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A New Classification of Convex Tetragons

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Abstract. This work is a continuation and expansion of Dalcín (2022). As a continuation of Dalcín (2022) a diagram is constructed with sixty-five different economical definitions of thirty-five families of convex tetragons defined hierarchically on the basis of equal sides, equal angles, and equal diagonal segments. The diagram allows us to appreciate the existing duality between the definitions that include equality of sides and angles, among those that include equality of sides and diagonal segments, as well as those that include equality of angles and diagonal segments. It will also be possible to appreciate the existing duality between the definitions that include equality of sides and angles with the definitions that include equality of sides and diagonal segments. As an extension of Dalcín (2022), we analyze which families are defined and the economic hierarchical definitions for these families when the tetragon has perpendicular diagonals. A diagram of these last families is constructed.

Key Words: tetragon definition, convex tetragon classification

MSC 2020: 51M04

1 Introduction

A distinction can be made between complete quadrangle, complete quadrilateral and tetragon. A *quadrangle* is a plane figure consisting of four points no three of which are collinear. A *complete quadrangle* is a plane figure determined by four points no three of which are collinear and the six lines determined by them (see Figure 1, top left). A *quadrilateral* is a plane figure consisting of four lines no three of which are concurrent. A *complete quadrilateral* is a plane figure determined by four lines no three of which are concurrent and their six points of intersection (see Figure 1, top right).

A *tetragon* is defined as a configuration of four points in a plane no three of which are collinear and four line segments joining these points in a cycle, where a cycle and its reverse cycle are considered the same. Four points can be cyclically ordered in three ways (A-B-C-D, A-B-D-C, A-C-B-D) so in tetragons we can differentiate between consecutive and opposite vertices/sides, which we couldn't do in complete quadrangles and complete quadrilaterals.

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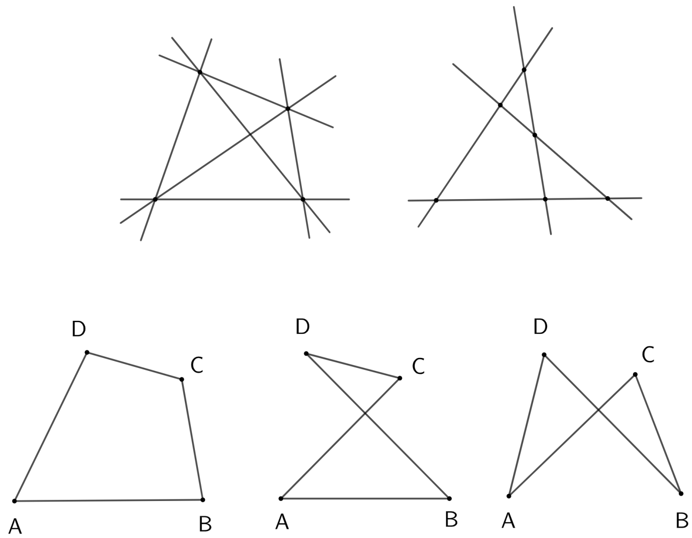

Figure 1: Complete quadrangle (top left), complete quadrilateral (top right), convex tetragon (bottom left) and non convex tetragons (bottom right)

The diagonals of a tetragon are line segments whose ends are opposite vertices. We can also differentiate between convex and non-convex tetragons. A tetragon is convex if its diagonals intersect in an inner point of the diagonal segments, otherwise it is non-convex. In what follows we will work exclusively with convex tetragons.

As notation we will use a, b, c, d for the measure of the sides of the tetragon and A, B, C, D for the measure of its angles. If diagonals AC and BD of a convex tetragon $ABCD$ intersect at O, we call diagonal segments to OA, OB, OC, OD. As notation for the measures of the diagonal segments we will use a', b', c', d' (Figure 2).

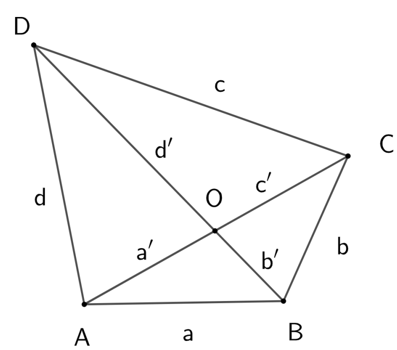

Figure 2: Notation for sides, angles and diagonal segments of the convex tetragon

2 The Families of Tetragons

In [1] thirty-five families of convex tetragons are hierarchically defined in economical way in sixty-five different forms. Definitions in geometry are non-economic if one part of the definition can be deduced from another part, and are economical if they do not contain superfluous information. The large number of families and the large number of definitions of each family make it difficult to have a global vision of the existing dualities between the families and between their definitions. The first objective of this article is the construction of a diagram that makes it possible to visualize all these families of tetragons with all their economic definitions.

In [3] de Villiers explicit the duality between sides and angles of tetragons. In [1] two new dualities were pointed out, the dualities between sides and diagonal segments, and between angles and diagonal segments and the existence of an analogy between the dualities of sides-angles and that of sides-diagonal segments was founded. The second objective of this article is to show in the diagram the three dualities and the analogy mentioned.

The third objective is to make explicit the inclusion relationships between the different families of tetragons.

As a fourth objective we will analyze which families are defined – and the economic hierarchical definitions for these families – when the thirty-five original families of tetragons have perpendicular diagonals.

As last objective, a diagram of these families and their economic hierarchical definitions will be made. This diagram will allow visualizing the aforementioned dualities for the case in which the diagonals of the tetragon are perpendiculars.

In the following table we include the thirty-five families of convex tetragons defined hierarchically in sixty-five different economical ways that will be included in the diagram.

The conditions that define each family do not make a difference if the tetragon is convex or not, but in this work we consider only convex tetragons.

Family	Definition(s)
1ops	At least one pair of opposite sides equal ($a = c$)
1op ∠	At least one pair of opposite angles equal ($A = C$)
1opd	At least one pair of opposite diagonal segments equal ($a' = c'$)
1cps	At least one pair of consecutive sides equal ($a = b$)
1cp ∠	At least one pair of consecutive angles equal ($A = B$)
1cpd	At least one pair of consecutive diagonal segments equal ($a' = b'$)
3s	At least three sides equal ($d = a = b$)
3 ∠	At least three angles equal ($D = A = B$)
3d	At least three diagonal segments equal ($d' = a' = b'$)
1ops-1cp ∠	At least one pair of opposite sides equal and at least one pair of consecutive angles equal ($a = c$ and $A = B$)
1ops-1op ∠	At least one pair of opposite sides equal and at least one pair of opposite angles equal ($a = c$ and $A = C$)
1cps-1op ∠	At least one pair of consecutive sides equal and at least one pair of opposite angles equal ($d = a$ and $A = C$)
1cps-1cp ∠ (1)	At least one pair of consecutive sides equal and at least one pair of consecutive angles equal ($d = a$ and $A = B$)
1cps-1cp ∠ (2)	At least one pair of consecutive sides equal and at least one pair of consecutive angles equal ($d = a$ and $C = D$)
1ops-1cpd	At least one pair of opposite sides equal and at least one pair of consecutive diagonal segments equal ($a = c$ and $a' = b'$)
1ops-1opd	At least one pair of opposite sides equal and at least one pair of opposite diagonal segments equal ($a = c$ and $a' = c'$)
1cps-1opd	At least one pair of consecutive sides equal and at least one pair of opposite diagonal segments equal ($d = a$ and $a' = c'$)
1cps-1cpd(1)	At least one pair of consecutive sides equal and at least one pair of consecutive diagonal segments equal ($d = a$ and $a' = b'$)
1cps-1cpd(2)	At least one pair of consecutive sides equal and at least one pair of consecutive diagonal segments equal ($d = a$ and $c' = d'$)
1opd-1cp ∠	At least one pair of opposite diagonal segments equal and at least one pair of consecutive angles equal ($a' = c'$ and $A = B$)
1cpd-1op ∠	At least one pair of consecutive diagonal segments equal and at least one pair of opposite angles equal ($a' = b'$ and $A = C$)
1cpd-1cp ∠	At least one pair of consecutive diagonal segments equal and at least one pair of consecutive angles equal ($d' = a'$ and $A = B$)
3s-1cp ∠	At least three sides equal and at least one pair of consecutive angles equal ($d = a = b$ and $A = B$)
1cps-3 ∠	At least one pair of consecutive sides equal and at least three angles equal ($a = b$ and $D = A = B$)
3s-1cpd	At least three sides equal and at least one pair of consecutive diagonal segments equal ($d = a = b$ and $a' = b'$)
1cps-3d	At least one pair of consecutive sides equal and at least three diagonal segments equal ($d = a$ and $a' = b' = c'$)
Parallelogram	Two pairs of opposite sides equal ($a = c$ and $b = d$)
	Two pairs of opposite angles equal ($A = C$ and $B = D$)
	Two pairs of opposite diagonal segments equal ($a' = c'$ and $b' = d'$)

