + \frac{\beta}{\alpha+\beta}d = d. \end{aligned}$$

We then say that $\mathbb{W}$ is an affine hyperplane *of $\mathbb{A}$. This example can be further modified by considering more than one such equation, so that more general affine subspaces of $\mathbb{A}$*

yield further examples. An important special case is the space $\mathbb{A}_1^n$ consisting of points $A = [a_1, a_2, \ldots, a_n]$ subject to the condition $a_1 + a_2 + \cdots + a_n = 1$. We could say that this is the canonical hyperplane *in $\mathbb{A}^n$. As we progress, we will see that this is in some sense the most important example.*

Example 5. *If $\mathbb{A}$ and $\mathbb{B}$ are affine spaces, then a function $\phi\colon \mathbb{A} \to \mathbb{B}$ is* affine *precisely when for any points A and B of $\mathbb{A}$ and any rational numbers α and β subject to $\alpha + \beta \neq 0$ we have $(\phi(A))^\alpha(\phi(B))^\beta = \phi(A^\alpha B^\beta)$. The space $\mathrm{Aff}(\mathbb{A}, \mathbb{B})$ of such affine functions is itself an affine space under the operation $(\phi^\alpha \varphi^\beta)(A) = (\phi(A))^\alpha(\varphi(A))^\beta$.*

Example 6. *If $\mathbb{A}$ is an Archimedean affine space then a function $f\colon \mathbb{A} \to \mathbb{F}$ is an* affine functional *precisely when*

$$f\left(\left[A^\alpha B^\beta\right]\right) = \frac{\alpha}{\alpha+\beta} f(A) + \frac{\beta}{\alpha+\beta} f(B).$$

Here the field $\mathbb{F}$ has additive and multiplicative structures along with the distinguished elements 0, 1 and -1, so is fundamentally distinct from the affine line $\mathbb{A}^1$. Let $\mathrm{Aff}(\mathbb{A})$ denote the space of such affine functionals; this is actually a vector space over $\mathbb{F}$, as we have natural operations of addition and scalar multiplication, defined by $(f + g)(C) = f(C) + g(C)$ and $(\lambda f)(C) = \lambda f(C)$.

2.3 Conjugates

It is perhaps surprising that the notion of harmonic conjugates, which plays a central role in projective geometry, already appears here in a simple way in this physical manifestation of affine geometry. If A and B are affine points and $\alpha \pm \beta \neq 0$, then we define the points $C = [A^\alpha B^\beta]$ and $D = [A^\alpha B^{-\beta}]$ to be *conjugate points* with respect to A and B.

Proposition 7. *If C and D are conjugate points with respect to A and B, then A and B are conjugate points with respect to C and D. More specifically, if $C = [A^\alpha B^\beta]$ and $D = [A^\alpha B^{-\beta}]$ then*

$$A = [C^{\alpha+\beta} D^{\alpha-\beta}] \quad \textit{and} \quad B = [C^{\alpha+\beta} D^{-\alpha+\beta}].$$

Proof. We compute that $C^{\alpha+\beta} D^{\alpha-\beta} = (A^\alpha B^\beta)(A^\alpha B^{-\beta}) = A^{2\alpha}$ so that $A = [C^{\alpha+\beta} D^{\alpha-\beta}]$ and similarly $C^{\alpha+\beta} D^{-\alpha+\beta} = (A^\alpha B^\beta)(A^{-\alpha} B^\beta) = B^{2\beta}$ so that $B = [C^{\alpha+\beta} D^{-\alpha+\beta}]$. □

Let us introduce the notation that in this case $\overline{AB}$ and $\overline{CD}$ are *conjugate sides*. This avoids the need to have a prior definition of cross ratio in order to flag the special relation between these four points, and indeed avoids the ambiguity in an associated cross ratio arising from different possible orderings.

Example 8. *If $A = [a]$, $B = [b]$, $C = [\frac{2}{3}a + \frac{1}{3}b]$, $D = [2a - b]$ are four points in $\mathbb{A}^1$ where $a \neq b$ are rational numbers, then $\overline{AB}$ and $\overline{CD}$ are conjugate sides because*

$$C = [A^2 B^1], \quad D = [A^2 B^{-1}], \quad \textit{and} \quad A = [C^3 D^1], \quad B = [C^3 D^{-1}].$$

3 Basis, dimension and dual functionals on an affine plane

A set $\{A_1 \ A_2 \ \cdots \ A_n\}$ of elements of an Archimedean affine space $\mathbb{A}$ is a *spanning set* precisely when any element B of $\mathbb{A}$ can be written as

$$B = [A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n}]$$

for some numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. A set $\{A_1 \ A_2 \ \cdots \ A_n\}$ is *independent* precisely when $A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_n^{\alpha_n} = 1$ implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

A set $\{A_1 \ A_2 \ \cdots \ A_n\}$ is a *basis* of $\mathbb{A}$ precisely when it is both a spanning set and independent, in which case an associated ordered set $\{A_1, A_2, \ldots, A_n\}$ is an *ordered basis* of $\mathbb{A}$. This implies that we can associate to an element B as above a unique projective vector $[\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n]$. We will use the convention that matrices or vectors enclosed in square brackets are defined projectively, that is, are invariant under multiplication by non-zero scalars. Now one should establish that if an Archimedean affine space $\mathbb{A}$ has a basis with n elements, then every basis has n elements. Adapting the usual similar result in linear algebra seems like an interesting challenge.

