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Concurrent Segments in a Tetrahedron – Applications of Ceva's and Carnot's Theorems

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Abstract. Ceva's theorem is about concurrence of three segments on a triangle with an affine ratio. Among the several theorems named after him, we are interested in Carnot's theorem that relates the concurrence of two segments in a skew quadrilateral in space, again, with an affine ratio. First, we apply these theorems to obtain a theorem on the concurrence of seven segments in a tetrahedron. Secondly, we show that the Steiner-Routh theorem implies Carnot's theorem, and obtain the volumes of the two parts of a tetrahedron separated by a planar quadrilateral. Thirdly, we consider a special case of Carnot's theorem (or an extension of Varignon's theorem) to determine when four points on a skew quadrilateral are to form a parallelogram. Finally, we give a new characterization of the centroid of a tetrahedron.

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1 Introduction

Let us begin with definitions.

Definition 1. Four non-co-planar points A, B, C, D are the vertices of a tetrahedron $ABCD$, the six segments joining these points are the edges of the tetrahedron, the four triangles built by any triple out of the vertices are the faces of the tetrahedron, and the tetrahedron carries (up to orientation and cyclically rearranged labels) three *skew quadrilaterals:* $DABC$, $DACB$, and $DBAC$. The *skew quadrilateral* $DABC$ consists of edges DA, AB, BC, and CD.

Theorems 1 and 2 in Section 2 were inspired by Ceva's theorem for a triangle and Carnot's theorem for a skew quadrilateral.

Theorem (Ceva's Theorem; see Theorem 1.21 on Page 4 of [4], or Theorem 326 on Page 159 of [2].)**.** *Let ABC be a triangle. Let C'', A'', B'' be points on the edges AB, BC, and CA, respectively. Then AA'', BB'', and CC'' concur if, and only if* $\frac{AB''}{B''C} \cdot \frac{CA''}{A''B} \cdot \frac{BC''}{C''A} = 1$.

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There are several theorems named after Carnot, but the one we are interested in is the following.

Theorem (Carnot's Theorem; see Page 111, Theorem 329 in [1].)**.** *Let $DABC$ be a skew quadrilateral. Let E, F, G, H be points on the edges DA, AB, BC, and CD, respectively. Then, $EFGH$ is a planar quadrilateral if, and only if, $\frac{DE}{EA}\cdot\frac{AF}{FB}\cdot\frac{BG}{GC}\cdot\frac{CH}{HD}=1$. In other words, the segments EG and FH concur (or E, F, G, H are coplanar) if and only if $\frac{DE}{EA}\cdot\frac{AF}{FB}\cdot\frac{BG}{GC}\cdot\frac{CH}{HD}=1$.*

Note. If a skew quadrilateral $DABC$ is *not* planar, edges EF and GH of the planar quadrilateral $EFGH$ in Carnot's theorem do not intersect since the edges EF and GH are on the two distinct triangles BAD and BCD, respectively, that share only the edge BD. Similarly, the edges EH and FG of the quadrilateral $EFGH$ do not intersect.

There are several generalizations of Ceva's theorem to a tetrahedron as in [9], and to n-dimensional ($n \geq 2$) simplices as in [3, 6, 8]. Our Theorem 1 has a resemblance to the one given by K. Witczynski [9], which states as follows:

Theorem (Witczynski's Theorem; see [9] or [3].)**.** *Let E, F, G, H, I, J be points on the edges DA, AB, BC, CD, AC, BD of a tetrahedron $ABCD$, respectively. Suppose that $\frac{DE}{EA}\cdot\frac{AF}{FB}\cdot\frac{BJ}{JD}=1$, $\frac{AF}{FB}\cdot\frac{BG}{GC}\cdot\frac{CI}{IA}=1$, $\frac{AI}{IC}\cdot\frac{CH}{HD}\cdot\frac{DE}{EA}=1$, and $\frac{DJ}{JB}\cdot\frac{BG}{GC}\cdot\frac{CH}{HD}=1$. Let A', B', C', D' be the intersection of (BH, CJ, and DG), (AH, CE, and DI), (AJ, BE, and DF), and (AG, BI, and CF), respectively. Then, the segments AA', BB', CC', and DD' concur.*

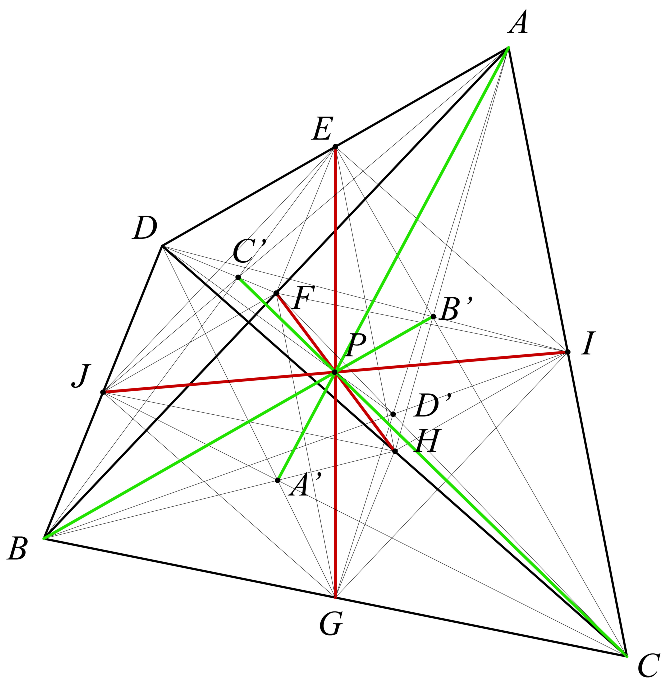

Figure 1: Explanation: The six thick black lines form the tetrahedron $ABCD$; the three red lines EG, FH, and IJ connect opposing pairs of edges in the tetrahedron; the four green lines AA', BB', CC', and DD' connect vertices to the points A', B', C', and D' in the faces of the tetrahedron; all the thin black lines lie in the faces of the tetrahedron connecting (A, B, D, E, F, J) in the face ABD, (A, B, C, F, G, I) in the face ABC, (A, C, D, I, E, H) in the face ACD, and (B, C, D, G, H, J) in the face BCD.