Family	Definition(s)
	At least a pair of opposite angles equal and at least a pair of opposite diagonal segments equal ($a' = c'$ and $B = D$)
Kite	Two pairs of consecutive sides equal ($a = b$ and $c = d$)
	At least one pair of consecutive sides equal and at least one pair of opposite angles equal ($a = b$ and $A = C$)
	At least one pair of consecutive sides equal and at least one pair of opposite diagonal segments equal ($a = b$ and $a' = c'$)
	At least a pair of opposite diagonal segments equal and at least a pair of opposite angles equal ($a' = c'$ and $A = C$)
Isosceles trapezium	Two pairs of consecutive angles equal ($A = B$ and $C = D$)
	Two pairs of consecutive diagonal segments equal ($a' = b'$ and $c' = d'$)
	At least one pair of opposite sides equal and at least one pair of consecutive angles equal ($d = b$ and $A = B$)
	At least one pair of opposite sides equal and at least one pair of consecutive diagonal segments equal ($d = b$ and $a' = b'$)
Kite 3 ∠	Two pairs of consecutive sides equal and at least one pair of consecutive angles equal ($a = b$, $c = d$ and $A = B$)
	At least one pair of consecutive sides equal and at least three angles equal ($a = b$ and $D = A = B$) (1)
	At least one pair of consecutive sides equal and at least three angles equal ($c = d$ and $D = A = B$) (2)
	At least one pair of opposite diagonal segments equal and at least three angles equal ($a' = c'$ and $D = A = B$)
Kite 3d	Two pairs of consecutive sides equal and at least one pair of consecutive diagonal segments equal ($a = b$, $c = d$ and $a' = b'$)
	At least one pair of consecutive sides equal and at least three diagonal segments equal ($a = b$ and $d' = a' = b'$) (1)
	At least one pair of consecutive sides equal and at least three diagonal segments equal ($c = d$ and $d' = a' = b'$) (2)
	At least three diagonal segments equal and at least one pair of opposite angles equal ($d' = a' = b'$ and $A = C$)
Isosceles trapezium 3s	Two pairs of consecutive angles equal and at least one pair of consecutive sides equal ($A = B$, $C = D$ and $d = a$)
	Two pairs of consecutive diagonal segments and at least one pair of consecutive sides equal ($a' = b'$, $c' = d'$ and $d = a$)
	At least three sides equal and at least one pair of consecutive angles equal ($d = a = b$ and $A = B$) (1)
	At least three sides equal and at least one pair of consecutive angles equal ($d = a = b$ and $C = D$) (2)
	At least three sides equal and at least one pair of consecutive diagonal segments equal ($d = a = b$ and $a' = b'$) (1)
	At least three sides equal and at least one pair of consecutive diagonal segments equal ($d = a = b$ and $c' = d'$) (2)
Rhombus	Four sides equal ($a = b = c = d$)
	At least three sides equal and at least one pair of opposite angles equal ($d = a = b$ and $A = C$)
	At least three sides equal and at least one pair of opposite diagonal segments equal ($d = a = b$ and $a' = c'$)
	Two pairs of consecutive sides equal and at least one pair of opposite angles equal ($a = b$, $c = d$ and $A = C$)
	Two pairs of consecutive sides equal and at least one pair of opposite diagonal segments equal ($a = b$, $c = d$ and $a' = c'$)
	At least one pair of consecutive sides equal and two pairs of opposite angles equal ($d = a$, $A = C$ and $B = D$)
	At least one pair of consecutive sides equal and two pairs of opposite diagonal segments equal ($d = a$, $a' = c'$ and $b' = d'$)
Rectangle	Four angles equal ($A = B = C = D$)
	Four diagonal segments equal ($a' = b' = c' = d'$)
	At least three angles equal and at least one pair of opposite sides equal ($A = B = C$ and $b' = d'$)
	At least three diagonal segments equal and at least one pair of opposite angles equal ($d = a = b$ and $B = D$)
	At least three angles equal and at least one pair of consecutive sides equal ($A = B = C$ and $a' = b'$) (1)
	At least three angles equal and at least one pair of consecutive sides equal ($A = B = C$ and $d' = a'$) (2)
	At least three diagonal segments equal and at least one pair of consecutive angles equal ($d = a = b$ and $A = B$) (1)
	At least three diagonal segments equal and at least one pair of consecutive angles equal ($d = a = b$ and $D = A$) (2)
	Two pairs of opposite sides equal and at least one pair of consecutive angles equal ($a = c$, $b = d$ and $A = B$)
	Two pairs of opposite sides equal and at least one pair of consecutive diagonal segments equal ($a = c$, $b = d$ and $a' = b'$)
	At least one pair of opposite sides equal and two pairs of consecutive angles equal ($a = c$, $A = B$ and $C = D$)
	At least one pair of opposite sides equal and two pairs of consecutive diagonal segments equal ($a = c$, $a' = b'$ and $c' = d'$)
	At least one pair of opposite sides equal and at least three angles equal ($a = c$, $A = B = C$)
	At least one pair of opposite sides equal and at least three diagonal segments equal ($a = c$, $a' = b' = c'$)
	Two pairs of consecutive angles equal and at least one pair of consecutive diagonal segments equal ($A = B$, $C = D$ and $a' = b'$)
	Two pairs of consecutive diagonal segments equal and at least one pair of consecutive angles equal ($a' = b'$, $c' = d'$ and $A = B$)

Family	Definition(s)
Square	Two pairs of consecutive angles equal and at least one pair of opposite diagonal segments equal ($A = B$, $C = D$ and $a' = c'$)
	Two pairs of consecutive diagonal segments equal and at least one pair of opposite angles equal ($a' = b'$, $c' = d'$ and $A = C$)
	At least one pair of consecutive sides equal and four angles equal ($d = a$, $A = B = C = D$)
	At least one pair of consecutive sides equal and four diagonal segments equal ($d = a$, $a' = b' = c' = d'$)
	At least three sides equal and three angles equal ($a = b = c$, $C = D = A$) (1)
	At least three sides equal and three angles equal ($a = b = c$, $A = B = C$) (2)
	At least three sides equal and three diagonal segments equal ($a = b = c$, $c' = d' = a'$) (1)
	At least three sides equal and three diagonal segments equal ($a = b = c$, $a' = b' = c'$) (2)
	Four sides equal and at least one pair of consecutive angles equal ($a = b = c = d$ and $A = B$)
	Four sides equal and at least one pair of consecutive diagonal segments equal ($a = b = c = d$ and $a' = b'$)
	At least three sides equal and two pairs of consecutive angles equal ($a = b = c$, $A = B$ and $C = D$)
	At least three sides equal and two pairs of consecutive diagonal segments equal ($a = b = c$, $a' = b'$ and $c' = d'$)
	Two pairs of consecutive sides equal and at least three angles equal ($a = b$, $c = d$ and $D = A = B$)
	Two pairs of consecutive sides equal and at least three angles equal ($a = b$, $c = d$ and $d' = a' = b'$)
	Two pairs of consecutive sides equal and two pairs of consecutive angles equal ($a = b$, $c = d$, $A = B$ and $C = D$)
	Two pairs of consecutive sides equal and two pairs of consecutive diagonal segments equal ($a = b$, $c = d$, $a' = b'$ and $c' = d'$)

Table 1: The families with their respective economical hierarchical definitions

3 The Diagram

The distribution of families in the diagram (Figure 3) will be in concentric circles and going from smallest to largest, in this order:

Circle 1. At least one pair of sides, angles or diagonal segments equal: six families with only one definition each.

Circle 2. At least three sides, angles or diagonal segments equal: three families with only one definition each.

Circle 3. At least one pair of sides, angles or diagonal segments equal and at least one pair of sides, angles or diagonal segments equal: thirteen families with only one definition each.

Circle 4. Parallelogram: four definitions.

Circle 5. Kite and isosceles trapezium: four and four definitions respectively.

Circle 6. At least one pair of sides, angles or diagonal segments equal and at least three sides, angles or diagonal segments equal: four families with only one definition each.

Circle 7. Kite three angles equal, kite three diagonal segments equal and isosceles trapezium with three sides equal: four, four and six definitions respectively.

Circle 8. Rhombus and rectangle: seven and eighteen definitions respectively.

Circle 9. Square: fourteen definitions

In three rays the families defined only by conditions referred to sides (vertically), angles (to the left) or to diagonal segments (to the right) will be located. In the lower left sector the families defined by sides and angles, in the lower right sector the families defined by sides and diagonal segments, and in the upper sector the families defined by angles and diagonal segments.

The diagram will be divided into different parts for a better visualization of the families, their definitions and the relationships between them.

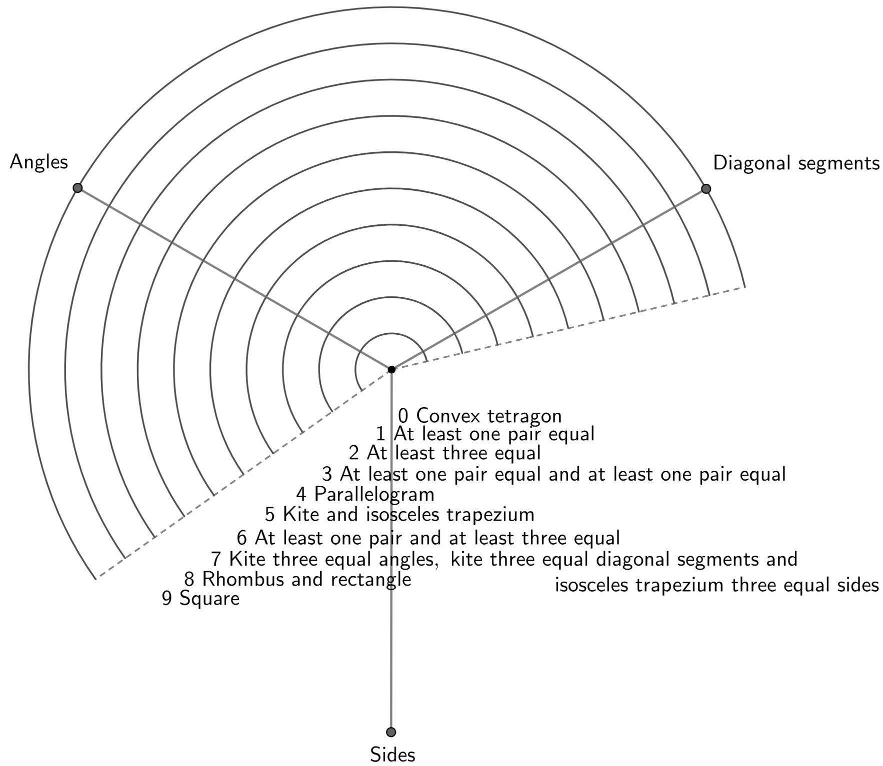

Figure 3: Location of tetragon families in the diagram

4 Families and Definitions in the Diagram

In Circle 1 (Figure 4) there are six families clearly separated into three pairs according to the definition referring to sides, angles or diagonal segments. In each pair there is a distinction according to whether the sides, angles or diagonal segments are consecutive or opposite. In [8] the family at least one pair of opposite angles equal is named tilted kite and five characterizations of the family are given. In [7] the family at least one pair of opposite diagonal segments equal is named bisect-diagonal tetragon and four characterizations of the family are given. Several properties of this family are demonstrated in [5]. Characterizing the other four families in Circle 1 is a pending task in Euclidean geometry. In [9] several properties of the family at least one pair of opposite sides are given but no characterization.

In Circle 2 (Figure 4) there are three families, each defined only by sides, by angles or by diagonal segments. Each family in this circle is part of two families in Circle 1. For example, the family at least three sides equal is part of the family at least two consecutive sides equal and the family at least two opposite sides equal.

The families of Circle 3 are defined by two conditions referring to sides and angles, sides and diagonal segments or angles and diagonal segments. Each family in this circle is part of two families in Circle 1. This inclusion is indicated by the lines that link each family in Circle 3 with two families in Circle 1. No references for any of the thirteen families in Circle 3 were found.

The side-angle duality, the side-diagonal segments duality and the angle-diagonal segments duality between two families is indicated in Figure 4 by arcs ending in arrowheads. The same will be done in the following figures. For example, the family “at least one pair of consecutive sides equal and at least one pair of opposite angles equal” and the family “at least one pair of consecutive angles equal and at least one pair of opposite sides equal" are dual changing sides by angles and angles by sides. The family "at least one pair of opposite sides equal and at least one pair of opposite angles equal" is self dual.

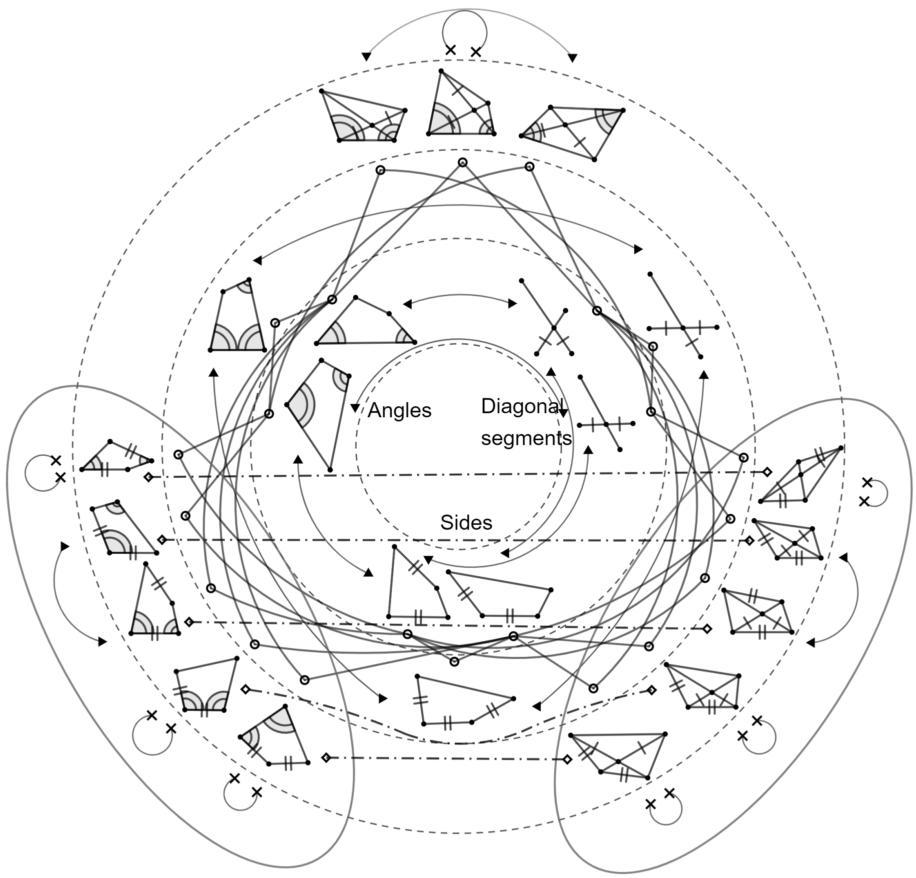

Figure 4: Families at least one pair equal, at least three equal, at least one pair equal and at least one pair equal

The analogy between the families defined by sides-angles conditions and the families defined by sides-diagonal segments conditions can be seen between each family of the oval on the left with a family of the oval on the right. For example the families "at least one pair of opposite sides equal and at least one pair of opposite angles equal" and "at least one pair of opposite sides equal and at least one pair of opposite diagonal segments equal" are analogous changing angles by diagonal segments and diagonal segments by angles.

The twenty two families of Circles 1, 2 and 3 are defined in only one way each.

In order not to overload the diagram, neither in Figure 5 nor in the following ones, lines indicating which other family each family is part of are not included.

In Circle 4 (Figure 5) are the parallelograms. The novelty of this circle with respect to the previous ones is that the same family appears defined in four different ways. Three of the definitions involve only sides, only angles, or only diagonal segments. Less frequent is the fourth definition – at least one pair of equal opposite angles and at least one pair of equal diagonal segments – which results from the intersection of two families of Circle 1.

The side-angle duality, the side-diagonal segments duality and the angle-diagonal segments duality between two dual or self dual definitions is indicated in Figure 3 to Figure 11 by arcs ending in symbol $\times$.

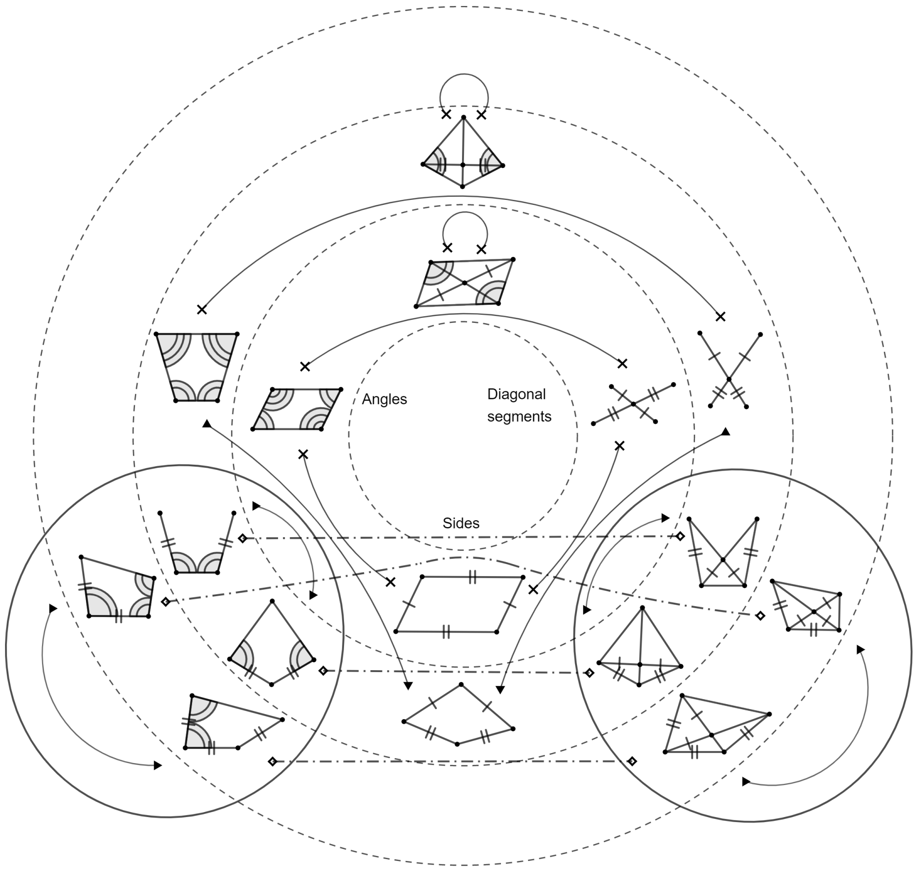

Figure 5: Families parallelogram, kite and isosceles trapezium, at least a pair equal and at least three equal

In Circle 5 (Figure 5) are kites and isosceles trapezium. Three of the definitions involve only two consecutive pairs of sides, angles or diagonal segments. In the first case they define kite and in the other two they define isosceles trapezium. This definition of kite admits two dual definitions of isosceles trapezium, depending on whether we consider the sides-angles duality or the sides-diagonal segments duality. It is worth noting the similarity between these three definitions and three of the parallelogram definitions: one becomes another by changing – whether sides, angles, or diagonal segments – consecutive ones into opposite ones. The definition of kite 'at least one pair of opposite angles equal and at least one pair of opposite diagonal segments equal' is self dual, in the same way as the definition of parallelogram. The difference between the two is the relative position between angles and diagonal segments.

Two dual definitions of kite and isosceles trapezium appear in this circle: "at least one pair of consecutive sides equal and at least one pair of opposite angles equal" and "at least one pair of consecutive angles equal and at least one pair of opposite sides equal" (sides-angles duality).

Also dual are the definitions of kite and isosceles trapezium "at least one pair of consecutive sides equal and at least one pair of opposite diagonal segments equal" and "at least one pair of consecutive diagonal segments equal and at least one pair of opposite sides equal" (sides-diagonal segments duality). The first two definitions are respectively analogous to the last two changing angles by diagonal segments and diagonal segments by angles.

This Circle 5 is the first in which two families appear – kite and isosceles trapezoid – defined in four different ways each. Both families and their definitions are linked by the aforementioned duality and analogy relationships.

In Circle 6 (Figure 5) there are four families. Two of them are linked by the sides-angles duality and the other two by the sides-diagonal segments duality. Angles and diagonal segments can be interchanged in the definitions of both pairs, exemplifying once again the analogy between families and definitions involving sides-angles and families and definitions involving sides- diagonal segments. No references were found to any of the four families.

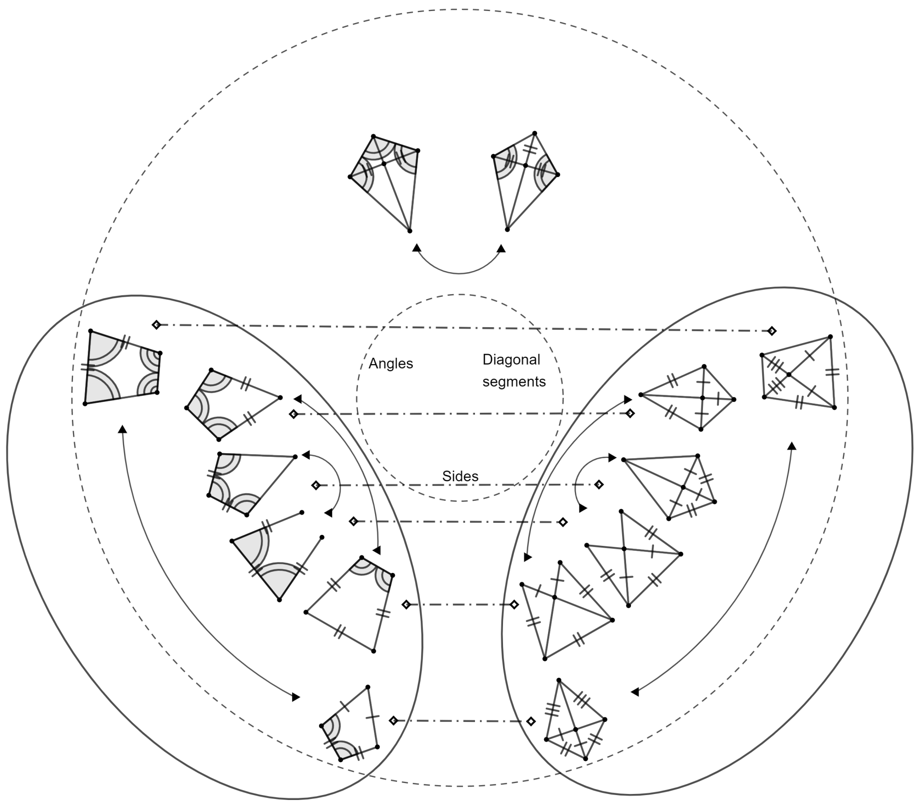

Figure 6: Families kite three equal angles, kite three equal diagonal segments and isosceles trapezium three equal sides

Circle 7 (Figure 6) shows the kite three equal angles family economically defined in four different ways, the kite three equal diagonal segments family economically defined in four different ways and the isosceles trapezium three equal sides family economically defined in six different ways. The first and third families appear in [4, p.154], and we have found no references for the third family.

Circle 8 (Figure7) include the rhombus defined in seven different ways and the rectangle defined in eighteen different ways. Six definitions involve sides and angles, another six involve sides and diagonal segments, and ten involve angles and diagonal segments.

In Circle 9 (Figure 8) the square appears defined in fourteen different ways. It is surprising that no definition involves only angles and diagonal segments.

5 First Conclusions

In [6] M. Josefsson makes a historical account of the different classifications of convex tetragons from Euclid to the present. It includes nine different classifications with their respective

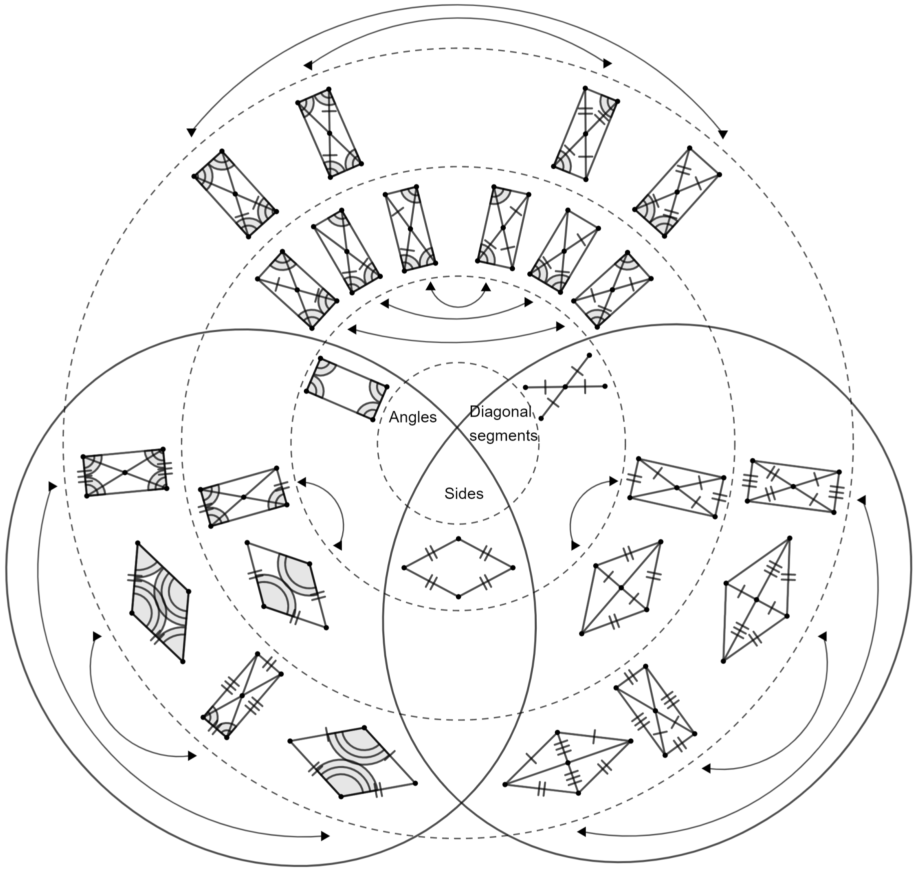

Figure 7: Families rhombus and rectangle

diagrams and all, more or less precisely, have a bilateral symmetry that responds to the side-angle duality. The novelty of the diagram presented here is that it has triple bilateral symmetry, which responds to the dualities sides-angles, sides-diagonal segments, angles-diagonal segments. The last two dualities allow the diagonal segments to be incorporated, maintaining symmetry, in the tetragons diagram (Figure 9). In addition to the dualities mentioned, this diagram also has bilateral symmetry that responds to the analogy between families and definitions that involve sides-angles and families and definitions that involve sides-diagonal segments. Another novelty of this diagram is that it takes into account the different economical definitions of each family of tetragons. It includes four definitions to parallelogram, four to kite, four to isosceles trapezium, four to kite three equal angles, four to kite three equal diagonal segments, six to isosceles trapezium three equal sides, seven to rhombus, eighteen to rectangle, fourteen to square.

6 New Families and New Definitions of Tetragons with Perpendicular Diagonals

In every tetragon the diagonals determine two pairs of equal angles. As a particular case, these four angles can be equal. In what follows, to the sixty-five economic definitions of the thirty-five families of convex tetragons we will add the condition that the diagonals be perpendicular. Which tetragon is defined in each case? The definition obtained in each case is economic? In the first and second columns of Table 2, the families with their definitions

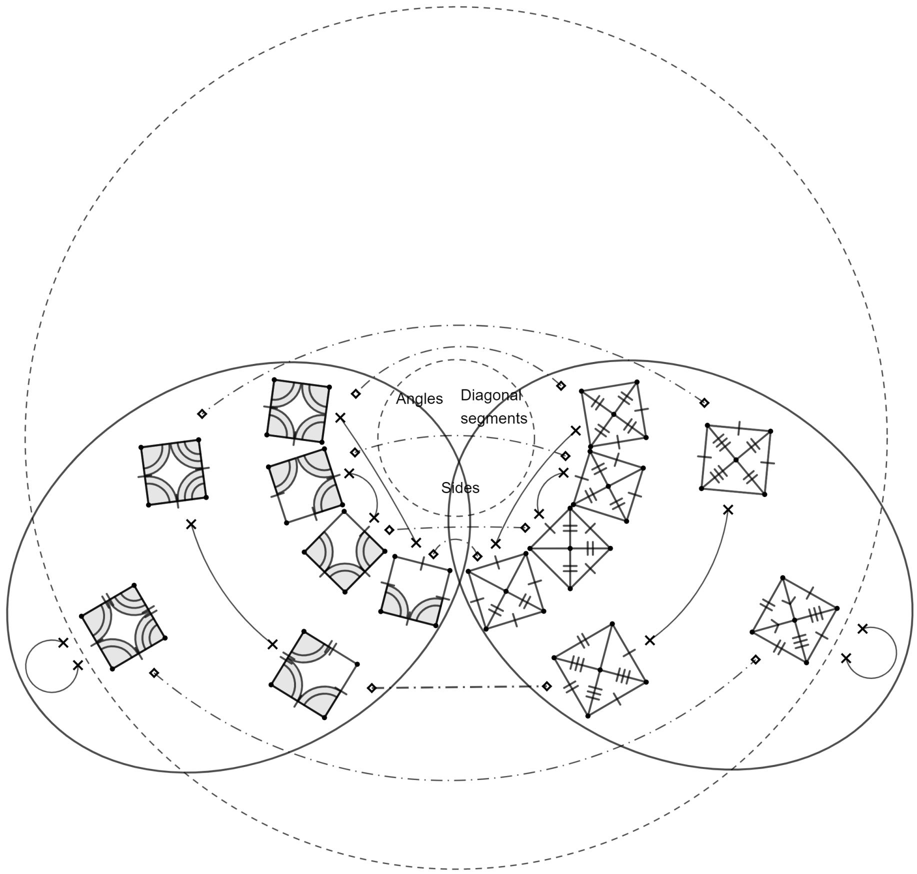

Figure 8: Family square

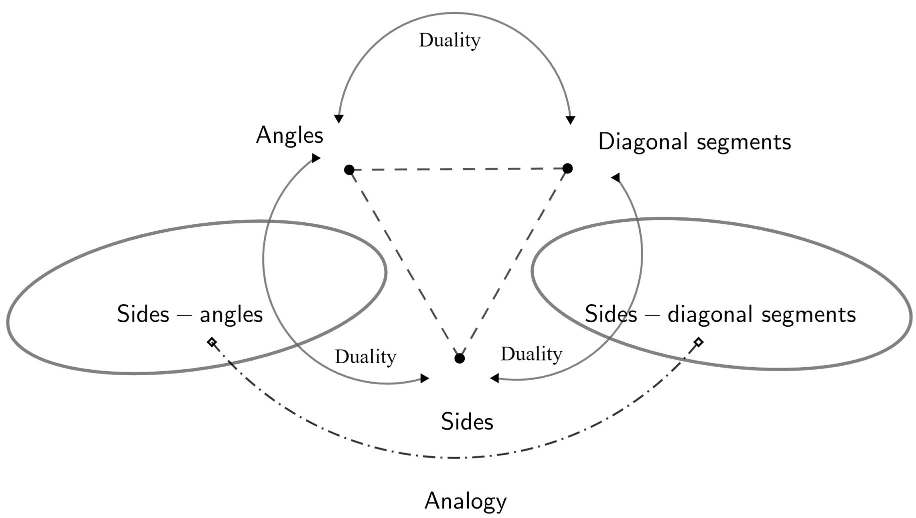

Figure 9: Dualities sides-angles, sides-diagonal segments, angles-diagonal segments and analogy between sides-angles and sides-diagonal segments

from Table 1 appear. The third column indicates the family that is obtained by adding the condition that the diagonals are perpendicular to each definition. The fourth column indicates whether the new definition is economic or not. As in Table 2, the conditions that define each family do not make a difference if the tetragon is convex or not, but in the following table we consider only tetragons with perpendicular diagonals and that are convex.

Family	Definition(s)	Family with $\perp$ diagonals	Economical definition
1ops	At least one pair of opposite sides equal ($a = c$)	1ops $\perp$ d	Yes
1op $\angle$	At least one pair of opposite angles equal ($A = C$)	Kite	Yes
1opd	At least one pair of opposite diagonal segments equal ($a' = c'$)	Kite	Yes
1cps	At least one pair of consecutive sides equal ($a = b$)	Kite	Yes
1cp $\angle$	At least one pair of consecutive angles equal ($A = B$)	1cp $\angle$ $\perp$ d	Yes
1cpd	At least one pair of consecutive diagonal segments equal ($a' = b'$)	1cpd $\perp$ d	Yes
3s	At least three sides equal ($d = a = b$)	Rhombus	Yes
3$\angle$	At least three angles equal ($D = A = B$)	Kite 3$\angle$	Yes
3d	At least three diagonal segments equal ($d' = a' = b'$)	Kite 3d	Yes
1ops-1cp $\angle$	At least one pair of opposite sides equal and at least one pair of consecutive angles equal ($a = c$ and $A = B$)	1ops-1cp $\angle$ $\perp$ d	Yes
1ops-1op $\angle$	At least one pair of opposite sides equal and at least one pair of opposite angles equal ($a = c$ and $A = C$)	Rhombus	Yes
1cps-1op $\angle$	At least one pair of consecutive sides equal and at least one pair of opposite angles equal ($d = a$ and $A = C$)	Rhombus	Yes
1cps-1cp $\angle$(1)	At least one pair of consecutive sides equal and at least one pair of consecutive angles equal ($d = a$ and $A = B$)	Kite 3$\angle$	Yes
1cps-1cp $\angle$(2)	At least one pair of consecutive sides equal and at least one pair of consecutive angles equal ($d = a$ and $C = D$)	Kite 3$\angle$	Yes
1ops-1cpd	At least one pair of opposite sides equal and at least one pair of consecutive diagonal segments equal ($a = c$ and $a' = b'$)	1ops-1cpd $\perp$ d	Yes
1ops-1opd	At least one pair of opposite sides equal and at least one pair of opposite diagonal segments equal ($a = c$ and $a' = c'$)	Rhombus	Yes
1cps-1opd	At least one pair of consecutive sides equal and at least one pair of opposite diagonal segments equal ($d = a$ and $a' = c'$)	Rhombus	Yes
1cps-1cpd(1)	At least one pair of consecutive sides equal and at least one pair of consecutive diagonal segments equal ($d = a$ and $a' = b'$)	Kite 3d	Yes
1cps-1cpd(2)	At least one pair of consecutive sides equal and at least one pair of consecutive diagonal segments equal ($d = a$ and $c' = d'$)	Kite 3d	Yes
1opd-1cp $\angle$	At least one pair of opposite diagonal segments equal and at least one pair of consecutive angles equal ($a' = c'$ and $A = B$)	Kite 3$\angle$	Yes
1cpd-1op $\angle$	At least one pair of consecutive diagonal segments equal and at least one pair of opposite angles equal ($a' = b'$ and $A = C$)	Kite 3d	Yes
1cpd-1cp $\angle$	At least one pair of consecutive diagonal segments equal and at least one pair of consecutive angles equal ($d' = a'$ and $A = B$)	1cpd-1cp $\angle$ $\perp$ d	Yes
3s-1cp $\angle$	At least three sides equal and at least one pair of consecutive angles equal ($d = a = b$ and $A = B$)	Square	Yes
1cps-3$\angle$	At least one pair of consecutive sides equal and at least three angles equal ($a = b$ and $D = A = B$)	Square	Yes
3s-1cpd	At least three sides equal and at least one pair of consecutive diagonal segments equal ($d = a = b$ and $a' = b'$)	Square	Yes
1cps-3d	At least one pair of consecutive sides equal and at least three diagonal segments equal ($d = a$ and $a' = b' = c'$)	Square	Yes
Parallelogram	Two pairs of opposite sides equal ($a = c$ and $b = d$)	Rhombus	Yes
	Two pairs of opposite angles equal ($A = C$ and $B = D$)	Rhombus	Yes
	Two pairs of opposite diagonal segments equal ($a' = c'$ and $b' = d'$)	Rhombus	Yes
	At least a pair of opposite angles equal and at least a pair of opposite diagonal segments equal ($a' = c'$ and $B = D$)	Rhombus	Yes
Kite	Two pairs of consecutive sides equal ($a = b$ and $c = d$)	Kite	No
	At least one pair of consecutive sides equal and at least one pair of opposite angles equal ($a = b$ and $A = C$)	Kite	No
	At least one pair of consecutive sides equal and at least one pair of opposite diagonal segments equal ($a = b$ and $a' = c'$)	Kite	No
	At least a pair of opposite diagonal segments equal and at least a pair of opposite angles equal ($a' = c'$ and $A = C$)	Kite	No
Isosceles trapezium	Two pairs of consecutive angles equal ($A = B$ and $C = D$)	Isosceles Trapezium $\perp$ d	Yes
	Two pairs of consecutive diagonal segments equal (a' = b' and c' = d')	Isosceles Trapezium $\perp$ d	Yes
	At least one pair of opposite sides equal and at least one pair of consecutive angles equal ($d = b$ and $A = B$)	Isosceles Trapezium $\perp$ d	Yes
	At least one pair of opposite sides equal and at least one pair of consecutive diagonal segments equal ($d = b$ and $a' = b'$)	Isosceles Trapezium $\perp$ d	Yes

Family	Definition(s)	Family with $\perp$ diagonals	Economical definition
Kite 3$\angle$	Two pairs of consecutive sides equal and at least one pair of consecutive angles equal ($a = b$, $c = d$ and $A = B$)	Kite 3$\angle$	No
	At least one pair of consecutive sides equal and at least three angles equal ($a = b$ and $D = A = B$) (1)	Kite 3$\angle$	No
	At least one pair of consecutive sides equal and at least three angles equal ($c = d$ and $D = A = B$) (2)	Kite 3$\angle$	No
	At least one pair of opposite diagonal segments equal and at least three angles equal ($a' = c'$ and $D = A = B$)	Kite 3$\angle$	No
Kite 3d	Two pairs of consecutive sides equal and at least one pair of consecutive diagonal segments equal ($a = b$, $c = d$ and $a' = b'$)	Kite 3d	No
	At least one pair of consecutive sides equal and at least three diagonal segments equal ($a = b$ and $d' = a' = b'$) (1)	Kite 3d	No
	At least one pair of consecutive sides equal and at least three diagonal segments equal ($c = d$ and $d' = a' = b'$) (2)	Kite 3d	No
	At least three diagonal segments equal and at least one pair of opposite angles equal ($d' = a' = b'$ and $A = C$)	Kite 3d	No
Isosceles trapezium 3s	Two pairs of consecutive angles equal and at least one pair of consecutive sides equal ($A = B$, $C = D$ and $d = a$)	Square	Yes
	Two pairs of consecutive diagonal segments and at least one pair of consecutive sides equal ($a' = b'$, $c' = d'$ and $d = a$)	Square	Yes
	At least three sides equal and at least one pair of consecutive angles equal ($d = a = b$ and $A = B$) (1)	Square	Yes
	At least three sides equal and at least one pair of consecutive angles equal ($d = a = b$ and $C = D$) (2)	Square	Yes
	At least three sides equal and at least one pair of consecutive diagonal segments equal ($d = a = b$ and $a' = b'$) (1)	Square	Yes
	At least three sides equal and at least one pair of consecutive diagonal segments equal ($d = a = b$ and $c' = d'$) (2)	Square	Yes
Rhombus	Four sides equal ($a = b = c = d$)	Rhombus	No
	At least three sides equal and at least one pair of opposite angles equal ($d = a = b$ and $A = C$)	Rhombus	No
	At least three sides equal and at least one pair of opposite diagonal segments equal ($d = a = b$ and $a' = c'$)	Rhombus	No
	Two pairs of consecutive sides equal and at least one pair of opposite angles equal ($a = b$, $c = d$ and $A = C$)	Rhombus	No
	Two pairs of consecutive sides equal and at least one pair of opposite diagonal segments equal ($a = b$, $c = d$ and $a' = c'$)	Rhombus	No
	At least one pair of consecutive sides equal and two pairs of opposite angles equal ($d = a$, $A = C$ and $B = D$)	Rhombus	No
	At least one pair of consecutive sides equal and two pairs of opposite diagonal segments equal ($d = a$, $a' = c'$ and $b' = d'$)	Rhombus	No
Rectangle	Four angles equal ($A = B = C = D$)	Square	Yes
	Four diagonal segments equal ($a' = b' = c' = d'$)	Square	Yes
	At least three angles equal and at least one pair of opposite sides equal ($A = B = C$ and $b' = d'$)	Square	Yes
	At least three diagonal segments equal and at least one pair of opposite angles equal ($d = a = b$ and $B = D$)	Square	Yes
	At least three angles equal and at least one pair of consecutive sides equal ($A = B = C$ and $a' = b'$) (1)	Square	Yes
	At least three angles equal and at least one pair of consecutive sides equal ($A = B = C$ and $d' = a'$) (2)	Square	Yes
	At least three diagonal segments equal and at least one pair of consecutive angles equal ($d = a = b$ and $A = B$) (1)	Square	Yes
	At least three diagonal segments equal and at least one pair of consecutive angles equal ($d = a = b$ and $D = A$) (2)	Square	Yes
	Two pairs of opposite sides equal and at least one pair of consecutive angles equal ($a = c$, $b = d$ and $A = B$)	Square	Yes
	Two pairs of opposite sides equal and at least one pair of consecutive diagonal segments equal ($a = c$, $b = d$ and $a' = b'$)	Square	Yes
	At least one pair of opposite sides equal and two pairs of consecutive angles equal ($a = c$, $A = B$ and $C = D$)	Square	Yes
	At least one pair of opposite sides equal and two pairs of consecutive diagonal segments equal ($a = c$, $a' = b'$ and $c' = d'$)	Square	Yes
	At least one pair of opposite sides equal and at least three angles equal ($a = c$, $A = B = C$)	Square	Yes

Family	Definition(s)	Family with $\perp$ diagonals	Economical definition
Square	At least one pair of opposite sides equal and at least three diagonal segments equal ($a = c$, $a' = b' = c'$)	Square	Yes
	Two pairs of consecutive angles equal and at least one pair of consecutive diagonal segments equal ($A = B$, $C = D$ and $a' = b'$)	Square	Yes
	Two pairs of consecutive diagonal segments equal and at least one pair of consecutive angles equal ($a' = b'$, $c' = d'$ and $A = B$)	Square	Yes
	Two pairs of consecutive angles equal and at least one pair of opposite diagonal segments equal ($A = B$, $C = D$ and $a' = c'$)	Square	Yes
	Two pairs of consecutive diagonal segments equal and at least one pair of opposite angles equal ($a' = b'$, $c' = d'$ and $A = C$)	Square	Yes
	At least one pair of consecutive sides equal and four angles equal ($d = a$, $A = B = C = D$)	Square	No
	At least one pair of consecutive sides equal and four diagonal segments equal ($d = a$, $a' = b' = c' = d'$)	Square	No
	At least three sides equal and three angles equal ($a = b = c$, $C = D = A$) (1)	Square	No
	At least three sides equal and three angles equal ($a = b = c$, $A = B = C$) (2)	Square	No
	At least three sides equal and three diagonal segments equal ($a = b = c$, $c' = d' = a'$) (1)	Square	No
	At least three sides equal and three diagonal segments equal ($a = b = c$, $a' = b' = c'$) (2)	Square	No
	Four sides equal and at least one pair of consecutive angles equal ($a = b = c = d$ and $A = B$)	Square	No
	Four sides equal and at least one pair of consecutive diagonal segments equal ($a = b = c = d$ and $a' = b'$)	Square	No
	At least three sides equal and two pairs of conse-cutive angles equal ($a = b = c$, $A = B$ and $C = D$)	Square	No
	At least three sides equal and two pairs of conse-cutive diagonal segments equal ($a = b = c$, $a' = b'$ and $c' = d'$)	Square	No
	Two pairs of consecutive sides equal and at least three angles equal ($a = b$, $c = d$ and $D = A = B$)	Square	No
	Two pairs of consecutive sides equal and at least three angles equal ($a = b$, $c = d$ and $d' = a' = b'$)	Square	No
	Two pairs of consecutive sides equal and two pairs of consecutive angles equal ($a = b$, $c = d$, $A = B$ and $C = D$)	Square	No
	Two pairs of consecutive sides equal and two pairs of consecutive diagonal segments equal ($a = b$, $c = d$, $a' = b'$ and $c' = d'$)	Square	No

Table 2: The families with perpendicular diagonals and their respective economical hierarchical definitions

The families square, rhombus, kite 3d, kite 3$\angle$ and kite that appear in the first column of Table 2 have perpendicular diagonals as a consequence of the conditions that define them (second column). If we add the condition that the diagonals are perpendicular, the same families are defined but with non-economic definitions (see third and fourth columns).

New definitions were obtained for the square: Eighteen from the definitions of rectangle, six from the definitions of isosceles trapezoid, and four from the families at least one pair of equal sides, angles, or diagonal segments and at least three sides or angles equal.

New definitions are obtained for the rhombus: Four from the parallelogram definitions, one from 3s, and four from at least one pair of sides equal and at least one pair of angles or diagonal segments equal (1ops-1cpd, 1ops-1opd, 1ops-1op $\angle$, 1cps-1op $\angle$).

The four definitions of isosceles trapezoid, with the addition of perpendicular diagonals define the isosceles trapezoid of perpendicular diagonals (Is tr $\perp$ d).

Four new definitions of kite 3$\angle$ arise from the families 3$\angle$, 1cps-1cp $\angle$(1), 1cps-1cp $\angle$(2), 1opd-1cp $\angle$.

Four new definitions of kite 3d arise from the families 3d, 1cps-1cpd(1), 1cps-1cpd(2), 1cpd-1op ∠.

Three definitions of kite arise from 1cps, 1op ∠, 1opd respectively.

Six new families are generated as subfamilies of 1ops, 1cp ∠, 1cpd, 1ops-1cp ∠, 1ops-1cpd, 1cpd-1cp ∠.

7 The Diagram to Families with Perpendicular Diagonals as Definition Condition

Figures 10, 11, 12, 13 are analogous to Figures 4, 5, 6, 7 respectively, with the addition of having perpendicular diagonals. Each of the families that appear in the figures that follow is included in the respective family of the previous figures.

The cases in which the definition is not economical when adding the condition of perpendicularity between the diagonals were left indicated with dotted circles.

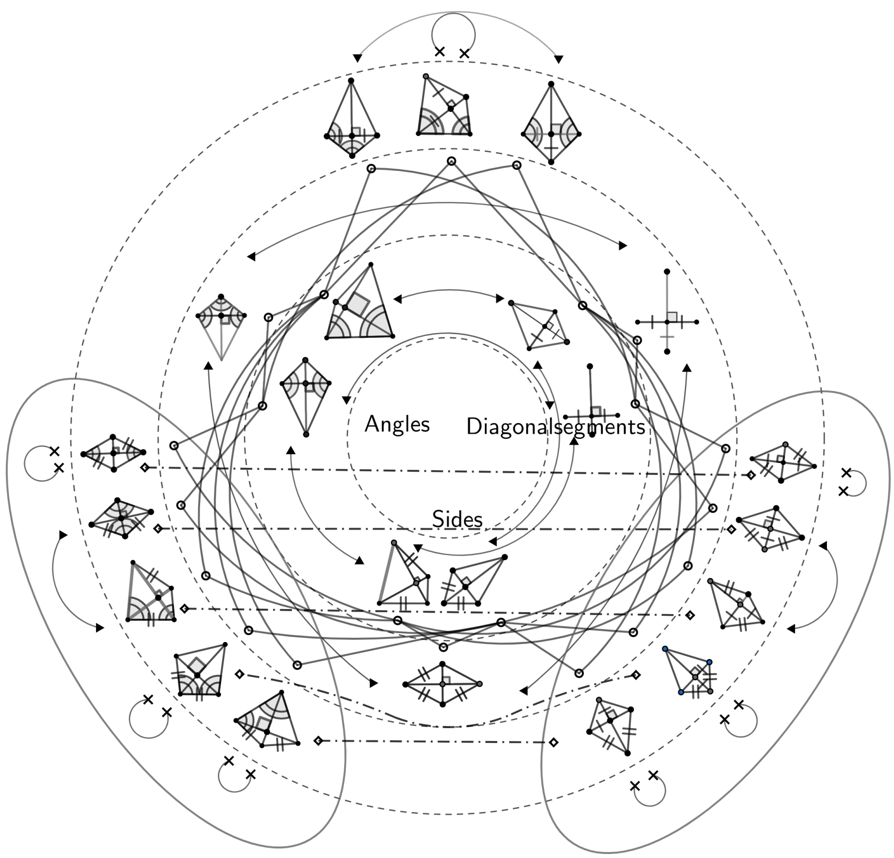

Figure 10: Families at least one pair equal with perpendicular diagonals, at least three equal with perpendicular diagonals, at least one pair equal and at least one pair equal with perpendicular diagonals

8 Second Conclusions

Just as the isosceles trapezoid with perpendicular diagonals is defined as a particular case of the isosceles trapezoid, new families with perpendicular diagonals are defined for the families 1ops, 1cp ∠, 1cpd, 1ops-1cp ∠, 1ops-1cpd, 1cpd-1cp ∠. Thus, seven new families