If n is the size of a basis of $\mathbb{A}$, then we may define the *dimension* of $\mathbb{A}$ to be $n-1$. An affine space of dimension 0 is called an *affine point*, an affine space of dimension 1 is called an *affine line*, and an affine space of dimension 2 is called an *affine plane*. More generally an affine space of dimension n is called an *affine n-space*.

From now on we suppose that we are dealing with an affine plane $\mathbb{A}$. We choose a distinguished ordered basis $\{A_1, A_2, A_3\}$ so that every point $A = [A_1^{\alpha_1} A_2^{\alpha_2} A_3^{\alpha_3}]$ can be associated with a projective 3-vector $[\alpha_1 \ \ \alpha_2 \ \ \alpha_3]$, which will necessarily satisfy $\alpha_1 + \alpha_2 + \alpha_3 \neq 0$, and for simplicity we write

$$A \simeq [\alpha_1 \ \ \alpha_2 \ \ \alpha_3].$$

In particular

$$A_1 \simeq [1 \ \ 0 \ \ 0], \quad A_2 \simeq [0 \ \ 1 \ \ 0], \quad A_3 \simeq [0 \ \ 0 \ \ 1].$$

The space of affine functionals on $\mathbb{A}$ is a vector space with ordered basis consisting of the dual functionals $\{f_1, f_2, f_3\}$ defined by $f_i(A_j) = \delta_{ij}$. A general affine functional will have the form $f = x_1 f_1 + x_2 f_2 + x_3 f_3$ and its value on $A \simeq [\alpha_1 \ \ \alpha_2 \ \ \alpha_3]$ will be

$$f(A) = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3} x_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} x_2 + \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} x_3.$$

This motivates us to write

$$f \simeq \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1 \ \ x_2 \ \ x_3)^T.$$

A functional of the form $f \equiv [x \ \ x \ \ x]^T$ is special and will be called a *vector affine functional.*

3.1 Affine lines, incidence and the cross product

An affine line, or just a line, $\mathbb{L}$ contained in $\mathbb{A}$ is then determined by any two points B and C which lie on it, and we write $\mathbb{L} = BC$. Such a line is also uniquely determined by a non-vector affine functional $f = (x_1 \ \ x_2 \ \ x_3)^T$ which is then unique up to a scalar, and so we write

$$\mathbb{L} \simeq [x_1 \ \ x_2 \ \ x_3]^T.$$

The incidence between an affine point $A \simeq [\alpha_1 \;\; \alpha_2 \;\; \alpha_3]$ and an affine line $\mathbb{L} \simeq [x_1 \;\; x_2 \;\; x_3]^T$ is then just given by the relation

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0.$$

It follows that the *join* AB of two points $A \simeq [\alpha_1 \;\; \alpha_2 \;\; \alpha_3]$ and $B \simeq [\beta_1 \;\; \beta_2 \;\; \beta_3]$ which is the line they determine, is given by a cross product

$$AB \simeq [\alpha_2\beta_3 - \alpha_3\beta_2 \;\; \beta_1\alpha_3 - \alpha_1\beta_3 \;\; \alpha_1\beta_2 - \alpha_2\beta_1]^T$$

and conversely the *meet* of two lines $\mathbb{L} \simeq [x_1 \;\; x_2 \;\; x_3]^T$ and $\mathbb{M} \simeq [y_1 \;\; y_2 \;\; y_3]^T$, which is the point they determine, is similarly given by

$$\mathbb{LM} \simeq [x_2y_3 - x_3y_2 \;\; x_3y_1 - x_1y_3 \;\; x_1y_2 - x_2y_1].$$

These two formulas lie at the heart of practical calculations in the affine plane.

The expression $\mathbb{LM}$ does not define a proper point when it is a vector expression, in other words when

$$x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1 = 0.$$

In this case we say that $\mathbb{L}$ and $\mathbb{M}$ are *parallel lines*. It should be equivalent to a relation between corresponding vectors of the form

$$(x_1 \;\; x_2 \;\; x_3) = (y_1 \;\; y_2 \;\; y_3) + (k \;\; k \;\; k).$$

So the space of all lines parallel to a given line forms a two dimensional subspace of 3 space through the vector $(1, 1, 1)$.

Over the rationals the space of affine points in an affine plane is topologically a plane, while the space of affine lines is topologically a Möbius band, as it is essentially a projective plane with a single point removed. So there is a slight assymmetry between affine points and affine lines. Note however that this setup extends naturally to more general fields also.

