Journal for Geometry and Graphics
Volume 26 (2022), No. 1, 139–146

An Interpretation of the Truncated Triangular Trapezohedron and the Sphere Depicted in "Melencolia I" by Albrecht Dürer

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Abstract. The truncated triangular trapezohedron depicted in the engraving "Melencolia I" I by Albrecht Dürer is a rarely used polyhedron. In addition to the meaning of this engraving, various interpretations have been made about the meaning of this polyhedron. In the interpretations, the composition of the dimensions of the polyhedron and its relationship with the projection were mainly argued, and Dürer's direct intention was not argued. This paper focuses on the oblique axis contained in the truncated triangular trapezohedron, consider the relationship with what is drawn around it and the historical background, and try to interpret the intention hidden in this polyhedron by Dürer. The meaning of the drawn sphere was also positioned as an extension of the intention.

Key Words: truncated triangular trapezohedron, Albrecht Dürer, applied descriptive geometry, history of engraving, projection

MSC 2020: 01A40 (primary), 51N05

1 Introduction

Albrecht Dürer (hereafter, Dürer) is a painter and mathematician born in 1471 and died in 1528. Dürer has a deep connection with descriptive geometry, such as explaining the principle of perspective projection in an easy-to-understand manner in his book, "Underweysung der messung mit dem zirckel un richtscheyt in linien ebnen unnd gantzen corporen" [1]. The copper print *"Melencolia I"* produced by Dürer in 1514 depicts truncated triangular trapezohedron (hereafter TTT) and a sphere without any particular decorative elements. The polyhedron and the sphere in this work cannot be considered as depictions of real tools or creatures. In addition to the meaning of expressing this purely mathematical shape, there are various interpretations of the meaning of this polyhedron itself. However, they analyzed the composition of the dimensions of the polyhedron and its relationship with the projection, and their relationship with Dürer's direct intention was not argued. This paper focuses on the

ISSN 1433-8157/\$ 2.50

oblique axis contained in the truncated triangular trapezohedron, consider the relationship with what is drawn around it and the historical background, and try to interpret the intention hidden in this polyhedron by Dürer. The meaning of the drawn sphere was also positioned as an extension of the intention.

2 *"Melencolia I"*, Truncated Triangular Trapezohedron and Existing works

2.1 The composition of *"Melencolia I"*

Fig. 1, right shows outline of *"Melencolia I"* and Fig. 1, left shows the composition of *"Melencolia I".* In this way, in addition to the above-mentioned polyhedron and sphere, women and a boy with wings, tools (a compass, an hourglass, and a scale), a light source and a rainbow, magic square with 4 columns and 4 rows, and the character strings of the title of the engraving "MELENCOLIA I" are drawn.

