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From Permutation Points to Permutation Cubics*

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Dedicated to Hellmuth Stachel on the occasion of his 80th birthday.

Abstract. The trilinear coordinates of a point V in the plane of a triangle can be permuted in six ways which yields the six permutation points of V. These six points always lie on a single conic, called the permutation conic. A natural variant or generalization seems to be: The six permutation points of V together with the six permutation points of V's image under a certain quadratic Cremona transformation γ comprise a set of twelve points that always lie on a single cubic which we shall call the permutation cubic of V with respect to γ. In the present paper we shall discuss especially the cases where γ is the isogonal or the isotomic conjugation. Properties and remarkable features of these cubics shall be elaborated.

Key Words: permutation point, triangle cubic, permutation cubic, triangle center, antiorthic axis, Mandart circumellipse

MSC 2020: 14H45 (primary), 51N35

1 Introduction

The geometry of the triangle is rich with centers, central lines, central triangles, central circles, and central conics (for details see [8], [13]). However, this wealth is not restricted to linear and quadratic objects. Many geometric and algebraic questions related to the triangle lead to cubic polynomials and forms, and thus, to cubic curves (see [1, 2, 6, 9, 11], [12]). Among the approximately 1300 cubics listed in B. GIBERT's *Catalogue of Triangle Cubics* [12], we find the Neuberg cubic, the Thomson cubic (sometimes referred to as the 17 point cubic), the MacCay cubic, the Darboux cubic, the Napoleon (or Feuerbach) cubic, the Orthocubic, and many more. All these curves are defined by means of geometric or algebraic properties. Some

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of them can be defined as the images of certain curves under special quadratic transformations related to the triangle, such as the isogonal or isotomic conjugation (cf. [12], [7]). Wide ranges of triangle cubics can also be defined by means of metric properties, cf. [12], [11].

In the following, we assume that we are given a triangle $\Delta = ABC$ in the Euclidean plane. Δ's side lengths are $a = \overline{BC}$, $b = \overline{CA}$, and $c = \overline{AB}$. The vertices A, B, C together with the incenter X_1 shall be the base points and the unit point of the projective coordinate frame. Here and in the following, we use X_i in order to denote the i-th triangle center according to C. Kimberling's list given in [8], [13]. With respect to the chosen frame, the vertices of Δ and its incenter X_1 are described by the homogeneous trilinear coordinates

$$A = 1:0:0, \quad B = 0:1:0, \quad C = 0:0:1, \quad X_1 = 1:1:1,$$

although we could also use barycentric coordinates or any other homogeneous coordinates with an arbitrarily chosen unit point. Any point X in the plane of Δ is uniquely described by the ratio $\xi:\eta:\zeta \neq 0:0:0$, and any such ratio defines one, and only one, point in the plane.

Following [10], we define the *permutation points* of a point $V = p:q:r$ by applying the six permutations of three elements (including the identity) to the coordinate triple (p,q,r). It can easily be shown that these six points lie on a single conic with the homogeneous equation

$$\mathcal{C}(V):\ \sum_{\text{cyc}} p^2 \sum_{\text{cyc}} \xi\eta = \sum_{\text{cyc}} \xi^2 \sum_{\text{cyc}} pq. \tag{1}$$

Herein, the symbol $\sum_{\text{cyc}}$ means the cyclic sum. This is to be understood literally: For example

$$\sum_{\text{cyc}} a = a+b+c, \qquad \sum_{\text{cyc}} \xi^2 = \xi^2+\eta^2+\zeta^2, \ldots,$$

i.e., the argument is replaced twice cyclically in alphabetic order

$$a \to b \to c \to a, \quad \xi \to \eta \to \zeta \to \xi, \quad p \to q \to r \to p, \ldots$$

The cyclic symmetry of the conic's equation displays the invariance of the conic under permutations of coordinates. Note that the representation of cyclic symmetric polynomials as cyclic sums is not always unique. For example: $\sum_{\text{cyc}} p(q^2+r^2) = \sum_{\text{cyc}} pq(p+q)$. Further, cyclic shifts in the argument do not change the cyclic sum: $\sum_{\text{cyc}} a = \sum_{\text{cyc}} b$ or $\sum_{\text{cyc}} \xi\eta = \sum_{\text{cyc}} \zeta\xi$.

The conic (1) is regular as long as V is not chosen on the octic cycle

$$\underbrace{(\xi-\eta)(\eta-\zeta)(\zeta-\xi)}_{\text{interior angle bisectors}} \cdot \underbrace{(\xi+\eta+\zeta)}_{\mathcal{L}_1} \cdot \underbrace{(\xi\eta+\varepsilon^2\eta\zeta+\varepsilon\zeta\xi)}_{c} \cdot \underbrace{(\xi\eta+\varepsilon\eta\zeta+\varepsilon^2\zeta\xi)}_{\overline{c}} = 0$$

(with $\varepsilon^2+\varepsilon+1=0$) consisting of the three interior angle bisectors, the antiorthic axis $\mathcal{L}_1$, and a pair $(c,\overline{c})$ of complex conjugate conics (since $\varepsilon^2 = \overline{\varepsilon}$) through the vertices and the incenter of Δ.

However, each permutation of the coordinates induces a collineation in the projectively closed plane of the triangle. Hence, the six permutations

$$\{\pi_0,\ldots,\pi_5\} = \{(1,2,3),(1,2,3),\ldots,(3,2,1)\}$$

correspond to six collineations

$$\mathrm{G}_6 = \{\mathrm{id}_{\mathbb{P}^2} = \kappa_0, \kappa_1, \ldots, \kappa_5\}$$

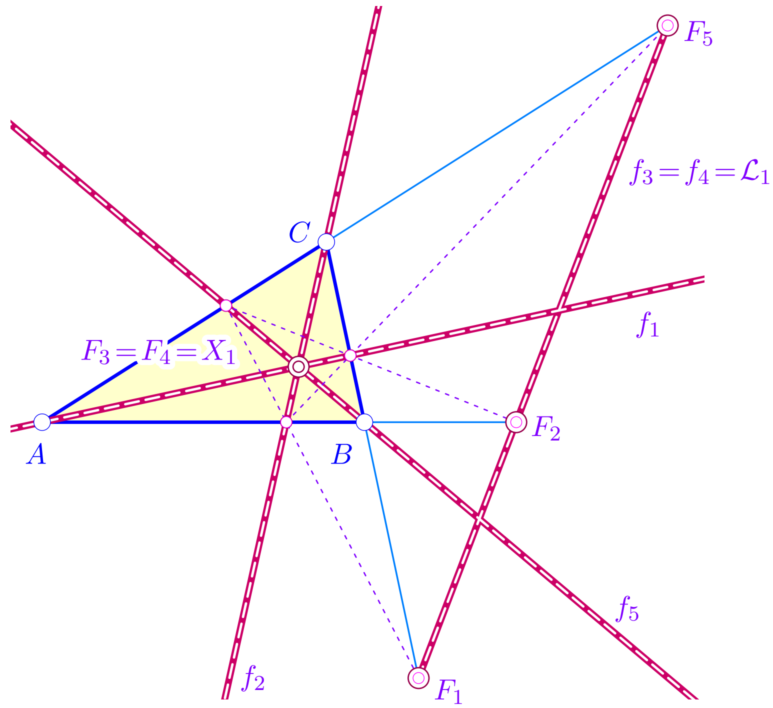

Figure 1: The self-dual figure of fixed lines and fixed points of the subgroup G_6 of automorphic collineations of permutation conics and permutation cubics.

which form a discrete group $\mathrm{G}_6 \subset \mathrm{Aut}(\mathcal{C}(V))$ and each element of this group transforms $\mathcal{C}(V)$ into itself. Except $\mathrm{id}_{\mathbb{P}^2}$, each collineation in G_6 has exactly one fixed line and exactly one non-incident fixed point. The set $\{F_1, \ldots, F_5\}$ of fixed points together with the set $\{f_1, \ldots, f_5\}$ of fixed lines forms a *self-dual figure* but not a configuration as can be seen in Fig. 1.

We can summarize in:

Lemma 1.1. *The permutation conics are fixed under each element of the group* G_6.

There are two quadratic Cremona transformations that occur regularly in triangle geometry and play an important role there: the *isogonal conjugation* ι and the *isotomic conjugation* τ (cf. [7, 8]). As is the case with any quadratic Cremona transformation with three base points, the isogonal conjugation and the isotomic conjugation are not defined at the base point, *i.e.*, at the vertices of Δ. Both mappings are degenerate on the sidelines (off the vertices). In terms of homogeneous trilinear coordinates, these mappings are given by

$$\iota(\xi, \eta, \zeta) = (\eta\zeta, \zeta\xi, \xi\eta), \quad \tau(\xi, \eta, \zeta) = (b^2c^2\eta\zeta, c^2a^2\zeta\xi, a^2b^2\xi\eta), \tag{2}$$

provided that the (Euclidean) side lengths of Δ are $a = \overline{BC}$, $b = \overline{CA}$, and $c = \overline{AB}$.

Independent of the choice of the quadratic transformation in (2), any pivot point $V = p : q : r$ defines besides its six permutation points six further permutation points of its quadratic image which makes in total twelve points. Note that permutation and isogonal conjugation commute, while permutation and isotomic conjugation do not commute.

In Section 2, we will show that the above mentioned twelve points lie on a single cubic which shall be called the *permutation cubic* of the pivot V with respect to the isogonal/isotomic conjugation. It turns out that *these triangle cubics are new* except the curve $\mathcal{K}(X_1)$ with pivot point X_1 which shows up in B. Gibert's list [12] as $\mathcal{K}_{228}$. This curve plays an exceptional role among the permutation cubics. Section 3 is devoted to the study of rational or degenerate permutation cubics, their inflection points, and permutation cubics with a triangle center for their pivot point. Finally, in Section 4 the isogonal conjugation will be replaced with the isotomic conjugation. The thus obtained curves show a similar but slightly different behaviour. They are not-self isotomic (with respect to the canonical isotomic transformation), for they do not pass through the vertices of Δ.

2 Permutation cubics defined by the isogonal transformation

The permutation points of a point $V = p : q : r$ lie on the permutation conic $\mathcal{C}(V)$ with the equation (1). The permutation conic $\mathcal{C}(\iota(V))$ of V's isotomic conjugate $\iota(V)$ has the equation

$$\sum_{\text{cyc}} \xi\eta \sum_{\text{cyc}} p^2q^2 = pqr \sum_{\text{cyc}} \xi^2 \sum_{\text{cyc}} p$$

which is regular if V is not chosen on

$$\underbrace{\xi\eta\zeta}_{\Delta} \cdot \underbrace{(\xi-\eta)(\eta-\zeta)(\zeta-\xi)}_{\text{interior angle bisectors}} \cdot \underbrace{\left(\sum_{\text{cyc}} \xi\eta\right)}_{m} \cdot \underbrace{\left(\sum_{\text{cyc}} \xi^2\eta^2 - \xi\eta\zeta \sum_{\text{cyc}} \xi\right)}_{c \,\cup\, \bar{c}} = 0$$

and does, in general, not coincide with $\mathcal{C}(V)$, see Fig. 2. This happens only if V is chosen on the ι-invariant sextic cycle

$$(\xi^2 - \eta\zeta)(\eta^2 - \zeta\xi)(\zeta^2 - \xi\eta) = 0$$

which is the union of three non-central conics passing through the unit point $X_1 = 1 : 1 : 1$. Each of these conics touches two sides of Δ and the third side plus the opposite vertex are conjugate with respect to the conic.

Now, we can show that the permutation points of V and the permutation points of $\iota(V)$, *i.e.*, twelve points lie on a single cubic:

Theorem 2.1. *The six permutation points of $V = p : q : r$ and the six permutation points of its isogonal conjugate $\iota(V) = qr : rp : pq$ are located on a self-isogonal cubic*

$$\mathcal{K}(V)\colon\ P\xi\eta\zeta - Q \sum_{cyc} \xi(\eta^2 + \zeta^2) = 0, \tag{3}$$

where $P = \sum_{cyc} p(q^2 + r^2)$ and $Q = pqr$.

Proof. The fact that the twelve above mentioned points lie on $\mathcal{K}(V)$ (with the equation (3)) can either be shown by inserting their trilinear coordinates into (3) or by computing the equation of $\mathcal{K}(V)$. For that purpose, we consider the coefficient vector of a generic planar cubic as a vector in $\mathbb{C}^{10}$. The monomials in the cubic's equation are identified with the base vectors, and then, the coefficients are the Veronese image of points incident with the cubic. The Veronese mapping $v\colon \mathbb{C}^3 \to \mathbb{C}^{10}$ sends a point $V = p : q : r$ to

$$v(V) = p^3 : q^3 : r^3 : p^2q : pq^2 : p^2r : pr^2 : q^2r : qr^2 : pqr.$$

Then, the 12×10-matrix $(v(\pi_0(V)), \ldots, v(\pi_5(V)), v(\pi_0(\iota(V))), \ldots, v(\pi_5(V)))$ is of rank 9 and its one-dimensional kernel is spanned by

$$(0, 0, 0, Q, Q, Q, Q, Q, Q, -P),$$

i.e., a multiple of the coefficient vector of the equation given in (3). The invariance of the cubic $\mathcal{K}(V)$ under the isogonal conjugation ι is veryfied by applying (2) to (3). $\square$

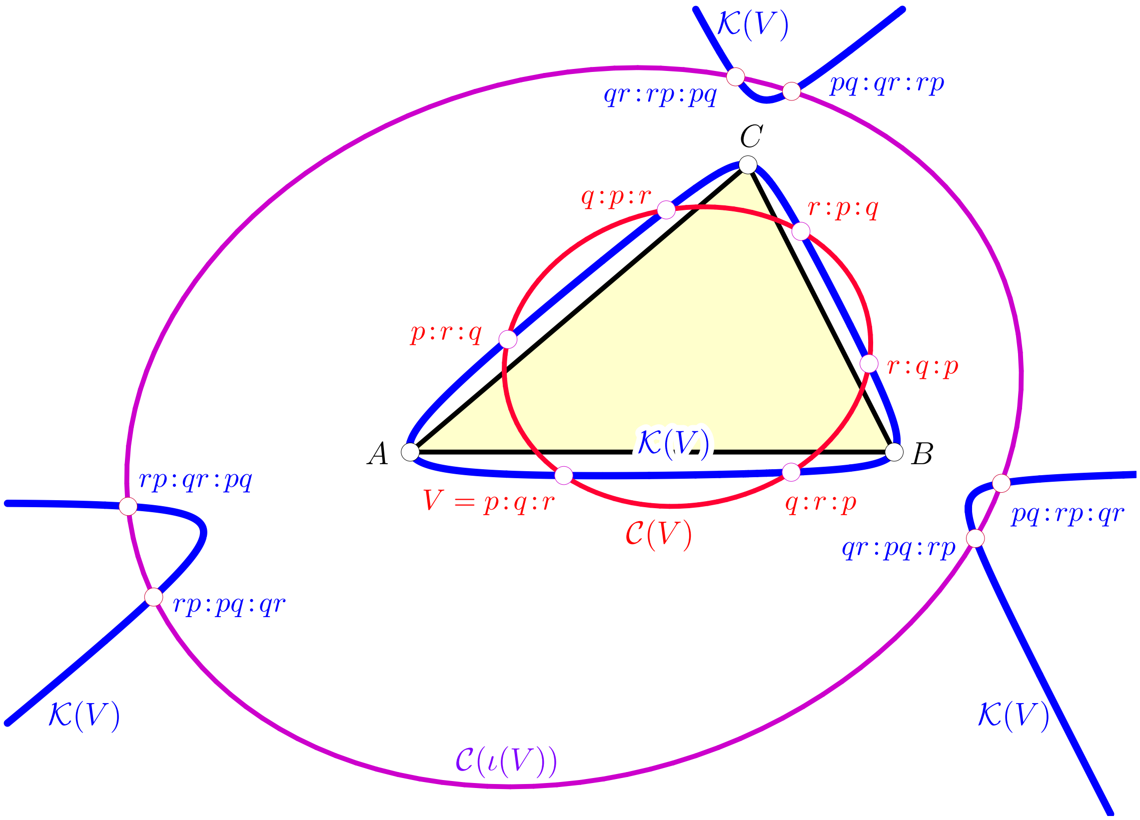

Figure 2: The permutation points of V lie on the permutation conic $\mathcal{C}(V)$ (red), the permutation points of $\iota(V)$ lie on the conic $\mathcal{C}(\iota(V))$ (violet), and all twelve points are located on the permutation cubic $\mathcal{K}(V)$.

In Fig. 2, the distribution of the permutation points of a point V and its isogonal image $\iota(V)$ are shown together with the permutations conics $\mathcal{C}(V)$ and $\mathcal{C}(\iota(V))$ of both points. Here, we remark that the pencil of conics spanned by $\mathcal{C}(V)$ and $\mathcal{C}(\iota(V))$ is a pencil of the third kind (cf. [7, p. 287]) with the antiorthic axis $\mathcal{L}_1$ (with multiplicity two) and the pair of complex conjugate lines through X_1 with the equation

$$\sum_{\text{cyc}} \xi^2 - \sum_{\text{cyc}} \xi\eta = (\xi + \varepsilon\eta + \varepsilon^2\zeta)(\xi + \varepsilon^2\eta + \varepsilon\zeta) = 0 \tag{4}$$

(again with with $\varepsilon^2 + \varepsilon + 1 = 0$, and thus, $\varepsilon^2 = \bar{\varepsilon}$) as singular conics.

Because of the cyclic symmetry of the equations (3) of the permutation cubics, these cubics are invariant under all six possible permutations of the homogeneous coordinates. Now, by virtue of Lem. 1.1, we can state:

Theorem 2.2. *Each of the permutation cubics* (3) *is transformed into itself under each element of the discrete group of collineations* G_6.

The permutation cubics (3) house 15 distinct points from the very beginning. As can be seen immediately from (3), each permutation cubic passes through the vertices of the base triangle which is a necessary (but by no means sufficient) condition on the cubics to be self-isogonal. A suitable coordinate frame can be attached to any cubic such that it is self-isogonal or self-isotomic, see [3, 4]. In the present paper, we always consider the standard form of ι and τ, *i.e.*, the vertices of Δ are the base points of the coordinate frame and the quadratic transformation as well.

3 Properties of the (isogonal transformation) permutation cubics

3.1 Rational and singular permutation cubics

In this section, we shall determine the set of pivot points such that the corresponding permutation cubics degenerate. For that purpose, we derive conditions on the trilinears $p:q:r$ of a generic pivot point V such that the gradient $\operatorname{grad}\mathcal{K}(V) = (\partial_\xi\mathcal{K}(V), \partial_\eta\mathcal{K}(V), \partial_\zeta\mathcal{K}(V))$ of the cubics' equations (3) vanishes. From the system of equations

$$\partial_\xi\mathcal{K} = 0, \quad \partial_\eta\mathcal{K} = 0, \quad \partial_\zeta\mathcal{K} = 0,$$

we eliminate η and ζ which yields

$$\begin{aligned} 0 = 81(pqr)^{15}\xi^{16}(p+q)^6(q+r)^6(r+p)^6\Big(\sum_{\text{cyc}} p\Big)^2\Big(\sum_{\text{cyc}} pq\Big)^2 \\ \cdot\Big(6pqr - \sum_{\text{cyc}} pq(p+q)\Big)^2 \cdot \Big(5pqr + 2\sum_{\text{cyc}} pq(p+q)\Big) \\ \cdot\Big(49p^2q^2r^2 + 5\sum_{\text{cyc}} p^4(q^2+r^2) + 10\sum_{\text{cyc}} p^3q(pr+q^2) + 30pqr\sum_{\text{cyc}} p^2(q+r)\Big). \end{aligned} \tag{5}$$

The elimination of any other pair of variables leads to the same trivariate homogeneous polynomial in terms of p, q, and r with either the factor η^{16} or ζ^{16}.

The geometric meaning of the different factors are:
(1) If at least one of p, q, or r equals zero, the pivot point is chosen on the side lines of Δ and the corresponding permutation cubic consists of the three side lines of Δ with the equation

$$\xi\eta\zeta = 0.$$

(2) If set equal to zero, the factors $p+q$, $q+r$, and $r+p$ deliver the side lines of the excentral triangle Δ_e. Thus, for a pivot point chosen on the degenerate cubic

$$(\xi+\eta)(\eta+\zeta)(\zeta+\xi) = 0,$$

the corresponding permutation cubic is also degenerate.

(3) The points $V = p:q:r$ with $p+q+r=0$ lie on the *antiorthic axis*, *i.e.*, the triangle polar of X_1 with respect to Δ (cf. [7, p. 209]). The permutation cubics of the points on $\mathcal{L}_1$ are given by the equation

$$\underbrace{(\xi+\eta+\zeta)}_{\mathcal{L}_1} \cdot \underbrace{(\xi\eta \;+\; \eta\zeta \;+\; \zeta\eta)}_{\text{Mandart's circumellipse } m} = 0. \tag{6}$$

The second factor of (6) is the equation of Mandart's circumellipse (cf. [8]). Under ι, points on $\mathcal{L}_1$ correspond to m, and *vice versa*, since $\iota(\mathcal{L}_1) = m$ and $\iota^2 = \mathrm{id}_{\mathbb{P}^2}$. Along the way, we have also disclosed the geometric meaning of the factor $pq+qr+rp$.

(4) The first cubic factor yields the equation of a rational (and non-degenerate) cubic

$$\mathcal{K}(X_1)\colon 6\xi\eta\zeta - \sum_{\text{cyc}} \xi\eta(\xi+\eta) = 0, \tag{7}$$

which is the permutation cubic of X_1 with respect to the isogonal transformation, since $P(1,1,1) = 6$ and $Q(1,1,1) = 1$. It has an acnode at $X_1 = 1:1:1$, and therefore, it admits a rational parametrization. In terms of a homogeneous parameter $t_0 : t_1 \neq 0 : 0$, the points on $\mathcal{K}(X_1)$ can be given as

$$t_0(t_1 - t_0)(2t_0 + t_1) : t_1(t_0 + 2t_1)(t_0 - t_1) : (t_0 + t_1)(t_0 + 2t_1)(2t_0 + t_1).$$

This cubic shows up in B. Gibert's catalogue of triangle cubics [12], where it is denoted by $\mathcal{K}_{228}$ and termed an isogonal circum-conico-pivotal cubic. It contains the triangle centers with Kimberling indices 1, 1022, 1023, 23889–23894. