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Construction of Hyperbolic Paraboloids According to a Prospective Outline in the Form of Hyperbola

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Abstract. The presented research is oriented toward developing constructive surface modeling methods in 3D-graphics program environment. First, the problem of hyperbolic paraboloid construction based on it's perspective outline is considered. The point of view and the outlined line defines a cone encircling a three-parameter set of hyperbolic paraboloid surfaces. Based on ideas and algorithms proposed in previous studies, the following conclusion was made: An arbitrary hyperbola, considered as the line of contact of a hyperbolic paraboloid with a cone whose vertex is chosen arbitrarily, is a determinant of a hyperbolic paraboloid. This determinant can be used to generate a two-parameter set of closed spatial four-link linear rings or a generator of a two-parameter set of parabolic cross-sections. They all belong to the same hyperbolic paraboloid. The axis of this paraboloid is parallel to the line connecting the cone's vertex with the center of the contact hyperbola. Redefinition of the determinant of a hyperbolic paraboloid is based on well-known theoretical principles: Each pair of intersecting generators determines the tangent plane of the paraboloid at the point of their intersection; the asymptotes of the hyperbola are parallel to the planes of parallelism of the hyperbolic paraboloid. The described cone specifies a two-parameter set of pairs of tangent planes according to the number of pairs of points on the hyperbola branches. Each such pair defines a spatial four-link linear ring having two vertices at selected points and two on the line of intersection of the tangent planes.

Key Words: second-order surface, hyperbolic paraboloid, computer simulation, enveloping cone, described cone, contact line, outline

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1 Introduction

One of the paradigms of working in the environment of three-dimensional computer graphics programs, from now on "programs", is the independence from the user's knowledge of analytical methods for solving particular problems. Instead, the constructive-geometric interpretation of the modeled object properties becomes more important, especially for tasks with no apparent solution algorithms. For example, one of such tasks is modeling surfaces, the determinant of which includes the outline of this surface on perspective images.

When constructing objects of design and architecture, assessing their artistic and aesthetic properties is essential, primarily determined by the visible contour of the modeled objects. If a significant part of an object is a curved surface, then the outline of this surface is a part of the general contour. It is the line of intersection of the plane of the picture with the cone described around the simulated surface. Such a cone is an envelope for a multiparameter set of surfaces. The specific surface from the set is selected using the specified design conditions.

The presented research is done to develop constructive methods of second-order surface definition at various initial conditions of their modeling. A hyperbolic paraboloid is one of the most widely used surfaces in design and architecture. It is the hyperbolic paraboloid that is the subject of this study.

1.1 Analysis of Primary Research and Publications

Second-order surfaces play a fundamental role in many theoretical and applied sciences, including design, architecture, and computer graphics. Therefore, the modern quadric theory reading is of great interest, on a systemic basis [9] and due to some research in this area. In particular: In [10] a synthetic approach to the creation of thread construction on quadrics is proposed. In all works, among others, hyperbolic paraboloid surfaces are considered. Finally, the work [5] is devoted to theoretical issues of modeling curved surfaces on the example of a hyperbolic paraboloid.

A significant amount of research is devoted to modeling quadric surfaces in the environment of 3D computer graphics programs. Possibilities of modern computer technologies for modeling linear surfaces, in particular, hyperbolic paraboloids, are demonstrated in [6]. In [1], some additional constructive properties of the hyperbolic paraboloid are proved, and the ways of solving design problems in 3D modeling based on these properties are shown.

The tasks of taking into account the perspective line of outline in modeling are also focused on their implementation in the environment of the program of three-dimensional graphics. This problem is proposed for linear surfaces [7, p. 86–89].

The spatial modeling of surfaces based on the outline is described in [8, p. 204–213]. In [2] the construction of central quadrics on one and two described cones is investigated, and in [4] the same problems are considered for surfaces of rotation of the second order. For a hyperbolic paraboloid, the problem of modeling along a perspective line of an outline of the second order is solved if the contact line between the enveloping cone and the hyperboloid is a parabola [3].