Theorem 1 and Theorem 2 in Section 2 are converse to each other, and should be treated as one theorem. Using notations in Witczynski's theorem, the essence of our Theorems 1 and

2 will be to prove that the segments EG, FH, and IJ are concurrent at P if, and only if, the four segments AA', BB', CC', and DD' concur at P as shown in Figure 1.

In Section 3, we will show that Carnot's theorem is a consequence of the Steiner-Routh theorem, and we will obtain the volumes of two parts of a tetrahedron separated by a planar quadrilateral in Theorem 3 using a similar idea used by Marko and Litvinov in [6] to prove the Steiner-Routh theorem.

The natural special case is when E, F, G, H are midpoints of the sides of a skew quadrilateral $DABC$. In this case, the planar quadrilateral $EFGH$ does not only divide the tetrahedron $ABCD$ into two equal volumes (by Theorem 3), but it is also a parallelogram (a special case of Carnot's theorem or an extension of Varignon's theorem). Motivated by this, we will obtain a necessary and sufficient condition when $EFGH$ on a skew quadrilateral $DABC$ is to form a parallelogram (Theorem 4).

Further, the midpoints of a skew quadrilateral also remind us of the centroids of a tetrahedron. Suppose P is a point inside of the tetrahedron $ABCD$, and suppose A', B', D', and C' are intersections of (the line AP and the face BCD), (BP and ACD), (CP and ABD), and (DP and ABC), respectively. If P is the centroid of the tetrahedron, then it is known that $\frac{PA'}{AP} = \frac{PB'}{BP} = \frac{PC'}{CP} = \frac{PD'}{DP} = \frac{1}{3}$ (see Theorem 170 in [1]). In Theorem 5, we will strengthen this result to prove that P is the centroid of a tetrahedron $ABCD$ if, and only if, $\frac{PA'}{AP} \cdot \frac{PB'}{BP} \cdot \frac{PC'}{CP} \cdot \frac{PD'}{DP} = \frac{1}{81}$. We will use Theorem 2 to prove this.

2 Concurrence Theorems

Theorems 1 and 2 below are converse to each other. Theorem 1 resembles to Witczynski's theorem, but we do not use any products of ratios in the statement nor in the proof.

Theorem 1. *Suppose E, F, G, H are points on the edges DA, AB, BC, and CD, respectively, of a tetrahedron $ABCD$ such that EG and FH intersect at a point P. Let A', B', C', and D' be the intersections of the segments (BH and DG), (AH and CE), (BE and DF), and (AG and CF), respectively. Then, the following are true:*

(1) The segments AA', BB', CC', and DD' concur at P.

(2) The lines AC' and CA' intersect, say at J, on the edge BD. The lines BD' and DB' intersect, say at I, on the edge AC.

(3) The lines (BC' and CB'), (DC' and CD'), (AD' and DA'), (AB' and BA') intersect at E, F, G, and H, respectively.

Proof of (1). The intersection of the triangles HAB and FCD is the segment FH, and the intersection of the triangles BCD and ADG is the segment EG. Since the segments EG and FH concur at P, the intersection of the four triangles HAB, FCD, EBC, and GAD is P.

Since the intersection of the triangles GAD and HAB is the segment AA', AA' intersects the triangle EBC at P. Since the intersection of the triangles HAB and EBC is the segment BB', BB' intersects the triangle FCD at P.

Since the intersection of the triangles EBC and FCD is the segment CC', CC' intersects the triangle GAD at P. And finally, since the intersection of the triangles FCD and GAD is the segment DD', DD' intersects the triangle HAB at P.

Therefore, we conclude that the segments AA', BB', CC', and DD' concur at P. □

Proof of (2). The lines AC' and CA' lie on the plane ACP, and they are not parallel. So, they intersect, say at J. Since CJ and AJ are on the tetrahedron $ABCD$, J must be on BD. Similarly, BD' and DB' intersect, say at I. □

Proof of (3). We know that C' is on BE so that the intersection of BC' with AD is E. The lines BC' and CB' are on the plane PBC, and they intersect. Since CB' intersects AD, the three lines BC', CB' and AD must intersect at E. Similarly, we can show that the segments (DC' and CD'), (AD' and DA'), (AB' and BA') intersect at F, G, and H, respectively. □

Remark 1. The condition "EG and FH intersect" in Theorem 1 can be replaced by the equation $\frac{DA}{ED} \cdot \frac{AF}{FB} \cdot \frac{BG}{GC} \cdot \frac{CH}{HD} = 1$ by Carnot's theorem. If we do this, the point P is not defined. This was an incentive to write the theorem using "EG and FH intersect".

Theorem 2. *Suppose P is a point inside a tetrahedron $ABCD$. Suppose A', B', C', D' are intersections of (the line AP and the face BCD), (BP and ACD), (CP and ABD), and (DP and ABC), respectively. Then the following are true:*

(1) The lines (BC' and CB'), (DC' and CD'), (AD' and DA'), (AB' and BA'), (AC' and CA'), and (BD' and DB') intersect, say at E, F, G, H, I, and J, respectively. The points E, F, G, H, I, and J are points on the edges DA, AB, BC, CD, AC, and BD, respectively.

(2) The segments EG, FH, and IJ intersect at P. Hence, $EFGH$, $EIGJ$, and $FJHI$ are planar quadrilaterals.