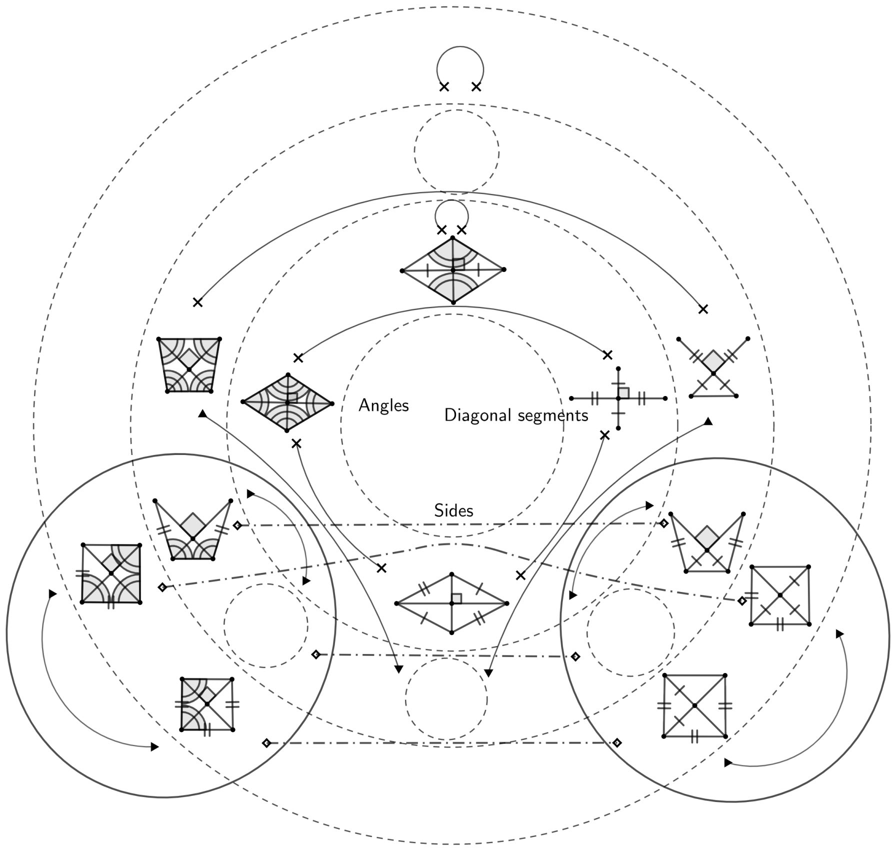

Figure 11: Families parallelogram with perpendicular diagonals, kite with perpendicular diagonals, isosceles trapezium with perpendicular diagonals, at least a pair equal and at least three equal with perpendicular diagonals

were generated with a definition each. Adding the condition of perpendicular diagonals to the original tetragons definitions too allowed generating this new economic definitions: twenty eight to square, nine to rhombus, four to kite 3d, four to kite 3∠, three to kite. Forty-eight new economic definitions were thus generated for these last five families.

9 Final Comments

The guide for this work was de Villiers' statement in [2, p.14] "...the most important function of an a priori classification is clearly the *discovery/creation* of new concepts." And new families of tetragons were defined by this classification and put in relation to each other and to families already known. Others well-known families like cyclic, tangential, extangential, trapezium, for example, do not appear in it. This is just one more possibility to classify convex tetragons and surely new classifications can be made that can illuminate new aspects of the fascinating world of tetragons.

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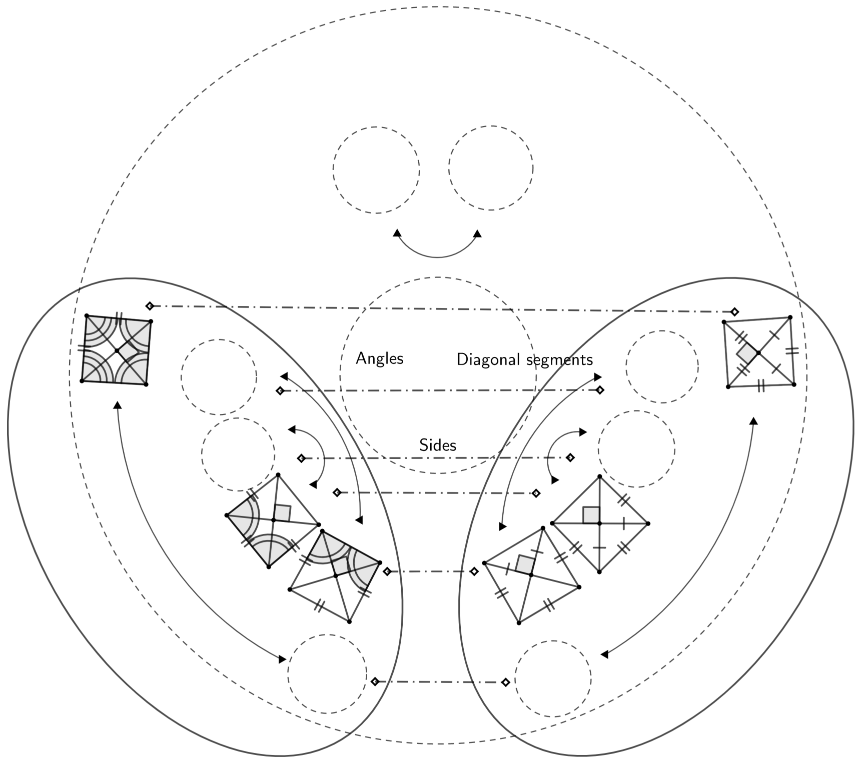

Figure 12: Families kite three equal angles with perpendicular diagonals, kite three equal diagonal segments with perpendicular diagonals and isosceles trapezium three equal sides with perpendicular diagonals

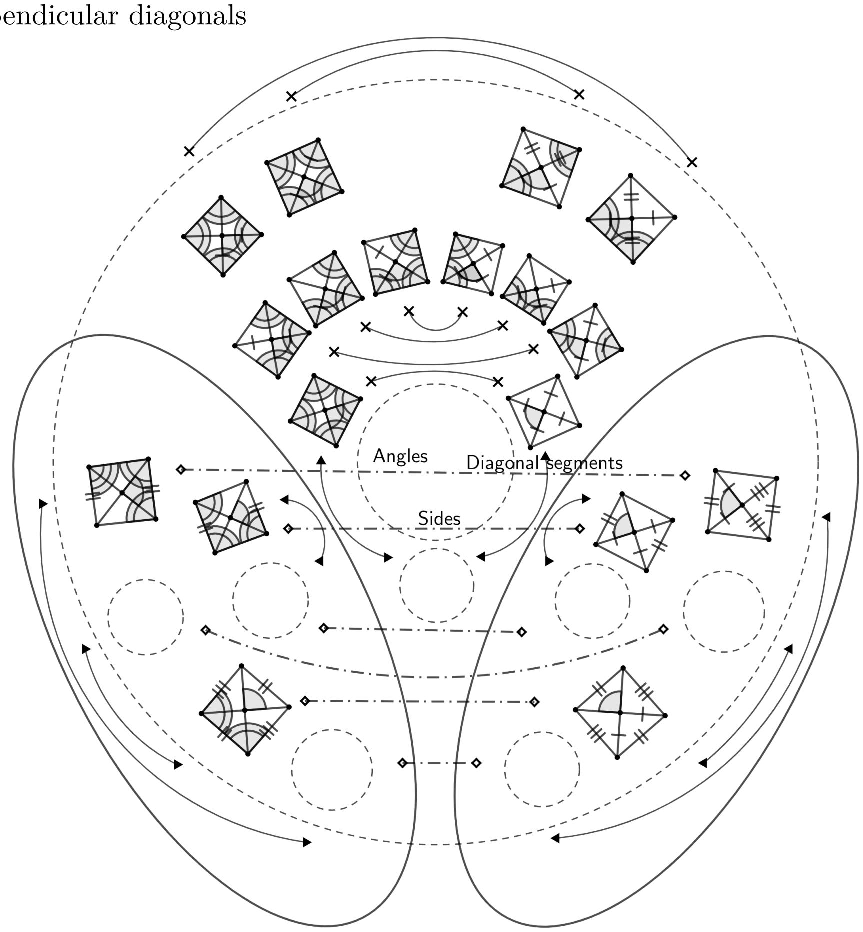

Figure 13: Families rhombus and rectangle with perpendicular diagonals

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