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Packings with geodesic and translation balls and their visualizations in $\widetilde{\mathrm{SL}_2\mathrm{R}}$ space

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To Professor Hellmuth Stachel on his 80th Birthday

Abstract. Remembering on our friendly cooperation between the Geometry Departments of Technical Universities of Budapest and Vienna (also under different names) a nice topic comes into consideration: the "Gum fibre model" (see Fig. 1).

One point of view is the so-called kinematic geometry by Vienna colleagues, e.g., as in [10], but also in very general context. The other point is the so-called $\mathbf{H}^2\times\mathbf{R}$ geometry and $\widetilde{\mathbf{SL}_2\mathbf{R}}$ geometry where – roughly – two hyperbolic planes as circle discs are connected with gum fibres, first: in a simple way, second: in a twisted way. This second homogeneous (Thurston) geometry will be our topic (initiated by Budapest colleagues, and discussed also in international cooperations).

We use for the computation and visualization of $\widetilde{\mathbf{SL}_2\mathbf{R}}$ its projective model, as in our previous papers [1–6, 8, 11, 12]. We found seemingly extremal geodesic ball packing for $\widetilde{\mathbf{SL}_2\mathbf{R}}$ group $\mathbf{pq}_k\mathbf{o}_\ell$ ($p=9$, $q=3$, $k=1$, $o=2$, $\ell=1$) with density ≈ 0.787758 (Table 2). Much better translation ball packing is for group $\mathbf{pq}_k\mathbf{o}_\ell$ ($p=11$, $q=3$, $k=1$, $o=2$, $\ell=1$) with density ≈ 0.845306 (Table 3).

Key Words: Thurston geometries, $\widetilde{\mathbf{SL}_2\mathbf{R}}$ geometry, density of ball packing under space group, regular prism tiling, volume in $\widetilde{\mathbf{SL}_2\mathbf{R}}$ space
MSC 2020: 51C17 (primary), 52C22, 52B15, 53A35, 51M20

1 Introduction

We received the nice Austro-Hungarian tradition of descriptive and projective geometry from our forerunners: Professors Walter Wunderlich and Gyula (Julius) Strommer. Then we continued in a conference series "Konstruktive Geometrie" jointly organized by Katalin Bognár Máthé, Emil Molnár, Hellmuth Stachel, Márta Szilvási Nagy (1993–2005, in Hotel Jogar, Balatonföldvár at lake Balaton). Then other Austrian, Croatian, Czechian, German, Hungarian,

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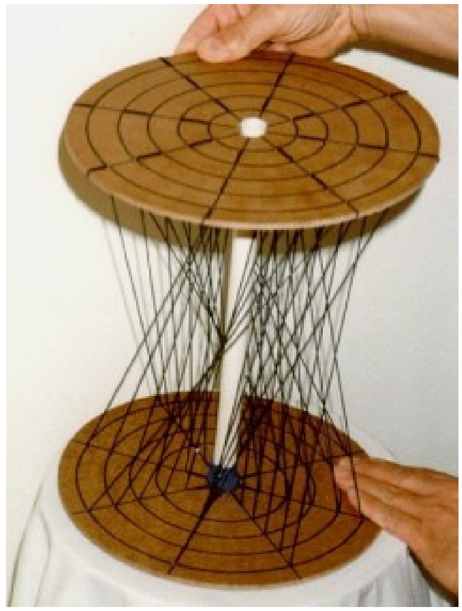

Figure 1: The unparted hyperboloid model of $\widetilde{\mathbf{SL_2R}} = \widetilde{\mathcal{H}}$ of skew line fibres growing in points of a hyperbolic base plane $\mathbf{H}^2$. Gum-fibre model of Hans Havlicek and Rolf Riesinger, used also by Hellmuth Stachel with other respects [10] (Vienna UT).

Italian, Polish, Romanian, Serbian, Slovakian colleagues joined into our organization. The topic is also extended to geometry and graphics.

Nowadays we have the International Society of Geometry and Graphics with the next conference in 2022 in Sao Paolo. Visualization by computer is based in the classical and modernized projective geometry. So theoretical development is also important. Such a new development is $\widetilde{\mathbf{SL_2R}}$ geometry and its applications in differentiable manifolds, crystallography, so in material sciences. E.g., we found extremely good packing density ≈ 0.787758 for geodesic balls by the $\widetilde{\mathbf{SL_2R}}$ space group in Table 2. A much better one is ≈ 0.845306 for translation ball packing by another space group in Table 3.

We thank the Referee Colleague for the important improvements.

2 The projective model for $\widetilde{\mathbf{SL_2R}}$

Real 2×2 matrices $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ with unit determinant $ad - bc = 1$ constitute a Lie transformation group by the standard product operation, taken to act on row matrices as point coordinates

$$(z^0, z^1) \begin{pmatrix} d & b \\ c & a \end{pmatrix} = (z^0 d + z^1 c, z^0 b + z^1 a) = (w^0, w^1) \tag{2.1}$$

with

$$w = \frac{w^1}{w^0} = \frac{b + (z^1/z^0)a}{d + (z^1/z^0)c} = \frac{b + za}{d + zc},$$

$z = z^1/z^0$, on the complex projective line $\mathbb{C}^\infty$. This group is a 3-dimensional manifold, because of its 3 independent real coordinates and with its usual neighbourhood topology [9, 13]. In order to model the above structure in the projective sphere $\mathcal{PS}^3$ and in the projective space $\mathcal{P}^3$ (see [2]), we introduce the new projective coordinates (x^0, x^1, x^2, x^3) where

$$a := x^0 + x^3, \quad b := x^1 + x^2, \quad c := -x^1 + x^2, \quad d := x^0 - x^3,$$

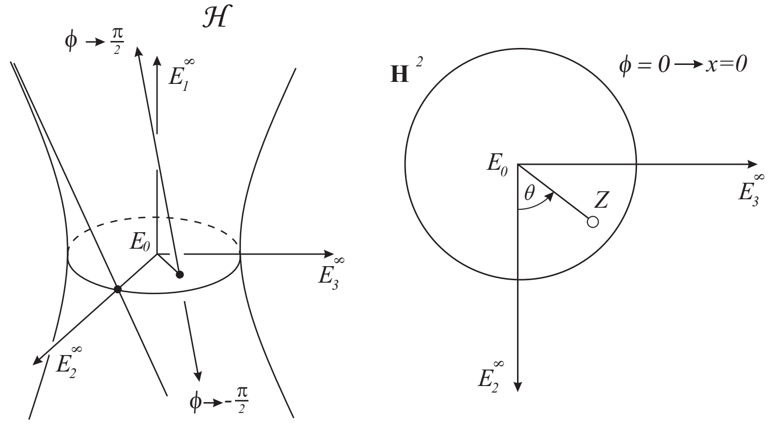

Figure 2: The hyperboloid model.

with positive, then the non-zero multiplicative equivalence as a projective freedom in $\mathcal{PS}^3$ and in $\mathcal{P}^3$, respectively. Then it follows that

$$0 > bc - ad = -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 \tag{2.2}$$

describes the interior of the above one-sheeted hyperboloid solid $\mathcal{H}$ in the usual Euclidean coordinate simplex, with the origin $E_0(1;0;0;0)$ and the ideal points of the axes $E_1^\infty(0;1;0;0)$, $E_2^\infty(0;0;1;0)$, $E_3^\infty(0;0;0;1)$. We consider the collineation group $\mathbf{G}_*$ that acts on the projective sphere $\mathcal{PS}^3$ and preserves a polarity, i.e. a scalar product of signature $(--++)$, as in (2.2). This group leaves the one sheeted hyperboloid solid $\mathcal{H}$ invariant. We have to choose an appropriate subgroup $\mathbf{G}$ of $\mathbf{G}_*$ as isometry group, then the universal covering group and space $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ will be the hyperboloid model of $\widetilde{\mathbf{SL_2R}}$ (see Fig. 1, 2 and [2]).

Consider isometries given by matrices

$$S(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \tag{2.3}$$

where $\phi \in [0; 2\pi)$. They constitute a one parameter group which we denote by $\mathbf{S}(\phi)$. The elements of $\mathbf{S}(\phi)$ are the so-called *fibre translations*. We obtain a unique fibre line to each $X(x^0;x^1;x^2;x^3) \in \widetilde{\mathcal{H}}$ as the orbit by right action of $\mathbf{S}(\phi)$ on X. The coordinates of points, lying on the fibre line through X, can be expressed as the images of X by $\mathbf{S}(\phi)$:

$$\begin{gathered}(x^0;x^1;x^2;x^3) \xrightarrow{\mathbf{S}(\phi)} (x^0\cos\phi - x^1\sin\phi; x^0\sin\phi + x^1\cos\phi; \\ x^2\cos\phi + x^3\sin\phi; -x^2\sin\phi + x^3\cos\phi).\end{gathered} \tag{2.4}$$

In (2.3) and (2.4) we can see the 2π-periodicity by ϕ. It admits us to introduce the extension to $\phi \in \mathbb{R}$, as real parameter, to obtain the universal covers $\widetilde{\mathcal{H}}$ and $\widetilde{\mathbf{SL_2R}}$, respectively, through the projective sphere $\mathcal{PS}^3$. The elements of the isometry group of $\mathrm{SL}_2(\mathbb{R})$ (and so by the above extension the isometries of $\widetilde{\mathbf{SL_2R}}$) can be described by the matrix (a_i^j) (see [2, 4])

$$(a_i^j) = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 & a_0^3 \\ \mp a_0^1 & \pm a_0^0 & \pm a_0^3 & \mp a_0^2 \\ a_2^0 & a_2^1 & a_2^2 & a_2^3 \\ \pm a_2^1 & \mp a_2^0 & \mp a_2^3 & \pm a_2^2 \end{pmatrix}, \tag{2.5}$$

where

$$\begin{cases} -(a_0^0)^2 - (a_0^1)^2 + (a_0^2)^2 + (a_0^3)^2 = -1, \\ -(a_2^0)^2 - (a_2^1)^2 + (a_2^2)^2 + (a_2^3)^2 = 1, \\ \quad -a_0^0 a_2^0 - a_0^1 a_2^1 + a_0^2 a_2^2 + a_0^3 a_2^3 = 0, \\ \quad -a_0^0 a_2^1 + a_0^1 a_2^0 - a_0^2 a_2^3 + a_0^3 a_2^2 = 0, \end{cases}$$

and we allow positive proportionality, of course, as projective freedom.

We define the *translation group* $\mathbf{G}_T$, as a subgroup of the isometry group of $\mathrm{SL}_2(\mathbb{R})$, those isometries acting transitively on the points of $\mathcal{H}$ and by the above extension on the points of $\widetilde{\mathcal{H}}$. $\mathbf{G}_T$ maps the origin $E_0(1;0;0;0)$ onto $X(x^0;x^1;x^2;x^3) \in \mathcal{H}$. These isometries and their inverses (up to a positive determinant factor) can be given by

$$\mathbf{T} = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix}, \quad \mathbf{T}^{-1} = \begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ -x^2 & -x^3 & x^0 & -x^1 \\ -x^3 & x^2 & x^1 & x^0 \end{pmatrix} \tag{2.6}$$

Horizontal intersection of the hyperboloid solid $\mathcal{H}$ with the plane $E_0E_2^\infty E_3^\infty$ provides the $\mathbf{H}^2$ hyperbolic *base plane* of the model $\widetilde{\mathcal{H}} = \widetilde{\mathbf{SL}_2\mathbf{R}}$.