Proof of (1). The plane PBC contains the segments BC' and CB', so they intersect and the intersection is on DA. We can prove the others similarly. □

Proof of (2). On the triangle ABD, since the intersection of AJ, BE, and DG is C', we have $\frac{DE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BJ}{JD} = 1$ by Ceva's theorem. Similarly, on the triangle BCD, we have $\frac{DJ}{JB} \cdot \frac{BG}{GC} \cdot \frac{CH}{HD} = 1$. By this latter equation, we have $\frac{BJ}{JD} = \left(\frac{DJ}{JB}\right)^{-1} = \frac{BG}{GC} \cdot \frac{CH}{HD}$.

Substituting this into the first equation, we have $\frac{DE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BG}{GC} \cdot \frac{CH}{HD} = 1$. Applying Carnot's theorem to the skew quadrilateral $DABC$, we know that $EFGH$ is a planar quadrilateral so that EG and FH intersect, say at Q. But by the part (1) of Theorem 1, we must have that $Q = P$, and EG and FH intersect at P. Similarly, we can show that $EIGJ$ is a planar quadrilateral since EG and IJ intersect at P. Therefore, the segments EG, FH, and IJ intersect at P. □

Corollary 1. *Let I and J be the points defined in Theorem 1(2). Then, EG, FH, and IJ concur at P.*

Proof. Since AA', BB', CC', and DD' concur at P by Theorem 1(1), we know that EG, FH, and IJ concur at P by Theorem 2(2). □

Corollary 2. *Let E, F, G, H, I, J be points on the edges DA, AB, BC, CD, AC, and BD, respectively, of a tetrahedron $ABCD$. Then, the following statements are equivalent:*

(1) EG, FH, and IJ concur.

(2) $\dfrac{DE}{EA} \cdot \dfrac{AF}{FB} \cdot \dfrac{BG}{GC} \cdot \dfrac{CH}{HD} = 1$, $\dfrac{AF}{FB} \cdot \dfrac{BJ}{JD} \cdot \dfrac{DH}{HC} \cdot \dfrac{CI}{IA} = 1$, *and* $\dfrac{DE}{EA} \cdot \dfrac{AI}{IC} \cdot \dfrac{CG}{GB} \cdot \dfrac{BJ}{JD} = 1.$

(3) $\dfrac{AF}{FB} \cdot \dfrac{BG}{GC} \cdot \dfrac{CI}{IA} = 1$, $\dfrac{AI}{IC} \cdot \dfrac{CH}{HD} \cdot \dfrac{DE}{EA} = 1$, *and* $\dfrac{DJ}{JB} \cdot \dfrac{BG}{GC} \cdot \dfrac{CH}{HD} = 1.$

(4) $\dfrac{DE}{EA} \cdot \dfrac{AF}{FB} \cdot \dfrac{BJ}{JD} = 1$, $\dfrac{AF}{FB} \cdot \dfrac{BG}{GC} \cdot \dfrac{CI}{IA} = 1$, *and* $\dfrac{DJ}{JB} \cdot \dfrac{BG}{GC} \cdot \dfrac{CH}{HD} = 1.$

Proof. (1) implies (2) by Carnot's theorem. Conversely, by Carnot's theorem, (2) implies that (EG and FH), (FH and IJ) and (EG and IJ) intersect. Thus, three lines EG, FH, and IJ intersect. Since these three lines are not in the same plane, EG, FH, and IJ must concur. Hence, (1) and (2) are equivalent.

By Theorem 1(2) and (3), and by Ceva's theorem, (1) implies (4). Clearly, (4) implies (3).

Suppose (3) holds. Since $\frac{AF}{FB} \cdot \frac{BG}{GC} \cdot \left(\frac{CI}{IA}\right) = 1$, we have $\frac{AI}{IC} = \frac{AF}{FB} \cdot \frac{BG}{GC}$. Substituting this into $\left(\frac{AI}{IC}\right) \cdot \frac{CH}{HD} \cdot \frac{DE}{EA} = 1$, we have

$$\left(\frac{AF}{FB} \cdot \frac{BG}{GC}\right) \cdot \left(\frac{CH}{HD} \cdot \frac{DE}{EA}\right) = \frac{DE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BG}{GC} \cdot \frac{CH}{HD} = 1.$$

From $\frac{DJ}{JB} \cdot \frac{BG}{GC} \cdot \left(\frac{CH}{HD}\right) = 1$, we have $\frac{BG}{GC} = \left(\frac{DJ}{JB} \cdot \frac{CH}{HD}\right)^{-1} = \frac{BJ}{JD} \cdot \frac{DH}{HC}$. Substituting this into $\frac{AF}{FB} \cdot \left(\frac{BG}{GC}\right) \cdot \frac{CI}{IA} = 1$, we have

$$\frac{AF}{FB} \cdot \left(\frac{BJ}{JD} \cdot \frac{DH}{HC}\right) \cdot \frac{CI}{IA} = \frac{AF}{FB} \cdot \frac{BJ}{JD} \cdot \frac{DH}{HC} \cdot \frac{CI}{IA} = 1.$$

From $\frac{DJ}{JB} \cdot \frac{BG}{GC} \cdot \left(\frac{CH}{HD}\right) = 1$, we have $\frac{CH}{HD} = \frac{BJ}{JD} \cdot \frac{CG}{GB}$. Substituting this into $\frac{AI}{IC} \cdot \left(\frac{CH}{HD}\right) \cdot \frac{DE}{EA} = 1$, we have

$$\frac{AI}{IC} \cdot \left(\frac{BJ}{JD} \cdot \frac{CG}{GB}\right) \cdot \frac{DE}{EA} = \frac{DE}{EA} \cdot \frac{AI}{IC} \cdot \frac{CG}{GB} \cdot \frac{BJ}{JD} = 1.$$

Hence, (3) implies (2).

Therefore, we have shown the equivalence of (1)–(4). □

Remark 2. Theorem 1 and Corollary 2 imply Witczynski's theorem.

Remark 3. Using the notations in Theorem 1 (or the ones in Theorem 2), it is interesting to note that the intersection of the three planes EBC, FCD, and IDB is P. Hence, Theorem 2 in [5] implies that $\frac{AP}{PA'} = \frac{AF}{FB} + \frac{AI}{IC} + \frac{AE}{ED}$.

3 Steiner-Routh Theorem Related Results

Carnot's theorem is proved using Ceva's theorem and Menelaus's theorem in [1]. However, it is also implied by the Steiner-Routh theorem which reads as follows:

Theorem (Steiner-Routh Theorem; see [7])**.** *Let $ABCD$ be a tetrahedron. Let E, F, G, H be points on the edges DA, AB, BC, and CD, respectively. Let $\frac{DE}{EA} = x$, $\frac{AF}{FB} = y$, $\frac{BG}{GC} = z$, $\frac{CH}{HD} = w$. Let the volumes of the two tetrahedra $EFGH$ and $ABCD$ be denoted by V_{EFGH} and V, respectively. Then*

$$V_{EFGH} = \frac{|1 - xyzw|}{(1+x)(1+y)(1+z)(1+w)} V.$$

Proof of Carnot's Theorem. Since $\frac{DE}{EA} = x$, $\frac{AF}{FB} = y$, $\frac{BG}{GC} = z$, $\frac{CH}{HD} = w$, the tetrahedron $EFGH$ has the volume zero if, and only if $xyzw = \frac{DE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BG}{GC} \cdot \frac{CH}{HD} = 1$. However, the tetrahedron $EFGH$ has volume zero if, and only if $EFGH$ is a planar quadrilateral, i.e., the segments EG and FH concur. (As we noted in the Introduction, two opposite edges of the planar quadrilateral $EFGH$ do not intersect.) □

Carnot lived from 1753 to 1823, and Steiner lived from 1796 to 1863. It is not clear when Carnot published his theorem. However, Steiner published his original paper in 1828 after Carnot's death, according to [7]. Hence, Carnot's theorem seems older than the Steiner-Routh theorem.

Next, we will find the two volumes of the divided tetrahedron by a quadrilateral.

Notation 1. Let the volumes of the tetrahedron $EFGH$ be denoted by V_{EFGH}.

Theorem 3. *Let $ABCD$ be a tetrahedron. Let E, F, G, H be points on the edges DA, AB, BC, and CD, respectively. Let $\frac{DE}{EA} = x$, $\frac{AF}{FB} = y$, $\frac{BG}{GC} = z$, $\frac{CH}{HD} = w$. Suppose that $xyzw = 1$. Then, there are two parts of the tetrahedron separated by the planar quadrilateral $EFGH$, one contains the edge AC (denote its volume by V_{AC}) and the other contains BD (denote its volume by V_{BD}). Then, we have the following identities:*

$$V_{DB} = \frac{1+x+z+xy+xz+zw+xyz+xzw}{(1+x)(1+y)(1+z)(1+w)}V, \quad \textit{and}$$
$$V_{AC} = \frac{1+y+w+xw+yz+yw+xyw+yzw}{(1+x)(1+y)(1+z)(1+w)}V,$$

where V is the volume of the tetrahedron $ABCD$.

Proof. We use similar ideas as used in [7] to prove the Steiner-Routh theorem. Let $\frac{AE}{AD} = a$, $\frac{AF}{AB} = b$, $\frac{BG}{GC} = c$, $\frac{CH}{CD} = d$. Then, $0 < a, b, c, d < 1$, and we have $a = \frac{1}{1+x}$; $b = \frac{y}{1+y}$; $c = \frac{z}{1+z}$; and $d = \frac{w}{1+w}$. Note that $V_{DABH} = (1-d)V$ since the tetrahedra $DABH$ and $DABC$ share the base DAB, and since the height of $DABH$ is shrunk by $1-d$ from $DABC$. Similarly, we have $V_{DAFH} = bV_{DABH}$, and $V_{DEFH} = (1-a)V_{DAFH}$. Hence, we have $V_{DEFH} = (1-a)V_{DAFH} = (1-a)bV_{DABH} = (1-a)b(1-d)V = \frac{xy}{(1+x)(1+y)(1+w)}V$.

Similarly, we can see that $V_{DFBH} = (1-b)V_{DABH} = (a-b)(a-d)V = \frac{1}{(1+y)(1+w)}V$, and $V_{BGHF} = cV_{BCHF} = cdV_{BCDF} = cd(1-b)V = \frac{zw}{(1+y)(1+z)(1+w)}V$.

Therefore, we have

$$\begin{aligned} V_{DB} &= V_{DEFH} + V_{DFBH} + V_{BGHF} \\ &= \frac{xy}{(1+x)(1+y)(1+w)}V + \frac{1}{(1+y)(1+w)}V + \frac{zw}{(1+y)(1+z)(1+w)}V \\ &= \frac{1+x+z+xy+xz+zw+xyz+xzw}{(1+x)(1+y)(1+z)(1+w)}V. \end{aligned}$$

Also, since $xyzw = 1$, we have

$$\begin{aligned} V_{AC} = V - V_{DB} &= \frac{y+w+xw+yz+xyw+yzw+xyzw}{(1+x)(1+y)(1+z)(1+w)}V \\ &= \frac{1+y+w+xw+yz+yw+xyw+yzw}{(1+x)(1+y)(1+z)(1+w)}V. \quad \square \end{aligned}$$

4 A Special Case of Carnot's Theorem/An Extension of Varignon's Theorem

If E, F, G, H are the midpoints of the edges DA, AB, BC, and CD, respectively, of a skew quadrilateral $ABCD$, the planar quadrilateral $EFGH$ splits the tetrahedron $ABCD$

into two equal volumes by Theorem 3. However, it is known that the midpoints of a planar quadrilateral are the vertices of a parallelogram (Varignon's theorem, see Theorem 3.11 of [4] or Theorem 249 of [2]). This is still true for a skew quadrilateral. That is, the midpoints of a skew quadrilateral are the vertices of a parallelogram. We can generalize this to the following as a special case of Carnot's theorem or as an extension of Varignon's theorem.

Theorem 4. *Let $DABC$ be a skew quadrilateral. Let E, F, G, H be points on the edges DA, AB, BC, and CD, respectively. Then $EFGH$ is a parallelogram if, and only if,*

$$\frac{DE}{EA}\cdot\frac{AF}{FB}=1,\quad \frac{DH}{HC}\cdot\frac{CG}{GB}=1,\quad \textit{and}\quad \frac{DE}{EA}=\frac{DH}{HC}.$$

(Note that $\frac{DE}{EA}\cdot\frac{AF}{FB}=1$, $\frac{DH}{HC}\cdot\frac{CG}{GB}=1$, and $\frac{DE}{EA}=\frac{DH}{HC}$ is equivalent to $\frac{DE}{EA}=\frac{DH}{HC}=\frac{BF}{FA}=\frac{BG}{GC}$.)

Proof. We use vectors. Let $\overrightarrow{DA}=\vec{a}$, $\overrightarrow{DB}=\vec{b}$, $\overrightarrow{DC}=\vec{c}$.

Suppose $\frac{DE}{EA}\cdot\frac{AF}{FB}=1$, $\frac{DH}{HC}\cdot\frac{CG}{GB}=1$, and $\frac{DE}{EA}=\frac{DH}{HC}$. Let $\frac{DE}{EA}=x$. Then $\frac{AF}{FB}=\frac{1}{x}$. Hence,

$$\overrightarrow{EA}=\frac{1}{x+1}\vec{a},\quad \overrightarrow{AF}=\frac{1}{x+1}\overrightarrow{AB}=\frac{1}{1+x}(\vec{b}-\vec{a}).$$

This shows that

$$\overrightarrow{EF}=\overrightarrow{EA}+\overrightarrow{AF}=\frac{1}{x+1}(\vec{b}-\vec{a})+\frac{1}{x+1}\vec{a}=\frac{1}{x+1}\vec{b}.$$

Similarly, we can show that $\overrightarrow{HG}=\frac{1}{x+1}\vec{b}$. Thus, $EFGH$ is a parallelogram.

Conversely, suppose $EFGH$ is a parallelogram. For some $0<s,t<1$, we have $\overrightarrow{DE}=s\vec{a}$, $\overrightarrow{AF}=t(\vec{b}-\vec{a})$, and $\overrightarrow{DF}=\vec{a}+t(\vec{b}-\vec{a})$. Hence, $\overrightarrow{EF}=\overrightarrow{DF}-\overrightarrow{DE}=(1-s-t)\vec{a}+t\vec{b}$. Similarly, for some $0<x,y<1$, we have $\overrightarrow{DH}=x\vec{c}$, $\overrightarrow{CG}=y(\vec{b}-\vec{c})$, and $\overrightarrow{DG}=\vec{c}+x(\vec{b}-\vec{c})$. Hence, $\overrightarrow{HG}=\overrightarrow{DG}-\overrightarrow{DH}=(1-x-y)\vec{c}+y\vec{b}$. Since $EFGH$ is a parallelogram, we must have $(1-s-t)\vec{a}+t\vec{b}=\overrightarrow{EF}=\overrightarrow{HG}=(1-x-y)\vec{c}+y\vec{b}$. Since $\vec{a}$ and $\vec{c}$ are not parallel, this shows that $1-s-t=0$, $1-x-y=0$, and $t=y$. Hence, $s=1-t=1-y=x$. Also, $\frac{DE}{DA}=s=x=\frac{DH}{DC}$. This implies $\frac{DE}{EA}=\frac{DH}{HC}=\frac{s}{1-s}$. And $\frac{AF}{AB}=t=y=\frac{CG}{CB}$ implies $\frac{AF}{FB}=\frac{CG}{GB}=\frac{t}{1-t}=\frac{1-s}{s}$. Therefore, we must have $\frac{DE}{EA}\cdot\frac{AF}{FB}=\frac{DH}{HC}\cdot\frac{CG}{GB}=1$ and $\frac{DE}{EA}=\frac{DH}{HC}$. □

Remark 4. Let E, F, G, H be points in Theorem 4 that form a parallelogram. Let A', B', C', and D' be the intersections of (BH and DG), (AH and CE), (BE and DF), and (AG and CF), respectively. We know that the segments AC' and CA' intersect, say at J, on the edge BD by Theorem 1. Further, the segments BD' and DB' intersect, say at I, on the edge AC. Since $\frac{DE}{EA}=\frac{DH}{HC}$, we have $\frac{AE}{ED}\cdot\frac{DH}{HC}=1$. Since we have $\frac{AE}{ED}\cdot\frac{DH}{HC}\cdot\frac{CI}{IA}=1$ be Ceva's theorem, we have $\frac{CI}{IA}=1$. Similarly, we have $\frac{BJ}{JD}=1$. Therefore, if $EFGH$ is a parallelogram, the points I and J are always the midpoints of AC and BD, respectively.

5 Centroids

We continue to investigate when E, F, G, H are the midpoints of the edges DA, AB, BC, and CD of a tetrahedron $ABCD$, respectively. Let A', B', C', and D' be the intersections of (BH and DG), (AH and CE), (BE and DF), and (AG and CF), respectively, as in Theorem 1. Then, A', B', C', and D' are the *centroids* of the triangular faces BCD, ACD,

ABD, and ABC, respectively (see Page 7, [4]). Moreover, the point of concurrency P of the segments AA', BB', CC', and DD' is the *centroid* of the tetrahedron $ABCD$. And we also have that $\frac{PA'}{AP} = \frac{PB'}{BP} = \frac{PC'}{CP} = \frac{PD'}{DP} = \frac{1}{3}$. (See [1], Theorem 170 on page 51.)

On the other hand, let P be a point inside the tetrahedron $ABCD$. Let A', B', C', and D' be the points of intersections of (the line AP and the plane BCD), (BP and ACD), (CP and ABD), and (DP and ABC), respectively, as in Theorem 2. Now, we raise the following question: If $\frac{PA'}{AP} = \frac{PB'}{BP} = \frac{PC'}{CP} = \frac{PD'}{DP} = \frac{1}{3}$, is P the centroid of the tetrahedron $ABCD$? The answer is yes. However, since we could not find this converse statement in the literature, we will prove this result for a tetrahedron in Lemma 1 using Theorem 2. Further, we will give a weaker characterization that P is the centroid of a tetrahedron $ABCD$ if, and only if, $\frac{PA'}{AP} \cdot \frac{PB'}{BP} \cdot \frac{PC'}{CP} \cdot \frac{PD'}{DP} = \frac{1}{81}$ in Theorem 5. We will obtain a similar result for a triangle in Corollary 3.

Notation 2. Let $\mathcal{A}_{ABC}$ denote the area of the triangle ABC.

Lemma 1. *Let P be a point inside the tetrahedron $ABCD$. Let A', B', C', D' be the intersection of (the line AP and the plane BCD), (BP and ACD), (CP and ABD), and (DP and ABC), respectively. Then, P is the centroid of the tetrahedron $ABCD$ if, and only if, $\frac{PA'}{AP} = \frac{PB'}{BP} = \frac{PC'}{CP} = \frac{PD'}{DP} = \frac{1}{3}$.*

Proof. If P is the centroid of the tetrahedron $ABCD$, then $\frac{PA'}{AP} = \frac{PB'}{BP} = \frac{PC'}{CP} = \frac{PD'}{DP} = \frac{1}{3}$ by Theorem 170 in [1].

Conversely, suppose $\frac{PA'}{AP} = \frac{PB'}{BP} = \frac{PC'}{CP} = \frac{PD'}{DP} = \frac{1}{3}$. By Theorem 2(1), we know that BC' and CB' intersect on the edge AD. Let E, F, G, H be the intersections of (BC' and CB' on AD), (CD' and DC' on AB), (DA' and AD' on BC), and (AB' and BA' on CD), respectively. We first show that E is the midpoint of AD. Since AA', DD', and EG intersect by Theorem 2, we have $\frac{DA'}{A'G} \cdot \frac{GD'}{D'A} \cdot \frac{AE}{ED} = 1$ by Ceva's theorem applied to the triangle ADG. Hence, we have

$$\frac{AE}{ED} = \frac{GA'}{A'D} \cdot \frac{AD'}{D'G}. \tag{$*$}$$

On the other hand, since $\frac{3}{4}AA' = AP$ and $\frac{3}{4}DD' = DP$ from $\frac{PA'}{AP} = \frac{PD'}{DP} = \frac{1}{3}$, we have $\frac{3}{4}\mathcal{A}_{AA'D} = \mathcal{A}_{APD} = \frac{3}{4}\mathcal{A}_{ADD'}$ so that $\mathcal{A}_{AA'D} = \mathcal{A}_{AD'D}$.

But $\mathcal{A}_{AA'D} = \frac{DA'}{DG}\mathcal{A}_{AGD}$ and $\mathcal{A}_{AD'D} = \frac{AD'}{AG}\mathcal{A}_{AGD}$. Thus, $\frac{DA'}{DG} = \frac{AD'}{AG}$. Since $DG = DA' + A'G$ and $AG = AD' + D'G$, we have $\frac{AD'}{D'G} = \frac{DA'}{A'G}$.

Substituting this into $(*)$, we have $\frac{AE}{ED} = \frac{GA'}{A'D} \cdot \frac{DA'}{A'G} = 1$. Therefore, $AE = ED$, i.e., E is the midpoint of the edge AD.

Similarly, we can show that F, G, H are the midpoints of the edges AB, BC, and CD. These show that A', B', C', D' are the centroids of the triangles BCD, ACD, ABD, and ABC, respectively. Since P is the intersection of the segments AA', BB', CC', and DD', P is the centroid of the tetrahedron $ABCD$. □

The proof of the next theorem uses the method of Lagrange multipliers.

Theorem 5. *Let P be a point inside or on the face of a tetrahedron $ABCD$ different from the vertices A, B, C, and D. Let A', B', C', and D' be intersections of (the line AP and the face BCD), (BP and ACD), (CP and ABD), and (DP and ABC), respectively. Then,*

(1) $\dfrac{PA'}{AA'} + \dfrac{PB'}{BB'} + \dfrac{PC'}{CC'} + \dfrac{PD'}{DD'} = 1$, *and*

(2) $\dfrac{PA'}{AP}\cdot\dfrac{PB'}{BP}\cdot\dfrac{PC'}{CP}\cdot\dfrac{PD'}{DP}\le\dfrac{1}{81}$.

(3) The point P is the centroid of the tetrahedron $ABCD$ if, and only if,

$$\frac{PA'}{AP}\cdot\frac{PB'}{BP}\cdot\frac{PC'}{CP}\cdot\frac{PD'}{DP}=\frac{1}{81}$$

Proof of (1). Let H_A, H_B, H_C, H_D be the feet of the altitudes from A, B, C, D to the planes BCD, ACD, ABD, ABC, respectively. Then the volume $V=V_{ABCD}$ of the tetrahedron $ABCD$ is given by $V=\frac{1}{3}\mathcal{A}_{BCD}\cdot AH_A=\frac{1}{3}\mathcal{A}_{BCD}\cdot AA'\cdot\sin\sphericalangle AA'H_A$. Similarly,

$$V=\frac{1}{3}\mathcal{A}_{ACD}\cdot BB'\cdot\sin\sphericalangle BB'H_B=\frac{1}{3}\mathcal{A}_{ABD}\cdot CC'\cdot\sin\sphericalangle CC'H_C=\frac{1}{3}\mathcal{A}_{ABC}\cdot DD'\cdot\sin\sphericalangle DD'H_D.$$

From these, we have

$$\text{(i)}\ \frac{1}{3}\mathcal{A}_{BCD}\cdot\sin\sphericalangle AA'H_A=\frac{V}{AA'},\qquad \text{(ii)}\ \frac{1}{3}\mathcal{A}_{ACD}\cdot\sin\sphericalangle BB'H_B=\frac{V}{BB'},$$
$$\text{(iii)}\ \frac{1}{3}\mathcal{A}_{ABD}\cdot\sin\sphericalangle CC'H_C=\frac{V}{CC'},\quad\text{and}\quad \text{(iv)}\ \frac{1}{3}\mathcal{A}_{ABC}\cdot\sin\sphericalangle DD'H_D=\frac{V}{DD'}.$$

Since $V_{BCDP}=\frac{1}{3}\mathcal{A}_{BCD}\cdot PA'\cdot\sin\sphericalangle AA'H_A$, $V_{CADP}=\frac{1}{3}\mathcal{A}_{ACD}\cdot PB'\cdot\sin\sphericalangle BB'H_B$, $V_{ABDP}=\frac{1}{3}\mathcal{A}_{ABD}\cdot PC'\cdot\sin\sphericalangle CC'H_C$, $V_{ABCP}=\frac{1}{3}\mathcal{A}_{ABC}\cdot PD'\cdot\sin\sphericalangle DD'H_D$, and since $V=V_{BCDP}+V_{CADP}+V_{ABDP}+V_{ABCP}$, we have

$$\text{(v)}\quad V=\frac{1}{3}\mathcal{A}_{BCD}\cdot PA'\cdot\sin\sphericalangle AA'H_A+\frac{1}{3}\mathcal{A}_{ACD}\cdot PB'\cdot\sin\sphericalangle BB'H_B$$
$$+\frac{1}{3}\mathcal{A}_{ABD}\cdot PC'\sin\sphericalangle CC'H_C+\frac{1}{3}\mathcal{A}_{ABC}\cdot PD'\cdot\sin\sphericalangle DD'H_D.$$

Substituting (i)–(iv) into (v), we have $V=\frac{PA'\cdot V}{AA'}+\frac{PB'\cdot V}{BB'}+\frac{PC'\cdot V}{CC'}+\frac{PD'\cdot V}{DD'}$. Dividing both sides by V, we obtain $\frac{PA'}{AA'}+\frac{PB'}{BB'}+\frac{PC'}{CC'}+\frac{PD'}{DD'}=1$. □

Proof of (2) and (3). If $PA'\cdot PB'\cdot PC'\cdot PD'=0$, then the inequality (2) holds since we always have the inequality $AP\cdot BP\cdot CP\cdot DP>0$. So, we assume that $PA'\cdot PB'\cdot PC'\cdot PD'\neq 0$. That is, we assume that P is inside the tetrahedron $ABCD$.

Let $AP=a$, $BP=b$, $CP=c$, $DP=d$, $PA'=x$, $PB'=y$, $PC'=z$, $PD'=w$. Then, we want to

$$\begin{array}{ll}\text{maximize} & \frac{x}{a}\cdot\frac{y}{b}\cdot\frac{z}{c}\cdot\frac{w}{d}\\ \text{subject to} & \frac{x}{a+x}+\frac{y}{b+y}+\frac{z}{c+z}+\frac{w}{d+w}=1,\quad\text{and}\quad a,b,c,d,x,y,z,w>0.\end{array}$$

(This constraint is from the equation (1) of this theorem expressed in terms of a, b, c, d, x, y, z and w, as in $\frac{PA'}{AA'}=\frac{x}{a+x}$, for example.)

Let $s=\frac{x}{a}$, $t=\frac{y}{b}$, $u=\frac{z}{c}$, $v=\frac{w}{d}$. Then $\frac{x}{a+x}=\frac{as}{a+as}=\frac{s}{1+s}$. Similarly, we have $\frac{y}{b+y}=\frac{t}{1+t}$, $\frac{z}{c+z}=\frac{u}{1+u}$, $\frac{w}{d+w}=\frac{v}{1+v}$. Then we can restate our maximizing problem to

$$\begin{array}{ll}\text{maximize} & f(s,t,u,v)=stuv\\ \text{subject to} & \frac{s}{1+s}+\frac{t}{1+t}+\frac{u}{1+u}+\frac{v}{1+v}=1,\quad\text{and}\quad s,t,u,v>0.\end{array}$$

Let $g(s,t,u,v) = \frac{s}{1+s} + \frac{t}{1+t} + \frac{u}{1+u} + \frac{v}{1+v} - 1$. Then critical points (s,t,u,v) are given by $\nabla f(s,t,u,v) = \lambda \nabla g(s,t,u,v)$ for some constant λ, called a Lagrange multiplier. Here, ∇f is the gradient of the function f. Hence, we have

$$\langle tuv, suv, stv, stu\rangle = \lambda \Big\langle \frac{1}{(1+s)^2}, \frac{1}{(1+t)^2}, \frac{1}{(1+u)^2}, \frac{1}{(1+v)^2} \Big\rangle.$$

Thus, from the first two components of the above vector equation, we have $\lambda = (1+s)^2 tuv = (1+t)^2 suv$. This implies $(t-s)(st-1) = 0$. Hence, $t = s$ or $t = \frac{1}{s}$.