We have not found work on modeling hyperbolic paraboloid for which the line of contact with the envelope cone is a hyperbola.

1.2 The Purpose of the Research

The purpose of the research is a further development of constructive-geometric methods of construction of second-order surfaces in order to apply these methods in computer modeling in the environment of 3D graphics programs. In particular, to find a method that allows building only the required surface fragment without building the whole surface. This method is essential for constructing the surface fragment defined by the limited arcs of the outline and for the task of conjugating two second-order surfaces by a planar curve.

Problem 1. To carry out a parametric analysis of the problem of constructing a hyperbolic paraboloid along the hyperbolic line of contact with the envelope cone.

Problem 2. To formulate and prove the fundamental constructive property of a hyperbolic paraboloid given by a line of outline and a hyperbolic line of contact with an enveloping cone.

Problem 3. To develop a constructive way to construct a hyperbolic paraboloid with an arbitrary location in the design space.

This method must be represented as a sequence of operations, each of which is supported by 3D graphics software. In this research, SolidWorks is used as an example of such software.

2 Main Part

2.1 Parametric Analysis. Initial Conditions.

The scheme of the problem is shown in Fig. 1.

A user sets the hyperbolic paraboloid surface outline lines on the imaginary perspective plane K, which he wants to see from S viewpoint. The outline lines are given in the form of branches of the hyperbola g'. The user's objective is to obtain the surface of the hyperbolic paraboloid Γ, the outline of which on the plane K coincides with the given branches of the hyperbola.

The lines of cone $\{S, g'\}$ contact the hyperbolic paraboloid and can be either hyperbolas or parabolas because these are the only possible options for the hyperbolic paraboloid plain section.

Let us consider the case when the contact line is a hyperbola and show that for a given contact line in the form of a hyperbola, the cone $\{S, g'\}$ unambiguously defines a hyperbolic paraboloid.

The cone $\{S, g'\}$ envelops a multiparameter set of surfaces with a contact line. If it is a second-order cone and the surfaces are quadrics, this set is four-parametric. Indeed, previously in [2], it was shown that the enveloping cone specifies five parameters of the modeled surface for any second-order surface modeling.

Three parameters (number of cutting planes) are used to determine the contact line. Therefore, the line of contact (three parameters) and the envelope cone (five parameters) determine eight quadric parameters in the general case, the number of hyperbolic paraboloid parameters. It applies only to the case of the hyperbolic line of contact because only in such a case do both the second-order cone and the hyperbolic paraboloid contain a three-parameter set of hyperbolas. Parabolic sections on a hyperbolic paraboloid belong to a two-parameter set, so the parabolic line of contact defines a one-parameter set of hyperbolic paraboloids. This problem is solved in [3].

This is a parametric aspect. For complete proof, it is necessary to ensure a constructive way to move from the declared determinant to any other determinant. As the last, we choose a four-link spatial chain – $\{4l\}$-determinant – or an arbitrary one-parameter set of parabolas.

Note 1. As described below, algorithms require means of precise definition of the hyperbola.

In SolidWorks software, we have found only one method for getting such a precise

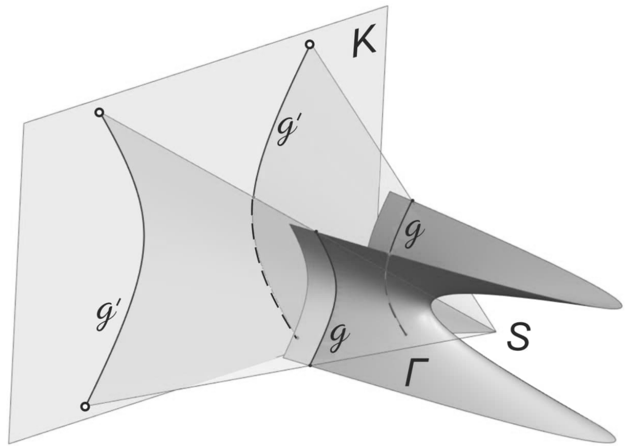

Figure 1: Outline model of the application of a perspective outline of the surface of a hyperbolic paraboloid for its modeling

definition – building a hyperbola as a section of a circular cone. Other methods result in spline approximation of the hyperbola and lead to errors in the construction of the tangents. With the construction of the hyperbola as a cone section, it is easy to find its asymptotes and, therefore, axis, center, and vertices. Therefore, all following cases are considered under the assumption that hyperbola is fully defined.