We generally introduce a so-called hyperboloid parametrization by [2] as follows

$$\begin{cases} x^0 = \cosh r \cos\phi, \\ x^1 = \cosh r \sin\phi, \\ x^2 = \sinh r \cos(\theta-\phi), \\ x^3 = \sinh r \sin(\theta-\phi), \end{cases} \tag{2.7}$$

where (r,θ) are the polar coordinates of the $\mathbf{H}^2$ base plane, and ϕ is the fibre coordinate. We note that

$$-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 = -\cosh^2 r + \sinh^2 r = -1 < 0.$$

The inhomogeneous coordinates, which will play an important role in the later $\mathbf{E}^3$-visualization of the prism tilings in $\widetilde{\mathbf{SL}_2\mathbf{R}}$, are given by

$$\begin{cases} x := \dfrac{x^1}{x^0} = \tan\phi, \\ y := \dfrac{x^2}{x^0} = \tanh r \dfrac{\cos(\theta-\phi)}{\cos\phi}, \\ z := \dfrac{x^3}{x^0} = \tanh r \dfrac{\sin(\theta-\phi)}{\cos\phi}. \end{cases} \tag{2.8}$$

3 Geodesic curves and geodesic balls

The infinitesimal arc-length-square can be derived by the standard pull back method. By T^{-1}-action presented by (2.6) on differentials $(\mathrm{d}x^0;\mathrm{d}x^1;\mathrm{d}x^2;\mathrm{d}x^3)$, we obtain the infinitesimal arc-length-square at any point of $\widetilde{\mathbf{SL}_2\mathbf{R}}$ in coordinates (r,θ,ϕ):

$$(\mathrm{d}s)^2 = (\mathrm{d}r)^2 + \cosh^2 r \sinh^2 r (\mathrm{d}\theta)^2 + \Big[(\mathrm{d}\phi) + \sinh^2 r (\mathrm{d}\theta)\Big]^2.$$

Hence we get the symmetric metric tensor field g_{ij} on $\widetilde{\mathbf{SL_2R}}$ by components:

$$g_{ij} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r(\sinh^2 r + \cosh^2 r) & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}. \tag{3.1}$$

Therefore,

$$\mathrm{d}\,\mathrm{vol} := \sqrt{\det(g_{ij})}\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi = \frac{1}{2}\sinh(2r)\mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi \tag{3.2}$$

is the volume element in hyperboloid coordinates.

The geodesic curves of $\widetilde{\mathbf{SL_2R}}$ are generally defined as having locally minimal arc length between any two of their (close enough) points.

By (3.1) the second order Levi-Civita differential equation system of the $\widetilde{\mathbf{SL_2R}}$ geodesic curve is the following:

$$\begin{cases} \ddot{r} = \sinh(2r)\,\dot{\theta}\,\dot{\phi} + \frac{1}{2}\Big[\sinh(4r) - \sinh(2r)\Big]\dot{\theta}\,\dot{\theta}, \\ \ddot{\theta} = \dfrac{-2\dot{r}}{\sinh(2r)}\Big[(3\cosh(2r) - 1)\dot{\theta} + 2\dot{\phi}\Big], \\ \ddot{\phi} = 2\dot{r}\tanh(r)\,[2\sinh^2(r)\,\dot{\theta} + \dot{\phi}]. \end{cases} \tag{3.3}$$

We can assume, by the homogeneity, that the starting point of a geodesic curve is the origin $(1,0,0,0)$. Moreover,

$$\begin{cases} r(0) = 0, \\ \theta(0) = 0, \\ \phi(0) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{r}(0) = \cos(\alpha), \\ \dot{\theta}(0) = -\sin(\alpha), \\ \dot{\phi}(0) = \sin(\alpha) \end{cases}$$

are the initial values in Table 1 for the solution of (3.3), and so the unit velocity will be achieved (see details in [1]). The solutions are parametrized by the arc-length s and the angle α from the initial condition: $(r(s,\alpha), \theta(s,\alpha), \phi(s,\alpha))$.

Table 1: Geodesic curves.

direction	parametrization of a geodesic curve
$0 \le \alpha < \frac{\pi}{4}$ ($\mathbf{H}^2$-like)	$r(s,\alpha) = \operatorname{arsinh}\Big(\frac{\cos\alpha}{\sqrt{\cos 2\alpha}}\sinh(s\sqrt{\cos 2\alpha})\Big)$ $\theta(s,\alpha) = -\arctan\Big(\frac{\sin\alpha}{\sqrt{\cos 2\alpha}}\tanh(s\sqrt{\cos 2\alpha})\Big)$ $\phi(s,\alpha) = 2s\sin\alpha + \theta(s,\alpha)$
$\alpha = \frac{\pi}{4}$ (light-like)	$r(s) = \operatorname{arsinh}\Big(\frac{\sqrt{2}}{2}s\Big)$ $\theta(s) = -\arctan\Big(\frac{\sqrt{2}}{2}s\Big)$ $\phi(s) = \sqrt{2}s + \theta(s)$
$\frac{\pi}{4} < \alpha \le \frac{\pi}{2}$ (fibre-like)	$r(s,\alpha) = \operatorname{arsinh}\Big(\frac{\cos\alpha}{\sqrt{-\cos 2\alpha}}\sin(s\sqrt{-\cos 2\alpha})\Big)$ $\theta(s,\alpha) = -\arctan\Big(\frac{\sin\alpha}{\sqrt{-\cos 2\alpha}}\tan(s\sqrt{-\cos 2\alpha})\Big)$ $\phi(s,\alpha) = 2s\sin\alpha + \theta(s,\alpha)$

The parametrization of a geodesic curve in the hyperboloid model with the geographical sphere coordinates (λ, α), as longitude and altitude, $(-\pi < \lambda \leq \pi, \ -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2})$, and the arc-length parameter $s \geq 0$, has been determined in [1]. The Euclidean coordinates $X(s,\lambda,\alpha)$, $Y(s,\lambda,\alpha)$, $Z(s,\lambda,\alpha)$ of the geodesic curves can be written by substituting the results of Table 1 (see also [1]) into formula (2.8) as follows

$$\begin{cases} X(s,\lambda,\alpha) = \tan\left(\phi(s,\alpha)\right), \\ Y(s,\lambda,\alpha) = \dfrac{\tanh\left(r(s,\alpha)\right)}{\cos\left(\phi(s,\alpha)\right)} \cos\Big[\theta(s,\alpha) - \phi(s,\alpha) + \lambda\Big], \\ Z(s,\lambda,\alpha) = \dfrac{\tanh\left(r(s,\alpha)\right)}{\cos\left(\phi(s,\alpha)\right)} \sin\Big[\theta(s,\alpha) - \phi(s,\alpha) + \lambda\Big]. \end{cases} \tag{3.4}$$

As a standard, the *geodesic distance* $d(P,Q)$ between points $P, Q \in \widetilde{\mathbf{SL_2R}}$ is defined as the arc length of the geodesic curve from P to Q.