3.2 Vectors and null weighted expressions

We have seen that the meaning of the expression $A^\alpha B^\beta C^\gamma$ depends on the nature of the weights α, β and γ. We want to use the associative property to enlarge the meaning when a sum of weights equals zero.

In the generic situation where $\alpha + \beta + \gamma \neq 0$, at least one of the numbers α, β and γ is non-zero, and hence at least one of the sums $\beta + \gamma, \alpha + \gamma$ and $\alpha + \beta$ are non-zero. In this case the corresponding term in (1) defines a weighted point $D^{(\alpha+\beta+\gamma)}$ and we can take this to be the common value of all three terms.

If $\alpha + \beta + \gamma = 0$ and if one of the weights is 0, say $\alpha = 0$, then $\beta + \gamma = 0$ and the common value of the terms in (1) is $B^\beta C^\gamma = B^\beta C^{-\beta}$ and this does not define a weighted point. We see that if we want a consistent associative property which is independent of the nature of the weights involved, then we need to augment our framework to include not just weighted points of the form A^α but also new objects — which we will call *vectors* — of the form $B^\beta C^{-\beta}$.

This is certainly a surprising way of introducing vectors, but the arithmetic of weighted means almost forces it upon us. It turns out that the expression B^1C^{-1} should be interpreted as the multiplicative form of what we usually write in linear algebra as $\overrightarrow{CB}$ or sometimes as the displacement $B - C$. The more general expression $B^\beta C^{-\beta}$ is then interpreted as β times the vector $\overrightarrow{CB}$.

If $\alpha+\beta+\gamma = 0$ and none of the weights is 0 then also $\beta+\gamma, \alpha+\gamma$ and $\alpha+\beta$ are all non-zero. That means that the weighted points

$$D^{(\beta+\gamma)} = D^{-\alpha} \equiv B^\beta C^\gamma \quad E^{(\alpha+\gamma)} = E^{-\beta} \equiv A^\alpha C^\gamma \quad \text{and} \quad F^{(\alpha+\beta)} = F^{-\gamma} = A^\alpha B^\beta$$

are all well-defined and then the associative property implies that we require an equality of vectors

$$A^\alpha D^{-\alpha} = B^\beta E^{-\beta} = C^\gamma F^{-\gamma}.$$

To ensure this, we declare two vectors $A^\alpha B^{-\alpha}$ and $C^\gamma D^{-\gamma}$ to be *equal*, written $A^\alpha B^{-\alpha} = C^\gamma D^{-\gamma}$, precisely when $\alpha + \gamma \neq 0$ and

$$A^\alpha D^\gamma = B^\alpha C^\gamma.$$

Example 9. *In Figure 4 we see an example where* $P^3 = A^1 D^2 = B^1 C^2$*, which implies that we have an equality of vectors*

$$A^1 B^{-1} = C^2 D^{-2}.$$

This is to be interpreted as a multiplicative version of the more familiar $\overrightarrow{BA} = 2\overrightarrow{DC}$ *in linear algebra, and allows us to introduce the concept of parallel vectors, such as* A^1B^{-1} *and* C^1D^{-1} *as shown.*

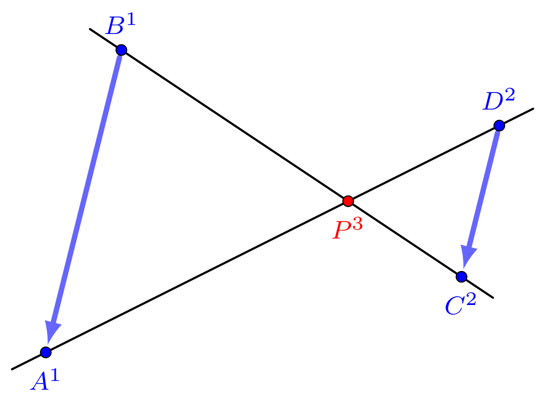

Figure 4: Parallel vectors

More generally an expression of the form $A_1^{\alpha_1} \cdots A_n^{\alpha_n}$ where $\alpha_1 + \cdots + \alpha_n = 0$ will be called a *null weighted expression.* Such an expression can always be reduced to a vector.

3.3 Menelaus' theorem

Let's use the concepts and notation so far to establish some familiar results in an affine plane $\mathbb{A}$, perhaps in a somewhat novel fashion. We start with an ordered triangle $\overrightarrow{ABC}$, which we will regard as a basis triangle for the affine plane, so that $A \simeq [1 \ \ 0 \ \ 0]$, $B \simeq [0 \ \ 1 \ \ 0]$ and $C \simeq [0 \ \ 0 \ \ 1]$. The lines of the triangle $\overrightarrow{ABC}$ may be labelled $a = BC \simeq [1 \ \ 0 \ \ 0]^T$, $b = AC \simeq [0 \ \ 1 \ \ 0]^T$ and $c = AB \simeq [0 \ \ 0 \ \ 1]^T$.

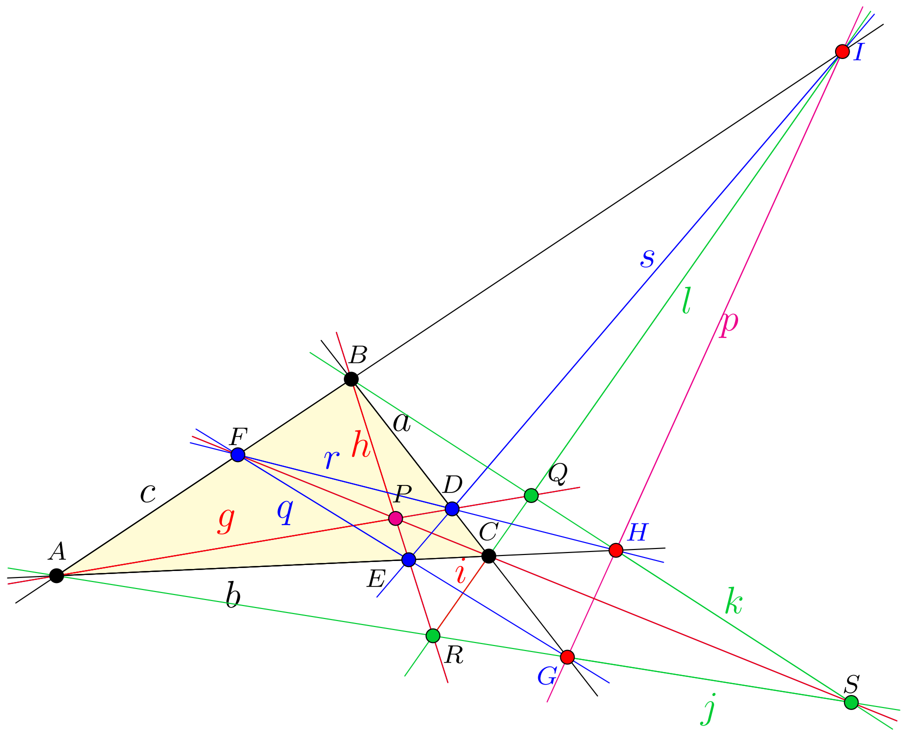

Figure 5: Points and lines from Menelaus/Ceva duality

Suppose that $p \simeq [x \quad y \quad z]^T$ is an additional line which is not parallel to either a, b or c. Then the respective meets G, H and I of p with the lines a, b and c of $\overrightarrow{ABC}$, which we call the *Menelaus points*, are directly computed to be

$$G \simeq [0 \quad -z \quad y] \qquad H \simeq [z \quad 0 \quad -x] \qquad I \simeq [-y \quad x \quad 0]. \tag{2}$$

Theorem 10 (Menelaus)**.** *If the Menelaus points of a line p with respect to a triangle $\overrightarrow{ABC}$ are expressed as weighted means of the points A, B and C by*

$$G = [B^{\delta_B} C^{\delta_C}], \quad H = [C^{\varepsilon_C} A^{\varepsilon_A}], \quad I = [A^{\phi_A} B^{\phi_B}]$$

then

$$\delta_B \varepsilon_C \phi_A = -\delta_C \varepsilon_A \phi_B.$$

Proof. The formulas (2) for G, H and I imply that there are non-zero scalars δ, ε and ϕ such that

$$\delta_B = \delta(-z), \quad \delta_C = \delta y, \quad \varepsilon_C = \varepsilon(-x), \quad \varepsilon_A = \varepsilon z, \quad \phi_A = \phi(-y), \quad \text{and} \quad \phi_B = \phi x.$$

The result then follows, as $\delta_B \varepsilon_C \phi_A = -(\delta\varepsilon\phi)(xyz)$ and $\delta_C \varepsilon_A \phi_B = (\delta\varepsilon\phi)(xyz)$. $\square$

If $p \simeq [x \quad y \quad z]^T$ is a line as above, then we may identify some additional lines and points. First we have the lines

$$j = AG \simeq [0 \quad y \quad z]^T, \quad k = BH \simeq [x \quad 0 \quad z]^T, \quad l = CI \simeq [x \quad y \quad 0]^T$$

which determine a triangle $\overrightarrow{QRS}$ where

$$Q = kl \simeq [-yz \quad xz \quad xy] \quad R = lj \simeq [yz \quad -xz \quad xy] \quad S = jk \simeq [yz \quad xz \quad -xy].$$

Notice that these are all obtained from $[yz \;\; xz \;\; xy]$ by simply negating one of the entries. Now as each of Q, R and S are naturally associated respectively with the points A, B and C, we may form the lines

$$g = AQ \simeq [0 \;\; -y \;\; z]^T \quad h = BR \simeq [x \;\; 0 \;\; -z]^T \quad i = CS \simeq [-x \;\; y \;\; 0]^T$$

and deduce the following.