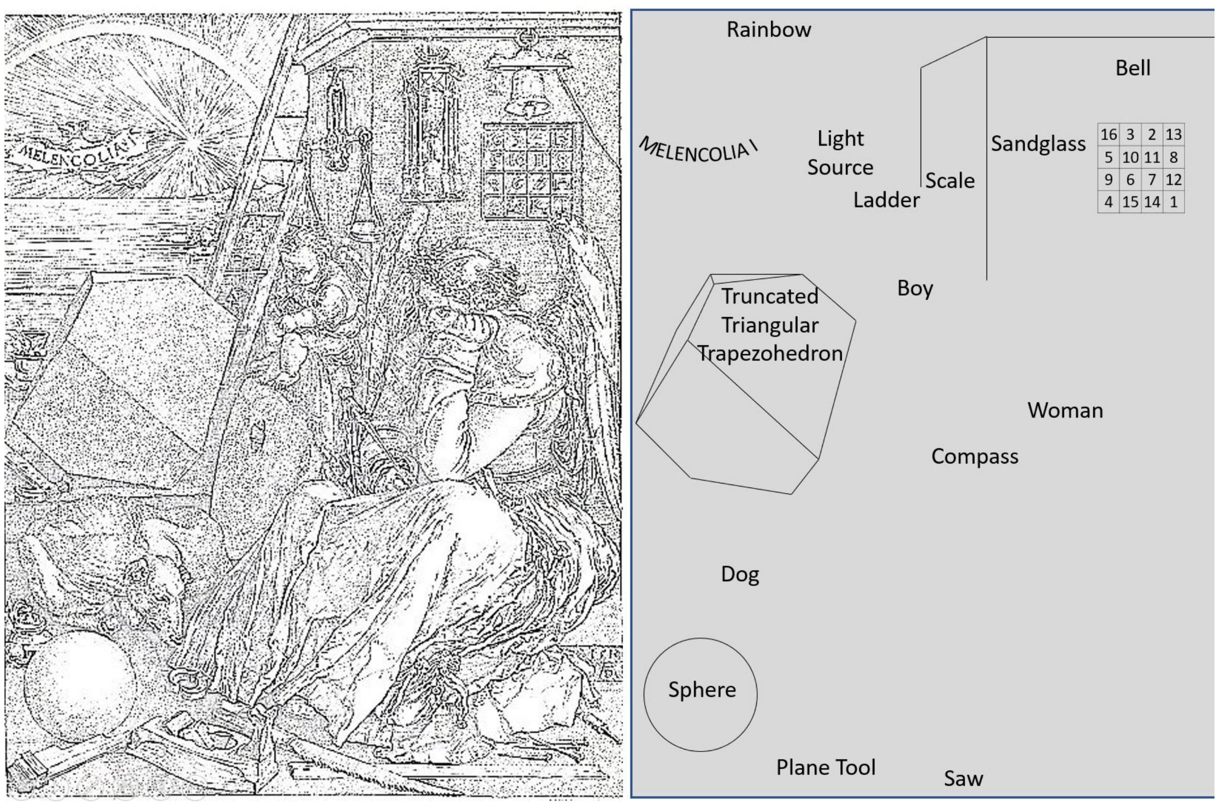

Figure 1: The outline of *"Melencolia I"* (left) and the composition of *"Melencolia I"* (right)

2.2 The explanation of truncated triangular trapezohedron

Of a TTT and a sphere which are focused on in this paper, the sphere is well-known shape, so the properties of the TTT are summarized here. Many researchers concluded that the polyhedron drawn in *"Melencolia I"* is a truncated rhombohedron [2, 3, 5–7, 10]. As shown in the Fig. 2, left, the rhombohedron is a shape composed of six congruent rhomboses. Of

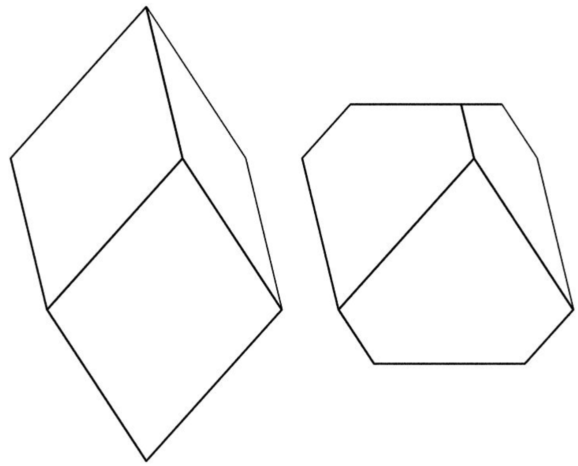

Figure 2: Rhombohedron (left) and truncated rhombohedron (right)

the two paired vertices of the rhombohedron, the farthest pair is cut off to obtain a truncated rhombohedron as shown in Fig. 2, right.

At the same time, the truncated rhombohedron can be considered as a TTT. When congruent two triangular pyramids with equilateral base are pasted together so that the bases of the two pyramids overlap, a triangular bipyramid with equilateral base is obtained (see Fig. 3, left).