The *geodesic sphere* centered in P of radius ρ is the set $S_P(\rho) = \{Q \in \widetilde{\mathbf{SL_2R}} : d(P,Q) = \rho\}$. Moreover, we require that the geodesic sphere is a simply connected surface without selfintersection. Fig. 3 shows the geodesic and the next translation halfspheres (left and right, respectively) with $E_0 = (1,0,0,0)$ as centre, and with the three "direction-domains" by Table 1.

The *geodesic ball* centered in P of radius ρ is the set $B_P(\rho) = \{Q \in \widetilde{\mathbf{SL_2R}} : d(P,Q) \leq \rho\}$. It follows from (3.4) that sphere of radius ρ, where $\rho \in [0, \frac{\pi}{2})$, is a simply connected surface in $\mathbb{E}^3$ and $\widetilde{\mathbf{SL_2R}}$ both. If $\rho \geq \frac{\pi}{2}$ then the universal cover should be discussed. Thus, we will consider geodesic spheres and balls with radii $\rho \in [0, \frac{\pi}{2})$ only.

The following theorem (see [5]) gives the volume formula for a geodesic ball $B(\rho)$.

Theorem 3.1. *The following formula holds:*

$$\begin{aligned} \mathrm{vol}(B(\rho)) = 4\pi \int_0^{\rho} \int_0^{\frac{\pi}{4}} \frac{1}{2} \sinh(2r(s,\alpha)) \cdot |J_1| \; d\alpha \; ds \\ + 4\pi \int_0^{\rho} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} \sinh(2r(s,\alpha)) \cdot |J_2| \; d\alpha \; ds \end{aligned} \tag{3.5}$$

where Jacobian $|J_1| = \left|\dfrac{\partial(r,\phi)}{\partial(s,\alpha)}\right|$ *corresponds to the case* $0 \leq \alpha \leq \pi/4$ *in Table 1, and Jacobian* $|J_2| = \left|\dfrac{\partial(r,\phi)}{\partial(s,\alpha)}\right|$ *corresponds to the case* $\pi/4 \leq \alpha \leq \pi/2$ *in Table 1.*

Namely, in hyperboloid coordinates (r,θ,ϕ) the volume element is given by (3.2). The connection between the hyperboloid coordinates (r,θ,ϕ) and the geographical coordinates (s,λ,α) is presented by Table 1 and Formula (3.4).

4 Translation curves and translation balls

We recall briefly some basic facts about translation curves in $\widetilde{\mathbf{SL_2R}}$ following [3, 4, 8]. For any point $X(x^0;x^1;x^2;x^3) \in \mathcal{H}$ (and later also for points in $\widetilde{\mathcal{H}}$) the *translation map* from the origin $E_0(1;0;0;0)$ to X is defined by the *translation matrix* T and its inverse presented in (2.6).

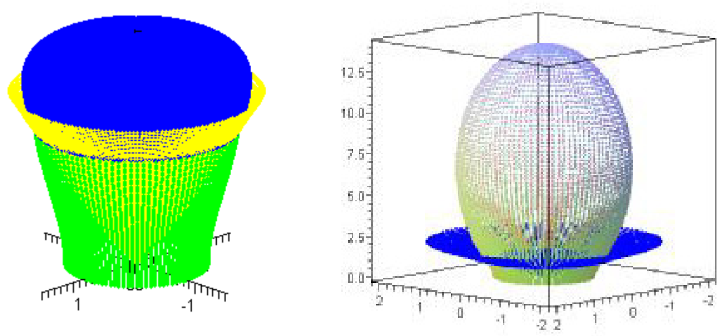

Figure 3: The geodesic ($R = 1.2$) and the translation ($R = 1.5$) halfspheres with "light-cones" in $\widetilde{\mathbf{SL_2R}}$ space.

Let us consider for a given vector $(q; u; v; w)$ a curve $\mathcal{C}(t) = (x^0(t); x^1(t); x^2(t); x^3(t))$, $t \geq 0$, in $\mathcal{H}$ starting at the origin: $\mathcal{C}(0) = E_0(1; 0; 0; 0)$ and such that

$$\dot{\mathcal{C}}(0) = (\dot{x}^0(0); \dot{x}^1(0); \dot{x}^2(0); \dot{x}^3(0)) = (q; u; v; w),$$

where $\dot{\mathcal{C}}(t) = (\dot{x}^0(t); \dot{x}^1(t); \dot{x}^2(t); \dot{x}^3(t))$ is the tangent vector at any point of the curve. For $t \geq 0$ there exists a matrix

$$T(t) = \begin{pmatrix} x^0(t) & x^1(t) & x^2(t) & x^3(t) \\ -x^1(t) & x^0(t) & x^3(t) & -x^2(t) \\ x^2(t) & x^3(t) & x^0(t) & x^1(t) \\ x^3(t) & -x^2(t) & -x^1(t) & x^0(t) \end{pmatrix}$$

which defines the translation from $\mathcal{C}(0)$ to $\mathcal{C}(t)$:

$$\mathcal{C}(0) \cdot T(t) = \mathcal{C}(t), \quad t \geqslant 0.$$

The t-parametrized family $T(t)$ of translations is used in the following definition.