Theorem 11. *The lines g, h, and i are concurrent at the point $P \simeq [yz \;\; xz \;\; xy]$.*

The point P is then defined to be the *pole* of the line p with respect to the triangle $\overrightarrow{ABC}$. We might call this *Menelaus / Ceva duality* in triangle geometry [5], and it has some similarities to the more familiar Apollonian duality / polarity between points and lines determined by a conic [7]. It can also be viewed as a special case of Desargues theorem, as the triangles $\overrightarrow{ABC}$ and $\overrightarrow{DEF}$ are perspective from the point P, and also from the line p.

3.4 Ceva's theorem

To more fully appreciate the above duality, we now reverse the discussion by starting with a point $P = [\alpha \;\; \beta \;\; \gamma]$ not lying on any of the three lines of $\overrightarrow{ABC}$. The Cevian lines, joining P to the points of the triangle, are

$$AP \simeq [0 \;\; -\gamma \;\; \beta]^T, \quad BP \simeq [-\gamma \;\; 0 \;\; \alpha]^T, \quad CP \simeq [-\beta \;\; \alpha \;\; 0]^T,$$

and these meet the lines of $\overrightarrow{ABC}$ at the Cevian base points

$$D \simeq [0 \;\; \beta \;\; \gamma], \quad E \simeq [\alpha \;\; 0 \;\; \gamma], \quad F \simeq [\alpha \;\; \beta \;\; 0]. \tag{3}$$

Theorem 12 (Ceva)**.** *If the Cevian base points of a point P with respect to a triangle $\overrightarrow{ABC}$ are expressed as weighted means of the points A, B and C by*

$$D = [B^{\delta_B} C^{\delta_C}], \quad E = [C^{\varepsilon_C} A^{\varepsilon_A}], \quad F = [A^{\phi_A} B^{\phi_B}]$$

then

$$\delta_B \varepsilon_C \phi_A = \delta_C \varepsilon_A \phi_B.$$

Proof. The formulas (3) for D, E and F imply that there are non-zero scalars δ, ε and ϕ such that

$$\delta_B = \delta\beta, \quad \delta_C = \delta\gamma, \quad \varepsilon_C = \varepsilon\gamma, \quad \varepsilon_A = \varepsilon\alpha, \quad \phi_A = \phi\alpha \quad \text{and} \quad \phi_B = \phi\beta.$$

The result then follows, as $\delta_B \varepsilon_C \phi_A = (\delta\varepsilon\phi)(\alpha\beta\gamma)$ and $\delta_C \varepsilon_A \phi_B = (\delta\varepsilon\phi)(\alpha\beta\gamma)$. □

Now carrying on, the triangle $\overrightarrow{DEF}$ has lines

$$DE \simeq [\beta\gamma \;\; \alpha\gamma \;\; -\alpha\beta]^T, \quad EF \simeq [-\beta\gamma \;\; \alpha\gamma \;\; \alpha\beta]^T, \quad FD \simeq [\beta\gamma \;\; -\alpha\gamma \;\; \alpha\beta]^T.$$

These meet the original lines a, b and c respectively at the points

$$G \simeq [0 \;\; -\beta \;\; \gamma], \quad H \simeq [-\alpha \;\; 0 \;\; \gamma], \quad I \simeq [-\alpha \;\; \beta \;\; 0].$$

and we obtain the same polarity as before, but now in the opposite direction.

Theorem 13. *The points G, H, and I are collinear, and lie on the line $p \simeq [\beta\gamma \;\; \alpha\gamma \;\; \alpha\beta]^T$.*

3.5 Symmetry of the Menelaus / Ceva 13 mixed configuration

The above discussion of Menelaus' and Ceva's theorems yields an interesting "mixed configuration" of thirteen points and thirteen lines as in Figure 5. Let us use the term degree symmetrically for points and lines in a diagram: the *degree of a point* is the number of lines that are incident with it, and the *degree of a line* is the number of points incident with it.

Then, of the thirteen points of Figure 5, nine are of degree 4 and the other four are of degree 3, and symmetrically of the thirteen lines, nine are of degree 4 and the other four are of degree 3.

Let us use the convention that if X, Y and Z are collinear points, then the line determined by them may be denoted by XYZ. Then with respect to the triangle $\overrightarrow{ABC}$ of in Figure 5, we have the following symmetries.

- If we start with the point P, then its Cevian triangle is $\overrightarrow{DEF}$ and associated line is $p = GHI$.
- If we start with the point Q, then its Cevian triangle is $\overrightarrow{DHI}$ and associated line is $q = EFG$.
- If we start with the point S, then its Cevian triangle is $\overrightarrow{FGH}$ and associated line is $s = DEI$.
- If we start with the point R, then its Cevian triangle is $\overrightarrow{EGI}$ and associated line is $r = DFH$.

We see four partitions of the set $\{D \quad E \quad F \quad G \quad H \quad I\}$ consisting of the six blue points formed by the quadrilateral $\overrightarrow{pqrs}$. These are obtained simply as follows: choose one of the four lines of the quadrilateral, then there are three points on this line, and three points formed by the other three lines. These are the partitions that appear.