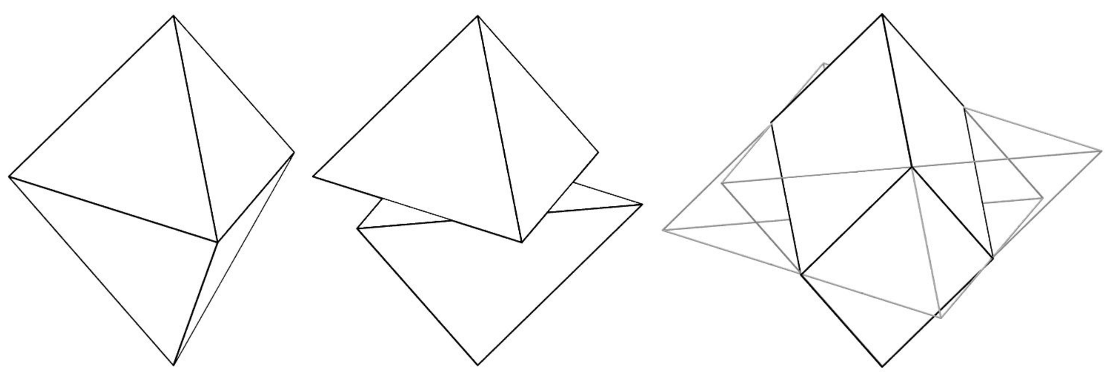

Figure 3: Triangular bipyramid with equilateral base (left), two triangular pyramids (center), and rhombohedron with extended faces of two triangular pyramids (right)

Here, if the straight line passing through the top and the bottom of the triangular bipyramid is used as the axis of rotation, and the upper and lower triangular pyramids are rotated so as to the location of the vertices of bases of two pyramids be in staggered position, two independent triangular pyramids are obtained as shown in the Fig. 3, center.

If the sides of each triangular pyramid are extended, a triangular trapezohedron with equilateral base is obtained and the polyhedron is corresponding to a rhombohedron as shown in the Fig. 3, right. Consequently, the truncated rhombohedron shown in the Fig. 2, right can also be considered as a TTT. A TTT is a shape in which triangular antiprism is sandwiched by two truncated triangular pyramids as well.

As shown on the Fig. 4, left, the paired polygons are parallel to each other. In the case of TTT, the polyhedron is an octahedron, and by putting a vertical line foot on each surface that constitutes TTT, as shown in Fig. 4, the four diagonal axes can be obtained. The polyhedron

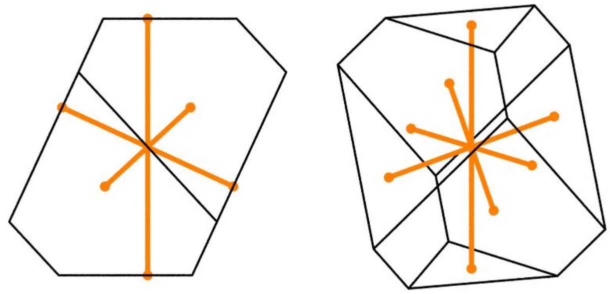

Figure 4: The truncated triangular trapezohedron and the four diagonal axes represented by orthogonal projection (left) and axonometric projection (right)

depicted in *"Melencolia I"* is the TTT, and the TTT is containing four diagonal axes. In the same way, the sphere also depicted in *"Melencolia I"* can be considered to contain an infinite number of diagonal axes, as shown in the Fig. 5.

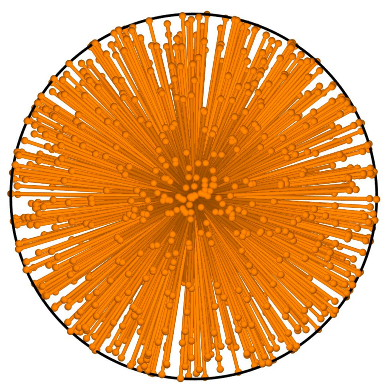

Figure 5: The sphere and an infinite number of diagonal axes

In the field of descriptive geometry, projection is the representation method of a three-dimensional solid shape on two-dimensional plane, and the three orthogonal axes can be represented in a plane. In the case of normal axonometric projection, the three orthogonal axes are converted into the three diagonally intersecting axes in the plane. By applying this principle of projection, it is possible to express a four-dimensional space in a three-dimensional space, that is, the four orthogonal axes in the four-dimensional space are converted into the four diagonally intersecting axes in the three-dimensional space by normal axonometric projection.

Though the systematization of the knowledge of descriptive geometry was completed by G. Monge [9] in later times, it is probable that Dürer had sufficient knowledge about projection and was fully aware that projection is a method of reducing one dimension. Therefore, he also understood that the shape in the four-dimensional space can be expressed in the three-dimensional space by projection, that is, the four orthogonal axes in the four-dimensional space can be converted into the four diagonal axes in the three-dimensional space. This paper hypothesizes that Dürer tried to express his understanding of the projection of a four-dimensional space into a three-dimensional space with the four diagonal axes contained in TTT. At the same time, the sphere depicted in the work can be considered as representation of the axis of infinite dimensions.

Here, let us consider the 3D space by the xyz coordinate system and consider the plane of $z = 0$ as projecting plane. Then, all kind of projection can be considered to movement from

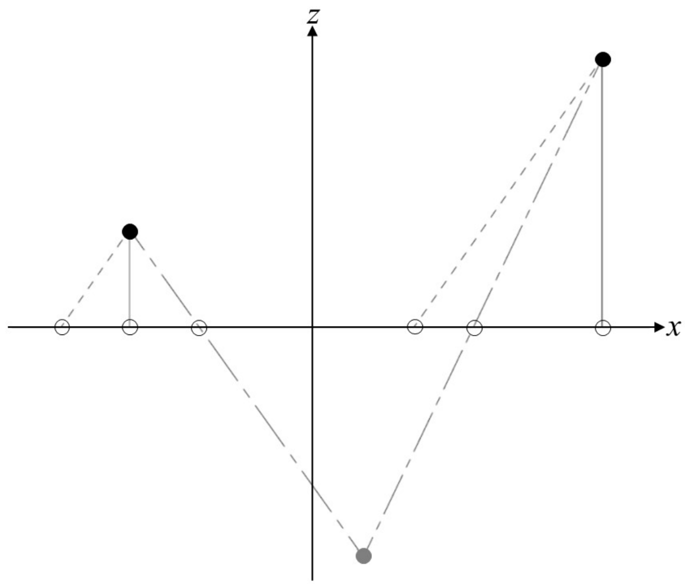

Figure 6: Movement of points by three projections in 3D space. The black circles are the points before the projection, the white circles are the points after the projection, the gray circle is the viewpoint, the solid line is corresponding to the normal projection, the dotted line, the oblique projection, and the dashed line, the perspective projection)

the point $\overrightarrow{P}(P_x, P_y, P_z)$ to the point $\overrightarrow{P'}(P'_x, P'_y, P'_z)$ where $P'_z = 0$. Then the three projections can be described as Fig. 6.

Let $\overrightarrow{E_x}(1,0,0,0)$, $\overrightarrow{E_y}(0,1,0,0)$, $\overrightarrow{E_z}(0,0,1,0)$, and $\overrightarrow{E_w}(0,0,0,1)$ be the unit vectors that represent the four axes. Let $\overrightarrow{E'_w}$ be the normal projection of $\overrightarrow{E_w}$ into projection cell $w = 0$. Then $\overrightarrow{E'_w}$ becomes $(0,0,0,0)$ and the size of $\overrightarrow{E'_w}$ becomes 0. By rotating the four unit vectors in the plane that include the w axis, the unit vectors after normal projection into projection cell $w = 0$ are not 0. Let P_n be the matrix corresponding to the normal projection into projection cell $w = 0$, and $R_{zw}(\theta)$ be the matrix corresponding to the rotation in the zw plane by angle θ. P_n and $R_{zw}(\theta)$ can be described as following equations:

$$P_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{zw}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

$\overrightarrow{E'_w}$, which is orthographically projected into projection cell $w = 0$ after rotating $\overrightarrow{E_w}$ in the zw plane by theta degree, is expressed by the following equation:

$$\overrightarrow{E'_w} = P_n R_{zw}(\theta) \overrightarrow{E_w}.$$

Then, magnitude of the vector $\overrightarrow{E'_w}$ is not zero. After the transformation by $P_n R_{zw}$, two vectors, $\overrightarrow{E'_z}$ and $\overrightarrow{E'_w}$, are parallel. By rotating each unit vectors again in another plane, it is possible to make $\overrightarrow{E'_z}$ and $\overrightarrow{E'_w}$ not parallel.

In Fig. 6, for the sake of simplicity, the projections are classified into three categories, normal projection, oblique projection, and perspective projection, based on the relationship between the projection line and the projection plane. Here, since the axes are rotated and then projected by normal projection, the result is that the four axes are projected into the projection cell of $w = 0$ by axonometric projection.

2.3 Existing works regarding the polyhedron depicted in *"Melencolia I"*

Existing works can be divided into those related to the dimensions of the polyhedron and those related to its meaning. First, existing works related to dimensions were examined.

Enomoto assumes that the polygons that make up the triangular trapezohedron that form the basis of the TTT in *"Melencolia I"* are diamonds with angles of 72 degrees and 108 degrees, and TTT is obtained by cutting the polyhedron so that it has a circumscribed sphere, and it is clarified that a reduced version of the triangular trapezohedron is inscribed [2]. MacGillavry pointed out that the shape of TTT is related to the structure of the crystal [6]. Schreiber explained that all vertices of TTT are located on one spherical surface [10]. Maekawa clarified that the ratio of the lengths of the two diagonals of the rhombus that constitutes the triangular trapezohedron that form the basis of the TTT in *"Melencolia I"* is $\sqrt{3} : \sqrt{5}$, and the ratio of length to width of the work *"Melencolia I"* is $\sqrt{3} : \sqrt{5}$ as well [7]. And Maekawa also pointed out that orthogonal projection of TTT in *"Melencolia I"* is inscribed in the magic square depicted in *"Melencolia I"* and intersects the intersection point on the grid in the magic square.

Lynch argued that the rhombohedron before the vertices are cut off is a cube, and that what is drawn on *"Melencolia I"* is based on an incorrect projection method [5]. Ishizu agreed with the theory that the triangular trapezohedron, which is the base of the TTT in *"Melencolia I"*, is a cube, and pointed out that Dürer drew the polyhedron using the anamorphosis method [3], which is why there is a difference between the cut shape of the cube and the TTT in *"Melencolia I"*.

As for existing works related to meaning, Enomoto associates the inscribed relationship as mentioned above with the melancholy of the title of the Dürer's work [2]. This finding infers Dürer's intentions, however does not contradict the hypothesis in this paper.

Although each of these findings is quite interesting, it holds true at the same time as the hypothesis in this paper, and does not contradict the hypothesis in this paper. In the following chapters, the basis for the hypothesis that Dürer represented the four-dimensional axes by TTT is discussed.

3 Historical Considerations

The existence of dimension beyond three dimensions has been known for a long time. Plato's *"The Allegory of the Cave"* in ancient Greece was based on the principle of projection, which converts three dimensions into two, so it was not unreasonable to think that three-dimensional space was a projection of four-dimensional space. For example, the philosopher Roger Bacon, in his *"Opus Ma-jus"* (1267), refers to the fourth direction as the flow of time [8]. In general, it is said that the first geometric representation of the shape of four-dimensional space was *"Der barycentrische Calculus"* (1827) by Möbius [8], and since then, it has been possible to treat four dimensions not only conceptually but also as an actual geometric shape.

On the other hand, the projection, which converts three dimensions into two dimensions, is earlier than that converts four dimensions into three dimensions, and as mentioned above, Dürer himself gave a detailed explanation of the projection in his book [1]. This book was published in 1525, but there is a work from 1514 called *"The Apparatus of Perspective Drawing"*, which suggests that Dürer had sufficient knowledge of projections when he was working on the work *"Melencolia I"* in 1514 [4].

Note that a polyhedron containing four diagonal axes can be any octahedron with two parallel pairs of faces. Among the various octahedra, the regular octahedron (see Fig. 7, left), which is known as "Platonic solid", is a polyhedron that were definitely known in Dürer's time. And the truncated tetrahedron (see Fig. 7, right), which is known as "Archimedean solid", is a polyhedron which Dürer might have known because the polyhedron had already been discovered at that time. Of these, the truncated tetrahedron has different shapes and sizes of paired polygons, making it difficult to consider that the two paired faces constitute one axis. The octahedron, on the other hand, is easy to understand because of its constitutive principle of four axes, and it had been judged inappropriate to hide the four diagonal axes in the polygon.

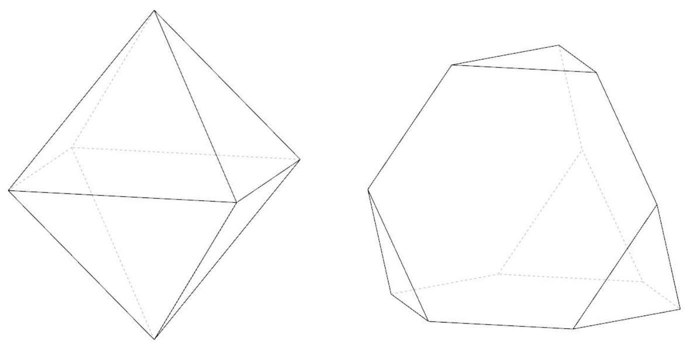

Figure 7: The octahedron (left) and truncated tetrahedron (right)

4 Consideration of the Subject Matter Depicted and the Title of the Work

Many of the subjects in *"Melencolia I"* remind the number four. The magic square is composed of 4×4. The name of the work, *"Melencolia I"* was originally spelled correctly as Melancolia, but the fourth letter was intentionally replaced. Melancolia is based on the theory that the human body is controlled by a balance of four basic body fluids (blood, mucus, yellow bile, and black bile), and Melancolia is a symptom that appears when black bile is abundant among the four.

On the other hand, many of the subjects depicted are tools for measuring some physical quantity. The seven modern SI basic units are the second (s) for time, the meter (m) for length, the kilogram (kg) for mass, the ampere (A) for electric current, the kelvin (K) for temperature, the mole (mol) for quantity of matter, and the candela (cd) for luminous intensity, and they are considered to be independent of each other, though there are some parts that are not independent in the current definition. Of these seven, electricity had not been discovered in Dürer's time, and no quantity of matter had been proposed. The thermometer had not been invented (invented by Galeria in 1592), nor had the illuminometer needed to measure luminosity been invented. The depiction of devices to measure the three remaining physical quantities (hourglass, compass, and scale) suggests that time and weight, which are independent of length, may have been considered by Dürer as candidates for the fourth axis.

5 Conclusions

This paper focuses on the polyhedron in Dürer's work *"Melencolia I"*, and assume that this polyhedron represents the four orthogonal axes in three-dimensional space by projection, and at the same time the sphere drawn beside it represents the axis of infinite dimensions. This paper discusses the hypothesis based on the historical consideration and the consideration of the subject matter depicted and the title of the work. At present, there is no definitive evidence for this hypothesis, and there are only weak evidences that such assumptions are consistent. In the future, more detailed examination should be conducted, including a survey of the technical and cultural background of the time when this work was completed.

The author does not believe that Dürer made this work only with an understanding of the four-dimensional axes, or with particular emphasis on this point. As is clear from previous studies, this work can be interpreted in multiple ways. The author believe that Dürer included the four-dimensional axes as one of such various intentions.

Acknowledgments

The author would like to thank the reviewers for their valuable comments. This work was supported by JSPS KAKENHI Grant Number 21K18506.

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Received June 1, 2022; final form July 22, 2022.

[1]In Japanese; the title was translated by the author.