Since $t = \frac{1}{s}$ implies $\frac{t}{1+t} = \frac{1/s}{1+1/s} = \frac{1}{1+s}$, and since $\frac{s}{1+s} + \frac{t}{1+t} + \frac{u}{1+u} + \frac{v}{1+v} = 1$, the equation $t = \frac{1}{s}$ implies that $\left(\frac{s}{1+s} + \frac{1}{1+s}\right) + \frac{u}{1+u} + \frac{v}{1+v} = 1$, i.e., $u = v = 0$. This is a contradiction to $u, v > 0$. Thus, we must have $t = s$. Similarly, we have $u = s$ and $v = s$. So, $\frac{s}{1+s} + \frac{t}{1+t} + \frac{u}{1+u} + \frac{v}{1+v} = 1$ gives us that $\frac{4s}{1+s} = 1$ or $s = \frac{1}{3}$. Hence, $t = u = v = \frac{1}{3}$. Since $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{81}$, and since we can make one of s, t, u, v as close to 0 as we want, we see that the maximum value of f is $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{81}$. Hence, we have $f(s,t,u,v) = stuv \leq \frac{1}{81}$ or $\frac{x}{a} \cdot \frac{y}{b} \cdot \frac{z}{c} \cdot \frac{w}{d} \leq \frac{1}{81}$. Therefore, we have shown that

$$\frac{PA'}{AP} \cdot \frac{PB'}{BP} \cdot \frac{PC'}{CP} \cdot \frac{PD'}{DP} \leq \frac{1}{81}.$$

The equality holds only when $s = t = u = v = \frac{1}{3}$, or when $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{w}{d} = \frac{PA'}{AP} = \frac{PB'}{BP} = \frac{PC'}{CP} = \frac{PD'}{DP} = \frac{1}{3}$. In other words, the equality in (2) holds if, and only if, P is the centroid of the tetrahedron $ABCD$ by Lemma 1. This proves the statement (3). □

We can obtain a similar theorem for a triangle from Theorem 5.

Corollary 3. *Let P be a point inside or on a triangle ABC different from the vertices A, B, and C. Let A'', B'', C'' be the intersections of (AP and BC), (BP and AC), and (CP and AB) respectively. Then, we have*

(1) $\frac{PA''}{AA''} + \frac{PB''}{BB''} + \frac{PC''}{CC''} = 1$, *and*

(2) $\frac{PA''}{AP} \cdot \frac{PB''}{BP} \cdot \frac{PC''}{CP} \leq \frac{1}{8}$.

(3) The point P is the centroid of the triangle ABC if, and only if, $\frac{PA''}{AP} \cdot \frac{PB''}{BP} \cdot \frac{PC''}{CP} = \frac{1}{8}$.

Proof of (1). Let P be an interior point or on the sides of the triangle ABC, different from A, B, and C. Let $ABCD$ be a tetrahedron having the triangle ABC as the base. Let $D' = P$. Then, $PD' = 0$ and $A'' = A$, $B'' = B'$, $C'' = C'$ in Theorem 5. Hence, we have

$$\frac{PA''}{AA''} + \frac{PB''}{BB''} + \frac{PC''}{CC''} = \frac{PA'}{AA'} + \frac{PB'}{BB'} + \frac{PC'}{CC'} + \frac{PD'}{DD'} = 1$$

by Theorem 5(1). □

Proof of (2) and (3). Unlike the proof of Theorem 5(2), we prove this without Lagrange multipliers.

Let $AP = a$, $BP = b$, $CP = c$, $PA'' = x$, $PB'' = y$, $PC'' = z$. Then, (1) becomes

$$\frac{x}{a+x} + \frac{y}{b+y} + \frac{z}{c+z} = 1.$$

This can be rewritten as

$$x(b+y)(c+z) + y(z+x)(c+z) + z(a+x)(b+y) = (a+x)(b+y)(c+z.)$$

This simplifies to $abc = 2xyz + ayz + bxz + cxy$.

By the Arithmetic-Geometric Mean Inequality applied to the right side of this equation, we have $abc = 2xyz + ayz + bxz + cxy \geq 4((2xyz)(ayz)(bxz)(cxy))^{1/4} = 4(2(xyz)^3abc)^{1/4}$. Hence, $(abc)^4 \geq 4^4\{2(xyz)^3abc\}$. This simplifies to $abc \geq 8xyz$, or

$$\frac{PA''}{AP} \cdot \frac{PB''}{BP} \cdot \frac{PC''}{CP} \leq \frac{1}{8}.$$

The equality holds when $2xyz = ayz = bxz = cxy$ so that $a = 2x$, $b = 2y$, $c = 2z$. The equation $a = 2x$ shows that $\frac{PA''}{PA} = \frac{x}{a} = \frac{1}{2}$. Similarly, $\frac{PB''}{PB} = \frac{PC''}{PC} = \frac{1}{2}$. Thus,

$$\frac{PA''}{PA} = \frac{PB''}{PB} = \frac{PC''}{PC} = \frac{1}{2}.$$

Similar to Lemma 1, we have that if P be a point inside the triangle ABC, and if A'', B'', C'' be the intersections of (AP and BC), (BP and AC), and (CP and AB), respectively, then P is the centroid of the triangle ABC if, and only if, $\frac{PA''}{AP} = \frac{PB''}{BP} = \frac{PC''}{CP} = \frac{1}{2}$. Therefore, the equality in (2) holds if, and only if, P is the centroid of the triangle ABC. This proves the statement (3). □

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