3.6 The 13 mixed configuration and quadrangle / quadrilateral duality

While the 13 mixed configuration arose from the duality between the Menelaus' and Ceva's theorems, it has another important role in the projective geometry of quadrangles and quadrilaterals, in which a different kind of duality arises. If we start with a quadrangle, say the four points (green) in Figure 6, and its six lines (green), and then construct its three diagonal points (black), then each of the three lines of the diagonal triangle (black) will meet the remaining two of the original lines also in two additional points (blue). There is an important duality here.

Theorem 14. *The six points (blue) in the above construction lie on a quadrilateral of four lines (blue).*

In total we have $4 + 3 + 2 \times 3 = 13$ points and also thirteen lines. The situation is symmetric, in that we can start with the quadrilateral and construct dually the quadrangle.

In Figure 6 there are 4 green points of degree 3, each incident with 3 green lines. Of the 9 points of degree 4, there are 3 black points incident with 2 green and 2 black lines, and 6 blue points incident with 1 green, 1 black and 2 blue lines. Dually there are 4 blue lines of degree 3, each incident with 3 blue points. Of the 9 lines of degree 4, there are 3 black lines incident with 2 blue and 2 black lines, and 6 green lines incident with 1 blue, 1 black and 2 green points. The symmetry here is represented by the pairings point / line, green / blue and black / black.

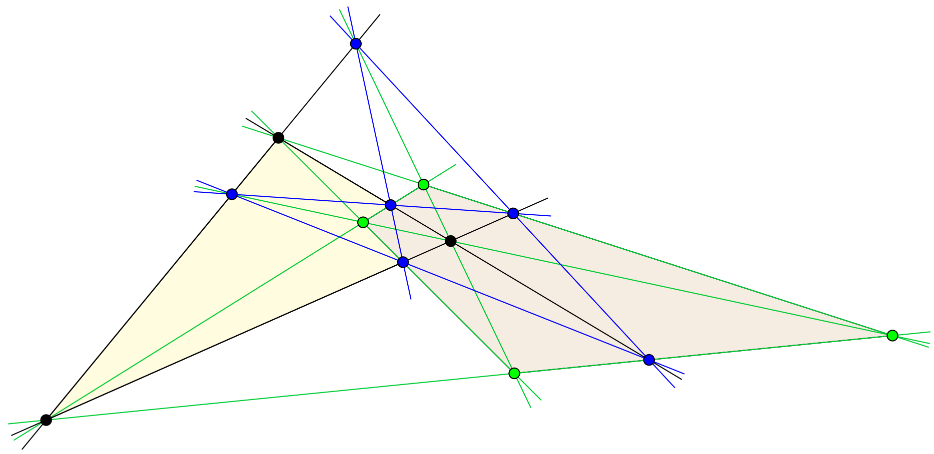

Figure 6: The 13 mixed configuration

We can now extend the usual definition of a *configuration* (p_γ, l_π) (see for example [13], [11], [8] and [9]) as a planar set of p points and l lines, where each point is incident with γ lines, and each line is incident with π points (so that necessarily $p\gamma = l\pi$). If $p = l$ then also $\gamma = \pi$; in this case the configuration is called *symmetric*, and the notation abbreviates to just (p_γ). Some elementary symmetric configurations [3] include the (3_3) triangle and the (7_3) Fano plane. More complicated examples include the (9_3) Pappus configuration (one of three such), the (10_3) Desargues configuration, and more recently (1990) the (21_4) Grunbaum-Rigby configuration [10], which however was known in some form earlier to F. Klein [12], W. Burnside [2] and H. S. M. Coxeter [4].

By just concentrating on the three types of points and lines, we could introduce the notation $(4_{0+0+3}3_{0+2+2}6_{2+1+1})$ to describe the above symmetric mixed configuration. The meaning is that there are three types of points, and three types of lines; with 4 points of the first type, incident with 0 lines of the first type, 0 lines of the second type, and 3 lines of the third type, and so on. But because this is a symmetric mixed combination, it follows that there are 4 lines of the first type, incident with 0 points of the first type, 0 points of the second type, and 3 points of the third type, and so on. This raises the possibility of studying such symmetric mixed configurations more generally.

3.7 Weighted means for an example quadrangle

Now we analyse the same 13 mixed configuration using weighted means, but starting from a quadrangle $\overrightarrow{ABCD}$, then moving to the diagonal triangle $\overrightarrow{XYZ}$, and then to the opposite quadrilateral $\overrightarrow{abcd}$ as in Figure 7.

The shape of a quadrangle $\overrightarrow{ABCD}$ in affine geometry is determined by a single weighted relation. Suppose that the diagonal point $X = (AC)(BD)$ can be expressed in weighted form as $X^{\alpha+\gamma} = A^\alpha C^\gamma$ and also as $X^{\beta+\delta} = B^\beta D^\delta$. After scaling, we may assume that $\alpha+\gamma = \beta+\delta$. This gives the *standard vector relation* for the quadrangle $\overrightarrow{ABCD}$, which is

$$A^\alpha B^{-\beta} C^\gamma D^{-\delta} = 1.$$

This can be interpreted physically as meaning that weights of α, $-\beta$, γ and $-\delta$ on A, B, C and D respectively will result in a completely balanced plane — effectively the same as having no weights at all. Now commutativity and associativity allow us to rewrite this equation as either $A^{\alpha}B^{-\beta} = C^{-\gamma}D^{\delta}$ or $A^{\alpha}D^{-\delta} = B^{\beta}C^{-\gamma}$ which determines the positions of the other diagonal points $Y = (AB)(CD)$ and $Z = (AD)(BC)$.

We illustrate the situation with Figure 7 where $X^5 = B^4D^1$ and $X^3 = A^1C^2$ so $X^{15} = B^{12}D^3 = A^5C^{10}$ and the standard vector relation for $\overrightarrow{ABCD}$ is $A^5B^{-12}C^{10}D^{-3} = 1$. That implies that $A^{-5}B^{12} = C^{10}D^{-3} = Y^7$ and $A^5D^{-3} = B^{12}C^{-10} = Z^2$.

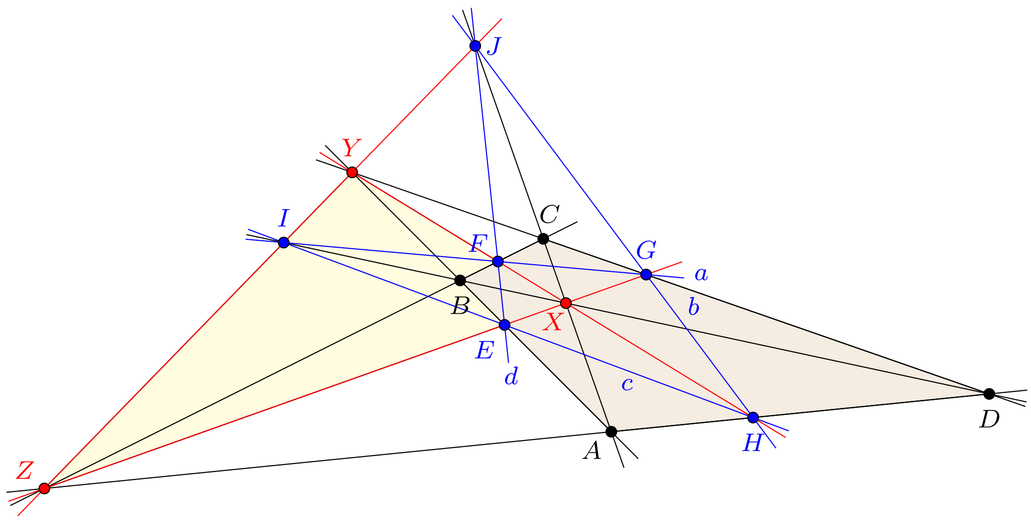

Figure 7: Quadrangle/quadrilateral configuration

In addition there are now additional points that we may obtain: for example $J = (YZ)(AC)$ so after rescaling we must have constants α, β, γ and δ, unique up to a scalar such that $J^{\alpha+\beta} = J^{\gamma+\delta} = A^{\alpha}C^{\beta} = Y^{\gamma}Z^{\delta}$. Substituting for Y and Z from the above equations we can write

$$A^{\alpha}C^{\beta} = \left(A^{-\frac{5\gamma}{7}}B^{\frac{12\gamma}{7}}\right)\left(A^{\frac{5\delta}{2}}D^{-\frac{3\delta}{2}}\right) \quad \text{or} \quad A^{\alpha+\frac{5\gamma}{7}-\frac{5\delta}{2}}B^{-\frac{12\gamma}{7}}C^{\beta}D^{\frac{3\delta}{2}} = 1$$

which establishes the relations

$$\alpha + \frac{5\gamma}{7} - \frac{5\delta}{2} = 5, \quad -\frac{12\gamma}{7} = -12, \quad \beta = 10, \quad \frac{3\delta}{2} = -3$$

with the unique solutions $\alpha = -5$,$\beta = 10$, $\gamma = 7$ and $\delta = -2$, so that $J^5 = A^{-5}C^{10} = Y^7Z^{-2}$.

It follows that $J^1 = A^{-1}C^2$ so that in fact C is the midpoint of $\overline{AJ}$. Recall also that $X^5 = B^4D^1$ and $X^3 = A^1C^2$ and so there is actually a pleasant relation between X and J, namely $X = [A^1C^2]$ and $J = [A^{-1}C^2]$, so they are conjugates with respect to $\overline{AC}$. The reader might like to calculate similarly the other points and weighted ratios in the diagram, and verify that other conjugate pairs arise similarly.

3.8 Relations in a general quadrangle

Let's see how this multiplicative algebra allows us to analyse the general quadrangle, simply by following the example above but with variables instead of numbers. These relations are quite useful in practice and ought to be more widely appreciated by students. We use the notation set out in Figure 7.

Theorem 15. *Suppose that we have the vector relation $A^\alpha B^\beta C^\gamma D^\delta = 1$ for the quadrangle $\overrightarrow{ABCD}$, with $\alpha+\beta+\gamma+\delta=0$. Then the diagonal points are*

$$Y^{\alpha+\beta} = A^\alpha B^\beta = C^{-\gamma} D^{-\delta}, \quad X^{\alpha+\gamma} = A^\alpha C^\gamma = B^{-\beta} D^{-\delta}, \quad Z^{\alpha+\delta} = A^\alpha D^\delta = B^{-\beta} C^{-\gamma}$$

and the further meets of the lines of the diagonal triangle with the lines of the quadrangle are

$$\begin{aligned} J^{\alpha-\gamma} &= A^\alpha C^{-\gamma} = Y^{\alpha+\beta} Z^{\alpha+\delta}, & I^{\beta-\delta} &= B^\beta D^{-\delta} = Y^{\alpha+\beta} Z^{-\alpha-\delta}, \\ G^{\gamma-\delta} &= C^\gamma D^{-\delta} = X^{\gamma+\alpha} Z^{\gamma+\beta}, & E^{\alpha-\beta} &= A^\alpha B^{-\beta} = X^{\gamma+\alpha} Z^{-\gamma-\beta}, \\ F^{\beta-\gamma} &= B^\beta C^{-\gamma} = X^{\beta+\alpha} Y^{\beta+\delta}, & H^{\alpha-\delta} &= A^\alpha D^{-\delta} = X^{\beta+\alpha} Y^{-\beta-\delta}. \end{aligned}$$

From these equations, we can quickly verify various conjugate pairs, such as (Y, Z) and (I, J), or (C, D) and (Y, G) etc. In fact each of the three red lines and each of the four black lines has a conjugate pair of points on it, so there are 7 such conjugate pairs over all.

3.9 Extensions to spherical and hyperbolic geometries

Let us note that Menelaus' theorem was first formulated in the spherical case, with the interest in astronomy in mind. The fact that this result holds in this context suggests the possibility of extending our Archimedean physical approach also to the geometry of the sphere, by establishing a similiar calculus of weighted means, with great circles playing the role of lines.

However clearly here the physical situation is somewhat different (likely we have to consider a central force to replace our uniform gravitational field). And then the challenge of formulating this also for hyperbolic geometry arises. These seem like interesting directions, especially as there will not be a familiar theory of linear algebra to connect to so readily.

Acknowledgments

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References

[1] M. Berger: *Geometry I.* Springer, Berlin, 4 ed., 2009. Translated by M. Cole and S. Levy.

[2] W. Burnside: *On A Configuration of 21 Points and 21 Lines Which Arises From the Complete Quadrilateral and Determines the Group of 168 Plane Collineations.* Proc. London Math. Soc. **14**(1), 106–110, 1915. doi: `10.1112/plms/s2_14.1.106`.

[3] H. S. M. Coxeter: *Self-Dual Configurations and Regular Graphs.* Bull. Amer. Math. Soc. **56**(5), 413–455, 1950.

[4] H. S. M. Coxeter: *My Graph.* Proc. London Math. Soc. **46**(1), 117–136, 1983. doi: `10.1112/plms/s3-46.1.117`.

[5] H. S. M. Coxeter: *Projective Geometry.* Springer-Verlag, New York, 2 ed., 1987.

[6] H. S. M. Coxeter: *Introduction to Geometry.* John Wiley & Sons, New York, 2 ed., 1989.

[7] H. S. M. Coxeter and S. L. Greitzer: *Geometry Revisited*. Mathematical Association of America, Washington, 1967.

[8] H. Gropp: *Enumeration of regular graphs 100 years ago*. Discrete Math **101**(1-3), 73–85, 1992. doi: 10.1016/0012-365x(92)90592-4.

[9] H. Gropp: *Configurations and their realization*. Discrete Math. **174**(1–3), 137–151, 1997. doi: 10.1016/s0012-365x(96)00327-5.

[10] B. Grünbaum and J. F. Rigby: *The Real Configuration* (21_4). J. London Math. Soc. **41**(2), 336–346, 1990. doi: 10.1112/jlms/s2-41.2.336.

[11] J. W. P. Hirschfeld: *Projective Geometries Over Finite Fields*. Clarendon Press, Oxford, 1979.

[12] F. Klein: *On the order-seven transformation of elliptic functions*. In *The Eightfold Way*, vol. 35, 287–331. Mathematical Sciences Research Institute Publications, 1999. Translated by S. Levy.

[13] F. Levy: *Geometrische Konfigurationen mit einer Einführung in die kombinatorische Flächentopologie*. Verlag von S. Hirzel, Leipzig, 1929.

[14] V. V. Prasolov and T. V. M.: *Geometry*. Amer. Math. Society, Rhode Island, 1997. Translated by O. V. Sipacheva.

[15] P. Scherk and R. Lingenberg: *Rudiments of Plane Affine Geometry*. University of Toronto Press, Toronto, 1975.

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