Definition 4.1. *The curve $\mathcal{C}(t)$, $t \geqslant 0$, is said to be a* translation curve *if*

$$\dot{\mathcal{C}}(0) \cdot T(t) = \dot{\mathcal{C}}(t), \quad t \geqslant 0.$$

In coordinates this relation can be written as follows: (more consequently)

$$(q; u; v; w) \cdot \begin{pmatrix} x^0(t) & x^1(t) & x^2(t) & x^3(t) \\ -x^1(t) & x^0(t) & x^3(t) & -x^2(t) \\ x^2(t) & x^3(t) & x^0(t) & x^1(t) \\ x^3(t) & -x^2(t) & -x^1(t) & x^0(t) \end{pmatrix} = (\dot{x}^0(t); \dot{x}^1(t); \dot{x}^2(t); \dot{x}^3(t)).$$

By rearranging the left part and introducing

$$Q = \begin{pmatrix} q & u & v & w \\ -u & q & -w & v \\ v & -w & q & -u \\ w & v & u & q \end{pmatrix},$$

we get

$$\mathcal{C}(t) \cdot Q = \dot{\mathcal{C}}(t), \quad t \geqslant 0. \tag{4.1}$$

This equation is said to be the *translation dynamical system*.

From the characteristic equation $\det(Q - \lambda I) = 0$, having the form

$$[(q-\lambda)^2 - (-u^2 + v^2 + w^2)]^2 = 0,$$

one can find solutions of (4.1), depending on the starting direction (q, u, v, w). It was done in [3], in details, where the straight line solutions split into the following three cases (compare also with Table 1).

Case I: $-\mathbf{u^2} + \mathbf{v^2} + \mathbf{w^2} > \mathbf{0}$. Let $0 < a \in \mathbb{R}$ be such that $a^2 = -u^2 + v^2 + w^2$. Then any solution of (4.1) can be presented as

$$(x^0(t); x^1(t); x^2(t); x^3(t)) = e^{qt}\left(\cosh(at); \frac{u}{a}\sinh(at); \frac{v}{a}\sinh(at); \frac{w}{a}\sinh(at)\right),$$

hence in Euclidean coordinates:

$$x(t) = \frac{u}{a}\tanh(at), \qquad y(t) = \frac{v}{a}\tanh(at), \qquad z(t) = \frac{w}{a}\tanh(at). \tag{4.2}$$

This solution can be extended for all $t \in \mathbb{R}$ along a straight line in $\widetilde{\mathbf{SL_2R}}$ (a segment of the hyperboloid solid $\mathcal{H}$). This is the case of a $\mathbb{H}^2$–direction curve.

Case II: $-\mathbf{u^2} + \mathbf{v^2} + \mathbf{w^2} < \mathbf{0}$. Let $0 < a \in \mathbb{R}$ be such that $-a^2 = -u^2 + v^2 + w^2$. Then any solution of (4.1) can be presented as

$$(x^0(t); x^1(t); x^2(t); x^3(t)) = e^{qt}\left(\cos(at); \frac{u}{a}\sin(at); \frac{v}{a}\sin(at); \frac{w}{a}\sin(at)\right),$$

hence in Euclidean coordinates:

$$x(t) = \frac{u}{a}\tan(at), \qquad y(t) = \frac{v}{a}\tan(at), \qquad z(t) = \frac{w}{a}\tan(at). \tag{4.3}$$

This solution can be extended for all $t \in \mathbb{R}$ along a straight line in $\widetilde{\mathbf{SL_2R}}$ (in the sense of the universal cover $\widetilde{\mathcal{H}}$ of the hyperboloid $\mathcal{H}$ in $\mathcal{PS}^3$). This is a fibre-direction curve.

Case III: $-\mathbf{u^2} + \mathbf{v^2} + \mathbf{w^2} = \mathbf{0}$. This case corresponds to an asymptotic direction. The unique solution of (4.1) is a straight line on the asymptotic cone of $\mathcal{H}$, passing through the origin E_0 and ideal points $(0; u; v; w)$ and $(0; -u; -v; -w)$ in the sense of $\mathcal{PS}^3$. In the Euclidean coordinates we have

$$x(t) = ut. \qquad y(t) = vt, \qquad z(t) = wt. \tag{4.4}$$

As we see, the translation curves are straight lines in the projective embedding of $\widetilde{\mathbf{SL_2R}}$. Summarizing the above cases the following result holds.

Proposition 4.2. *[3, 4, 8] Translation curves in the one-parted hyperboloid model of* $\widetilde{\mathbf{SL_2R}}$*-geometry are characterized by formulae* (4.2), (4.3), *and* (4.4).

It was observed above that for any $X(x^0; x^1; x^2; x^3) \in \widetilde{\mathcal{H}}$ there is a suitable transformation T^{-1}, given by (2.6), which sent X to the origin E_0 along a translation curve.

Definition 4.3. *A translation distance $\rho(E_0, X)$ between the origin $E_0(1;0;0;0)$ and the point $X(1;x;y;z)$ is the length of a translation curve connecting them.*

For a given translation curve $\mathcal{C} = \mathcal{C}(t)$ the initial unit tangent vector (u, v, w) (in Euclidean coordinates) at E_0 can be presented as

$$u = \sin\alpha, \quad v = \cos\alpha\cos\lambda, \quad w = \cos\alpha\sin\lambda, \tag{4.5}$$

for some $-\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}$ and $-\pi < \lambda \leqslant \pi$. In $\widetilde{\mathcal{H}}$ this vector is of length square $-u^2+v^2+w^2 = \cos 2\alpha$. We can always assume that $\mathcal{C}$ is parametrized by the translation arc-length parameter $t = s \geqslant 0$. Then coordinates of a point $X(x;y;z)$ of $\mathcal{C}$, such that the translation distance between E_0 and X equals s, depend on (λ, α, s) as geographic coordinates.

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cosh(s\sqrt{\cos 2\alpha}) \\ \frac{\sin\alpha}{\sqrt{\cos 2\alpha}}\sinh(s\sqrt{\cos 2\alpha}) \\ \frac{\cos\alpha\cos\lambda}{\sqrt{\cos 2\alpha}}\sinh(s\sqrt{\cos 2\alpha}) \\ \frac{\cos\alpha\cos\lambda}{\sqrt{\cos 2\alpha}}\sinh(s\sqrt{\cos 2\alpha}) \end{pmatrix} \sim \begin{pmatrix} 1 \\ \frac{\sin\alpha}{\sqrt{\cos 2\alpha}}\tanh(s\sqrt{\cos 2\alpha}) \\ \frac{\cos\alpha\cos\lambda}{\sqrt{\cos 2\alpha}}\tanh(s\sqrt{\cos 2\alpha}) \\ \frac{\cos\alpha\sin\lambda}{\sqrt{\cos 2\alpha}}\tanh(s\sqrt{\cos 2\alpha}) \end{pmatrix}$$

for $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$; and

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cos(s\sqrt{-\cos 2\alpha}) \\ \frac{\sin\alpha}{\sqrt{-\cos 2\alpha}}\sin(s\sqrt{-\cos 2\alpha}) \\ \frac{\cos\alpha\cos\lambda}{\sqrt{-\cos 2\alpha}}\sin(s\sqrt{-\cos 2\alpha}) \\ \frac{\cos\alpha\sin\lambda}{\sqrt{-\cos 2\alpha}}\sin(s\sqrt{-\cos 2\alpha}) \end{pmatrix} \begin{matrix} 1 \\ \frac{\sin\alpha}{\sqrt{-\cos 2\alpha}}\tan(s\sqrt{-\cos 2\alpha}) \\ \frac{\cos\alpha\cos\lambda}{\sqrt{-\cos 2\alpha}}\tan(s\sqrt{-\cos 2\alpha}) \\ \frac{\cos\alpha\sin\lambda}{\sqrt{-\cos 2\alpha}}\tan(s\sqrt{-\cos 2\alpha}) \end{matrix}$$

for $-\frac{\pi}{2} < \alpha < -\frac{\pi}{4}$ and $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$; and

$$(x^0; x^1; x^2; x^3) = (1; s\sin\alpha; , s\cos\alpha\cos\lambda; s\cos\alpha\sin\lambda)$$

for $\alpha = -\frac{\pi}{4}$ or $\alpha = \frac{\pi}{4}$.

Consider $\widetilde{\mathcal{H}}$ with geographic coordinates (s, λ, α), where λ is the longitude, $-\pi < \lambda \leq \pi$, α is the altitude, $-\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}$, and s is the distance. Calculating volume of a region in $\widetilde{\mathcal{H}}$ we need to divide the region in two parts according to cases $0 \leq |\alpha| \leq \frac{\pi}{4}$ and $\frac{\pi}{4} \leq |\alpha| \leq \frac{\pi}{2}$ as above. For these two case the volume formulae looks different (see[8]).

Theorem 4.4. *For a region in $\widetilde{\mathcal{H}}$ with geographic coordinates (s, λ, α) the volume formula depends on a parameter α. If $0 \leq |\alpha| \leq \frac{\pi}{4}$ then the volume is given by*

$$\mathrm{vol}_1 := \int_\lambda \int_\alpha \int_s \frac{\cos(\alpha)\left(\sinh\left(s\sqrt{\cos(2\alpha)}\right)\right)^2}{\cos(2\alpha)}\,\mathrm{d}s\,\mathrm{d}\alpha\,\mathrm{d}\lambda. \tag{4.6}$$

If $\frac{\pi}{4} \leq |\alpha| \leq \frac{\pi}{2}$, then the volume is given by

$$\mathrm{vol}_2 := \int_\lambda \int_\alpha \int_s \frac{\cos(\alpha)\left(\sin\left(s\sqrt{-\cos(2\alpha)}\right)\right)^2}{-\cos(2\alpha)}\,\mathrm{d}s\,\mathrm{d}\alpha\,\mathrm{d}\lambda. \tag{4.7}$$

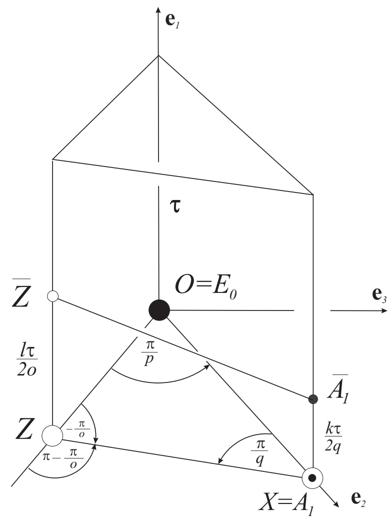

Figure 4: Points E_0, A_1, Z and the corresponding isometries.

5 The matrix presentation of groups $\mathbf{pq}_k\mathbf{o}_l$

In this section we find matrix presentations of a typical family of groups $\mathbf{pq}_k\mathbf{o}_\ell \subset \mathrm{Isom}(\widetilde{\mathbf{SL_2R}})$ given in [8] which we briefly reproduce here for the reader's convenience. The presentation (here better than representation) will be

$$\mathbf{pq}_k\mathbf{o}_\ell = \langle a, b, t | a^p = 1, b^q = t^k, (ab)^o = t^\ell, a^{-1}ta = t = b^{-1}tb \rangle. \tag{5.1}$$

That means this group is generated by a p-rotation $\mathbf{p}$ about the central fibre through $O = E_0$, then by $\mathbf{q}_k$ screw with q-rotation and $\frac{k}{q}$ translation, then by an $\mathbf{o}_l$ screw with o-rotation and $\frac{l}{o}$ translation, just by Euclidean analogy but exact projective computations.

Assume that p, q, $o \geq 2$. Let E_0, A_1, Z be such points that $E_0(1;0;0;0)$ is the origin, $A_1(\cosh r; 0; \sinh r; 0)$, and $Z(z^0; z^1 = 0; z^2; z^3)$ as in Fig. 4. The fibre translation $t(\tau)$ will be determined later.

For $X \in \widetilde{\mathbf{SL_2R}}$ we introduce two types of isometries. The first type is a rotation. Denote by $R_X(\omega)$ the ω–rotation about the fiber line through X. The ω–rotation, $-\pi \leq \omega \leq \pi$, about the fibre line through the origin E_0 can be expressed by

$$R_{E_0}(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega & \sin\omega \\ 0 & 0 & -\sin\omega & \cos\omega \end{pmatrix}.$$

For an arbitrary point $X(x^0; x^1; x^2; x^3)$ the rotation $R_X(\omega)$ can be obtained from $R_{E_0}(\omega)$ by the conjugacy $R_X(\omega) = \mathbf{T}^{-1} R_{E_0}(\omega) \mathbf{T}$, where the translation $\mathbf{T}$ from E_0 to X is given by (2.6). Observe that the fibre through X intersects the hyperbolic $\mathbf{H}^2$ base plane ($z^1 = x = 0$) in the foot point

$$Z(z^0 = x^0x^0 + x^1x^1; z^1 = 0; z^2 = x^0x^2 - x^1x^3; z^3 = x^0x^3 + x^1x^2).$$

If $-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 = -1$ stands, then the following formula holds

$$R_Z(\omega) = \begin{pmatrix} 1+(z^0-1) \\ \cdot(1-\cos\omega) & (z^0-1)\sin\omega & z^2(1-\cos\omega) \\ +z^3\sin\omega & -z^2\sin\omega \\ +z^3(1-\cos\omega) \\ (1-z^0)\sin\omega & \begin{matrix}1+(z^0-1)\\ \cdot(1-\cos\omega)\end{matrix} & \begin{matrix}-z^2\sin\omega\\ +z^3(1-\cos\omega)\end{matrix} & \begin{matrix}-z^2(1-\cos\omega)\\ -z^3\sin\omega\end{matrix} \\ \begin{matrix}-z^2(1-\cos\omega)\\ +z^3\sin\omega\end{matrix} & \begin{matrix}-z^2\sin\omega\\ -z^3(1-\cos\omega)\end{matrix} & \begin{matrix}1-z^0\\ \cdot(1-\cos\omega)\end{matrix} & z^0\cdot\sin\omega \\ \begin{matrix}-z^3(1-\cos\omega)\\ -z^2\sin\omega\end{matrix} & \begin{matrix}z^2(1-\cos\omega)\\ -z^3\sin\omega\end{matrix} & -z^0\cdot\sin\omega & \begin{matrix}1-z^0\\ \cdot(1-\cos\omega)\end{matrix} \end{pmatrix} \tag{5.2}$$

The second type of isometries is a screw motion. Recall that fiber translations $S(\phi)$ are given by (2.3). Denote by $H_X(\omega,\psi)$ a screw motion that is a composition of the ω-rotation about the fiber line through X with the fiber translation through ψ along the fiber line through X by a standard way as above and in [8].

E.g., we get a detail on the integer parameters p, q, k, o, ℓ to Fig. 4:

$$\begin{cases} \sinh^2 r = \dfrac{\cos(\pi/q+\pi/p-\varkappa\pi)-\cos(\pi(o-1)/o)}{2\sin(\pi/q)\sin(\pi/p-\varkappa\pi)} \\ z^0 = \dfrac{\cos(\pi/q)-\cos(\pi(o-1)/o+\pi/p-\varkappa\pi)}{2\sin(\pi(o-1)/o)\sin(\pi/p-\varkappa\pi)} \\ z^2 = \sinh r\cosh r\dfrac{\sin(\pi/q)}{\sin(\pi(o-1)/o)}\cos(\pi/p-\varkappa\pi), \\ z^3 = \sinh r\cosh r\dfrac{\sin(\pi/q)}{\sin(\pi(o-1)/o)}\sin(-\pi/p+\varkappa\pi), \\ \tau = \dfrac{\pi(o-1)/o-\pi/p-\pi/q+\varkappa\pi}{\ell/o-k/q}, \end{cases} \tag{5.3}$$

where $\varkappa$ is a non-negative integer parameter (also for the universal cover, not discussed here). The case $\varkappa = 0$ will play an important role below. These data lead to metric realizations in $\widetilde{\mathbf{SL_2R}}$ if $\pi > \pi/o+\pi/p+\pi/q$. and $l/o > k/q$ (compare with [9]). With corresponding matrix roles $\mathbf{A}\to a$, $\mathbf{B}\to b$, $\tau\to t$ the presentation (5.1) for the group $\mathbf{pq}_k\mathbf{o}_l$ holds indeed.

6 Prismatic tessellations and ball packings

In [5, 7, 11] we have defined and studied the regular prisms and their volumes. These results can be generalized to the considered tilings as well.

In this section we consider tilings $\mathcal{T}(p,(q,k),(o,\ell))$ for suitable integer positive parameters p, q, k, o, ℓ. Every tiling $\mathcal{T}$ is generated by discrete isometry group $\mathbf{pq}_k\mathbf{o}_\ell$ now for $o=2$, $l=1$.

Let $\mathcal{P}^b$ be a regular p-gon lying in the hyperbolic base plane. Following [11], we denote by $\mathcal{P}^i$ the infinite solid that is bounded by „side fibre lines” passing through the vertices of $\mathcal{P}^b$. The images of solids $\mathcal{P}^i$ by $\widetilde{\mathbf{SL_2R}}$ isometries are called *infinite regular p-sided prisms*. Here regular means that the side surfaces are congruent to each other under rotations about a fiber line (e.g. through the origin).

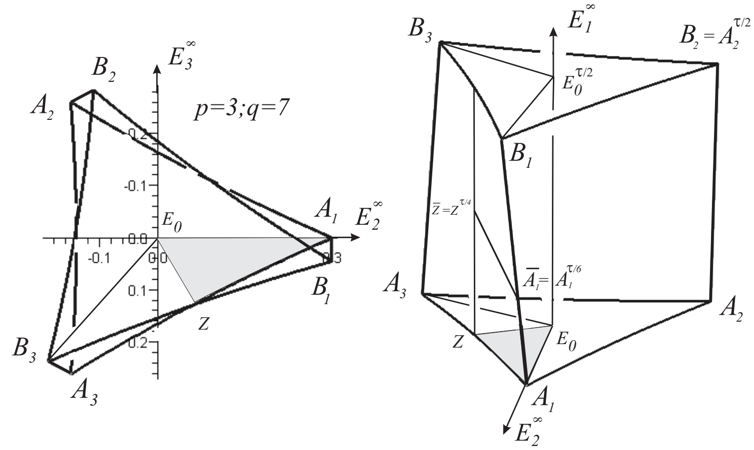

Figure 5: Bounded regular 3-sided prism related to parameters $p = 3$, $q = 7$.

The common part of $\mathcal{P}^i$ with the base plane is the *base figure* of $\mathcal{P}^i$ that is denoted by $\mathcal{P}$ and its vertices coincide with the vertices of $\mathcal{P}^b$, but $\mathcal{P}$ is not assumed to be a polygon.

A *bounded regular p-sided prism* is analogously defined if the face of the base figure $\mathcal{P}$ and its translated copy $\mathcal{P}^t$, under a fibre translation given by (2.3) and so (2.5), are also introduced. The faces $\mathcal{P}$ and $\mathcal{P}^t$ are called *cover faces*.

The volume formula of a *sector-like* 3-dimensional domain $\mathrm{vol}(D(\Phi))$ can be standardly computed by the metric tensor g_{ij} in hyperboloid coordinates (3.1). This is defined by the base figure D lying in the base plane (see Fig. 5) and by fibre translation τ given by (2.3) with the height parameter $\Phi = \tau$ (see [5] and [8]).

Theorem 6.1. *([5], [8]) Let D be a sector-like region in $\widetilde{\mathbf{SL_2R}}$ with the height parameter Φ and the radius $r = r(\theta)$ that is a continuous function of the polar angle $\theta \in [\theta_1, \theta_2]$. Then*

$$\mathrm{vol}(D(\Phi)) = \frac{\Phi}{4}\int_{\theta_1}^{\theta_2} (\cosh(2r(\theta)) - 1)\ \mathrm{d}\theta. \tag{6.1}$$

Theorem 6.2. *([5], [8]) Let $\mathcal{P}_p(q)$ be the bounded regular prism in $\widetilde{\mathbf{SL_2R}}$, where $p \geq 3$ and $q > \frac{2p}{p-2}$. Then*

$$\mathrm{vol}(\mathcal{P}_p(q)) = p \cdot \mathrm{vol}(D(p, q, \Phi)). \tag{6.2}$$

We indicate only one case of the multiply transitive geodesic or translation ball packings where the fundamental domains of the $\widetilde{\mathbf{SL_2R}}$ space groups $\mathbf{pq}_k\mathbf{o}_\ell$ are not prisms. Let the fundamental domains be derived by the Dirichlet-Voronoi cells, where their centers are images of the origin. The volume of the p-times fundamental domain and of the Dirichlet-Voronoi cell is the same, respectively, as in the prism case (for any above p, q, k, o, ℓ fixed).

These locally densest geodesic ball packings can be determined for all possible fixed integer parameters p, q, k, o, ℓ. The optimal radius ρ_{opt} is

$$\rho_{\mathrm{opt}} = \min\left\{\frac{\Phi}{2},\ \frac{d(E_0, ab(E_0))}{2}\right\}, \tag{6.3}$$

where $d(E_0, ab(E_0))$ is the geodesic or translation distance between E_0 and $ab(E_0)$.

The above presented formulae admit to compute the maximal density of the ball packings induced by the $\mathbf{pq}_k\mathbf{o}_l$ group action for any above parameters. In Tables 2 and 3 we have summarized some numerical results for $k = 1$, $o = 2$, $\ell = 1$, and small values of parameters p and q. The tables contain the optimal radius ρ_{opt}, the volume of the ball $B(\rho_{\text{opt}})$, the volume of the prism $\mathcal{P}_p$, and the packing density $\delta(\rho_{\text{opt}})$ that is the ratio of the preceding volumes.

Table 2: Geodesic ball packings with $k = 1, o = 2, \ell = 1$.

q	p	ρ_{opt}	$\text{vol}(B(\rho_{\text{opt}}))$	$\text{vol}(\mathcal{P}_p)$	$\delta(\rho_{\text{opt}})$
3	8	0.392699	0.266949	0.411234	0.635408
3	9	0.521044	0.647905	0.822467	**0.787758**
3	10	0.599849	1.017248	1.315947	0.773016
4	5	0.314159	0.134202	0.246740	0.543899
4	6	0.501354	0.573426	0.822467	0.697203
4	7	0.613204	1.092403	1.586186	0.688698
5	4	0.261799	0.076892	0.164493	0.467450
5	5	0.485013	0.516444	0.822467	0.627920
5	6	0.614925	1.102375	1.754596	0.628278

It can be seen from Table 2 that for $(k, o, \ell) = (1, 2, 1)$ the maximal density of translation balls packing $\delta(\rho_{\text{opt}}) = 0.787758$ corresponds to the case $(p, q) = (9, 3)$.

Table 3: Translation ball packings with $k = 1, o = 2, \ell = 1$.

q	p	ρ_{opt}	$\text{vol}(B(\rho_{\text{opt}}))$	$\text{vol}(\mathcal{P}_p)$	$\delta(\rho_{\text{opt}})$
3	9	0.523599	0.612686	0.822467	0.744936
3	10	0.628319	1.067834	1.315947	0.811456
3	11	0.713998	1.580082	1.869243	**0.845306**
4	5	0.314159	0.130745	0.246740	0.529888
4	6	0.523599	0.612686	0.822467	0.744936
4	7	0.673198	1.318939	1.586186	0.831516
5	4	0.261799	0.075508	0.164493	0.459033
5	5	0.523599	0.612686	0.822467	0.744936
5	6	0.698132	1.474648	1.754596	0.840449

It can be seen from Table 3 that for $(k, o, \ell) = (1, 2, 1)$ the maximal density of translation ball packing $\delta(\rho_{\text{opt}}) = 0.845306$ corresponds to the case $(p, q) = (11, 3)$. It can be conjectured that the limit maximal density of the translation balls packing generated by the action of the group $\mathbf{pq}_1\mathbf{2}_1$ corresponds to the case $(p, q) = (\infty, 3)$.

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