Note 2. In all subsequent figures, the hyperbola is located in a horizontal plane. Despite this fact, the algorithm remains generic because the vertex of the cone is chosen arbitrarily. In every specific case, the vertex occupies its respective location.

To implement a constructive approach, let us assume that a specific hyperbola g is defined in Fig. 1. Therefore, concerning the surface being modeled, the envelope cone is determined. We are interested in the general case in which the hyperbola and the vertex of the described cone positions do not depend on each other. First, however, it is essential to gradually consider a constructive way from a simple case Fig. 1 to general case.

2.2 A Variant of the Canonical Problem of the Surface

Fig. 2 shows the case in which the given hyperbola belongs to a plane that is perpendicular to the axis of the hyperbolic paraboloid. Let us assume that for this hyperbola vertices, axes, and asymptotes intersecting at the point T are defined. The vertex S of the described cone is located on the axis of the hyperbolic paraboloid. S is not located in the plane of the hyperbola, and ST is perpendicular to that plane. This statement corresponds to the surface definition in the canonical form.

From the initial condition through the vertices A and B of the hyperbola and the center of the coordinate system, point O (the vertex of the hyperbolic paraboloid), should pass a parabola, the tangents of which intersect at the point S so that $TO = OS$. The tangent to the hyperbolic paraboloid planes $\triangle A$ and $\triangle B$ at points A and B are defined by tangent lines SA and SB to the parabola and tangent lines to the hyperbola at points A and B (t_a and t_b). Tangents (t_a and t_b) are parallel to the imaginary axis of a given hyperbola. These planes intersect at the line (s), which is also parallel to the imaginary axis of the hyperbola and passes through the point S. Pairs of generators of hyperbolic paraboloid in the two tangent planes intersect on the line s in C, D.

Since the canonical case of the hyperbolic paraboloid problem is considered, the spatial

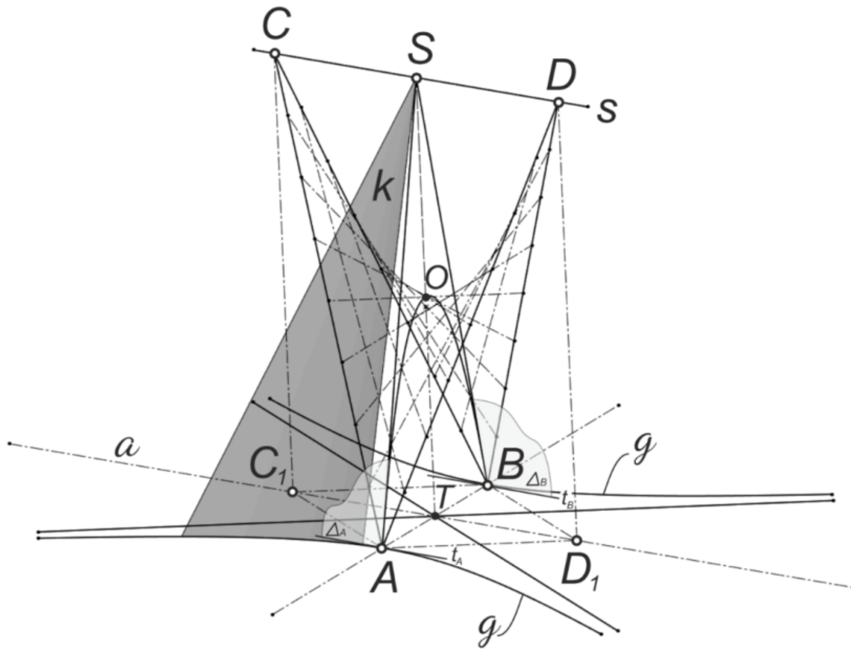

Figure 2: Surface modeling in the canonical version of the problem

linear ring of the central $\{4l\}$-determinant in parallel projection is projected as a rhombus, the vertices of which can be obtained as points of intersection of lines drawn parallel to the asymptotes of the hyperbola from vertices A and B [7, p. 27]. These lines intersect on the imaginary axis at points C_1 and D_1. They will be orthogonal projections of points C and D, located in the tangent to the hyperbolic paraboloid at points A and B planes $\triangle A$ and $\triangle B$ generators. Therefore points C and D lie in the tangent planes on their intersection line (s). Points C and D will be located in the projection direction parallel to ST.

In this case, we can conclude that the described second-order cone with a vertex at the point S' located on the axis of the hyperbolic paraboloid has a contact hyperbola with this paraboloid, located in a plane that is parallel to the main tangent plane and at a twice distance from the top of the hyperbolic paraboloid.

In this simple case, we simultaneously find the generating parabolas AOB and COD.

2.3 First Generalization

In this generalization (Fig. 3), the hyperbola g and the vertex of the described cone are given independently of each other and occupy an arbitrary position in the design space. However, as in the canonical case, the points for which the construction is performed belong to the vertices of the hyperbola.

The new position of the projection plane, together with the new position of the hyperbola, can be defined, in particular, by two asymptotes and one of the vertices. According to the problem statement, point S'(S'_1 is its orthogonal projection on the hyperbola's plane) can be chosen arbitrarily. Two groups of four points each $S'A'T'C'_1$ (Fig. 3) and $SATC_1$ (Fig. 2) define such an affine transformation of space that preserves parallelism, incidence, and the affine ratio of the shapes. Therefore, the properties mentioned above can be formally transferred from Fig. 2 to Fig. 3. For the affine transformation of three-dimensional space to correspond to the problem statement, it is necessary to declare that the angle between the lines $A'T'$ and $T'C'_1$

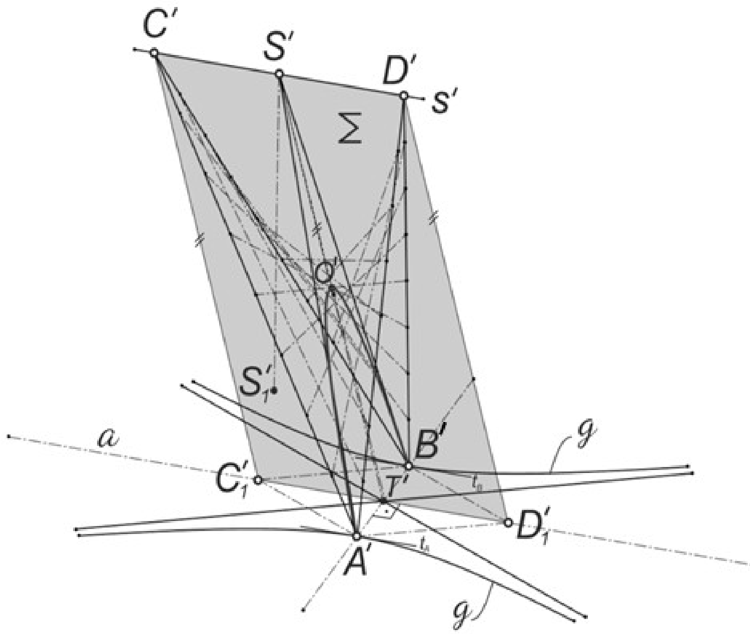

Figure 3: Simulation scheme for an arbitrarily given vertex of the described cone

must remain right. Then, under a given condition, only one hyperbola can be constructed with vertices at point A' and symmetric to it B'.

The line $T'C'_1$ defines the imaginary axis. The direction of the asymptotes is given by the lines $A'C'_1$ and $B'D'_1$.

Therefore, a hyperbola can be constructed on such a drawing, and therefore the whole algorithm for constructing the $\{4l\}$-determinant is preserved (namely, finding points C' and D' in space). An essential point of the construction algorithm is that the projection of points C'_1 and D'_1 on the line of intersection of tangent planes is made in a direction parallel to the line connecting the center of the cone with the center of the hyperbola and will belong to the plane Σ defined by the line $T'C'_1$ and vertex S'.

Tangent planes at points A' and B' intersect on a line (s') that goes through the point S' and is parallel to the imaginary axis of the hyperbola. On this line (s') points C' and D' are obtained by projection in a direction parallel to $T'S'$.

2.4 Second Generalization

Next, let us consider a generalization in which points A and B are chosen on the hyperbola, not at the vertices of the hyperbola (Fig. 4) but symmetrically about the imaginary axis.

In this case, the point of intersection of the tangents to the hyperbola at points A and B (t_a and t_b) becomes real but will belong to the imaginary axis (Fig. 4, point 1). The points C_1 and D_1 are vertices of the rombus created by the lines parallel to the asymptotes of the hyperbola at points A and B. They will also belong to the imaginary axis. The line of intersection of the tangent planes at points A and B will also belong to the plane $C_1D_1S_1$, and in the direction of projection, ST will be projected on the imaginary axis. In addition, it will definitely pass through point S, provided that it is the vertex of the second-order cone built on the hyperbola, and the cone is tangent to the modeled hyperbolic paraboloid. According to this scheme, a single-parameter set of parabolas with parallel chords A_iB_i can be constructed

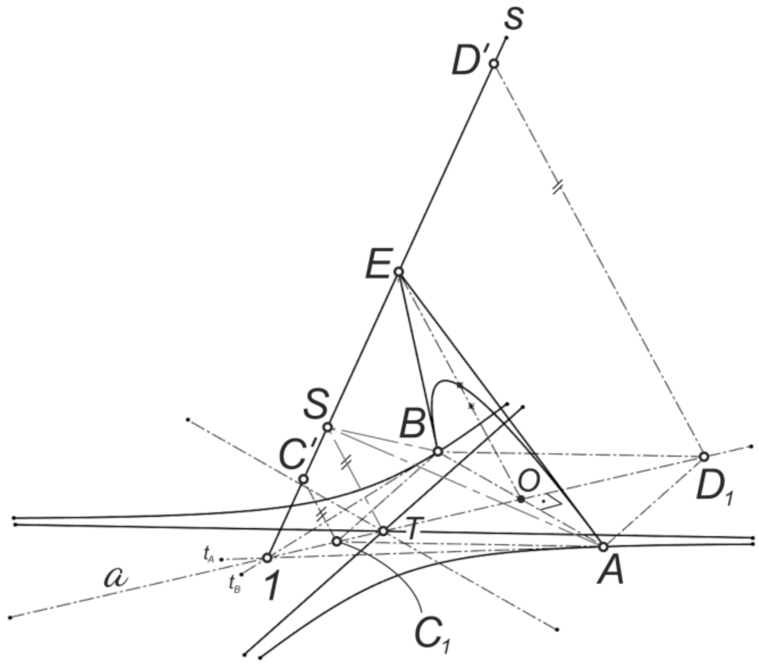

Figure 4: Scheme of modeling a hyperbolic paraboloid in the case of using a chord parallel to the real axis of the hyperbola

based on arbitrary points S and an arbitrary contact hyperbola. The chords will all belong to the same hyperbolic paraboloid.

Points E_i (Fig. 4 shows one of them – point E) of the intersection of tangents to these parabolas will belong to the line s, and can be found on it as projections of points O_i in the direction ST.

2.5 General Case

Let us consider the case when the chord AB is chosen arbitrarily. To do this, let us prove the property: If we set two arbitrary points (A and B) on different branches of the hyperbola and construct a parallelogram with vertices in these points and sides parallel to the hyperbola's asymptotes in these points, then the diagonal CD connecting the other pair of the parallelogram's vertices will pass through the center of the hyperbola, the asymptotes' intersection point.

Proof. Let an arbitrary hyperbola be given in the affine coordinate system xTy, the axis of which is its asymptotes. Then, its equation in this system has the form.

$$y = k/x \tag{1}$$

and the equations of lines passing through points $A(x_1, \frac{k}{x1})$ and $B(x_2, \frac{k}{x2})$, parallel to the coordinate axis (asymptotes) are: $AC\colon y = k/x_1$; $BC\colon x = x_2$; $BD\colon y = k/x_2$; $AD\colon x = x_1$. This defines the other two vertices $C(x_2, \frac{k}{x1})$ and $D(x_1, \frac{k}{x2})$, and the CD diagonal equation Fig. 5:

$$\frac{x - x_2}{x_1 - x_2} = \frac{y - \frac{k}{x_1}}{\frac{k}{x_2} - \frac{k}{x_1}}; \tag{2}$$

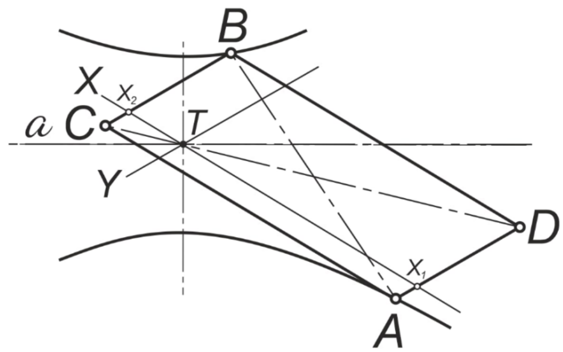

Figure 5: Scheme to explain the properties

Let us verify that center of the hyperbola (point T), which is located at zero coordinates, satisfies (2).

$$\frac{-x_2}{x_1 - x_2} = \frac{\frac{-k}{x_1}}{\frac{k}{x_2} - \frac{k}{x_1}}; \tag{3}$$

Transforming the right-hand side of equation (3), we have an identity

$$\frac{-x_2}{x_1 - x_2} = \frac{-x_2}{x_1 - x_2}. \tag{4}$$

So, for $x = 0$, $y = 0$ equation (2) does not depend on k and holds for any hyperbola. It also does not depend on the chosen points A and B on the hyperbola because it is fulfilled at any values of the coordinates x_1 and x_2. Thus the line CD always passes through the center of the hyperbola. □

The general scheme of modeling is shown in Fig. 6.

Points A and B are chosen on the hyperbola arbitrarily (asymmetric concerning its imaginary axis). Points C_1 and D_1 are no longer vertices of the rhombus but vertices of the parallelogram. O is the point of intersection of the diagonals of the parallelogram. Points (C_1, T, O, D_1) are located on one line by the proven above property.

Let 1 be the point of intersection of the tangents at points A and B (t_a and t_b), then the points (1, T, O) also belong to one line because for the central second-order curve, the line connecting the point of intersection of two tangents and the middle of the chord points of contact passes through the center of this curve. The two mentioned lines have a common segment OT, and therefore, all these points (1, C_1, T, O) belong to one line. In Fig. 6, it is denoted as b.

If points A and B are changed, the position of point 1 changes, and therefore the position of the lines b and s. However, the line ST remains unchanged. It is the axis of the pencil of planes $\{b_i, s_i\}$. The diameters of all parabolas belonging to one hyperbolic paraboloid are parallel to the axis of the hyperbolic paraboloid. In an axial pencil of planes, lines in distinct planes are parallel only if they are parallel with the axis.

Therefore, only ST can be the direction of the axis of the hyperbolic paraboloid.

Thus, the points C', D' and E belong to line s, which passes through the vertex of the cone and point 1. After re-projecting the points C_1, D_1, and O in the ST direction, they will be obtained. The closed spatial linear ring $AC'BD'$ determines the hyperbolic paraboloid modeled at any points A and B given on the hyperbola. The transition from such a task to the canonical environment of 3D graphics programs is shown in [1].

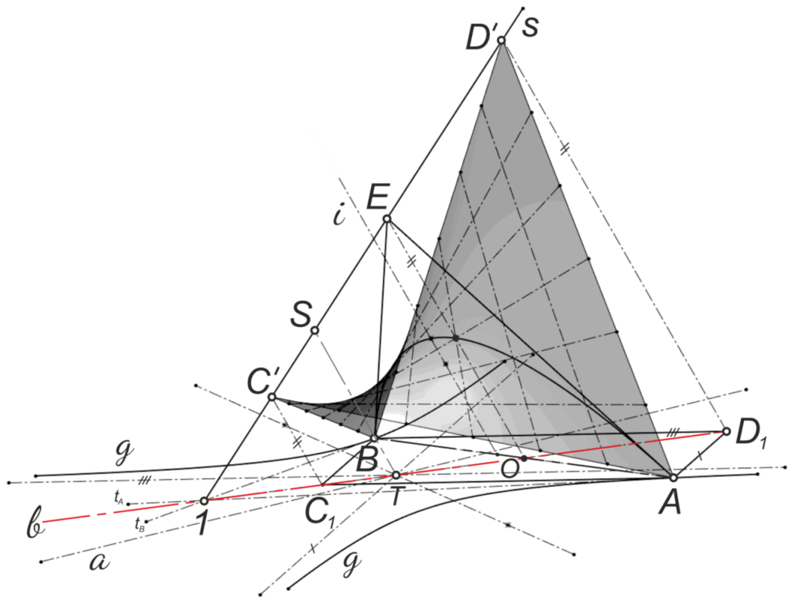

Figure 6: General scheme of modeling the surface of a hyperbolic paraboloid along the described cone when the contact line is a hyperbola

To construct a one-parameter set of parabolas, it is necessary to determine a one-parameter set of chords $\{A_iB_i\}$, $i = 0, 1, \ldots, n$ in the hyperbola plane. Chords A_0B_0 and A_nB_n limit the part of the surface to be modeled. The user in graphical dialog mode sets them. For $0 < i < n$ the bilinear interpolation of the quadrilateral $A_0A_nB_nB_0$ is used. It determines the lines p_i intersecting both branches of the hyperbola and, in the intersection with them, determines the points of A_iB_i chord. In special cases, these chords can be parallel or can form a bundle of lines with its center. The construction of a specific parabola is performed as in Section 2.4.

Parabolas can be extended beyond the plane of contact, such as the intersection with the plane K (Fig. 1). This makes it possible to model the desired surface sections in an interactive mode. Note that the preparation of input information (view point S, hyperbolas g and g', points A_0, A_n, B_n, B_0) and construction of the necessary sections should be performed in the interactive mode. To construct the desired set of parabolas, each defined by points A_i, B_i, O_i, 1_i, E_i, special macros should be created. All the operations can be performed in interactive mode or by a macro, a possibility provided by any general-purpose 3D graphics program.

3 Conclusions and Prospects

It is shown that a hyperbola, which is a line of contact of the hyperbolic paraboloid surface with a described cone, can be considered as one of the standard determinants of the hyperbolic paraboloid. Proven structural and geometric properties allow us to solve the problem of modeling of a hyperbolic paraboloid along the line of outline in the form of a hyperbola. Furthermore, the hyperbola and the center of the envelope cone can be located arbitrarily.

The operations required for the modeling of the hyperbolic paraboloid by a given determinant are fully supported by general-purpose 3D graphics programs.

The research can be continued and developed to model hyperboloids and solve problems of their conjugation with each other and with a hyperbolic paraboloid. In addition, these tasks are relevant in the computer modeling of design objects.

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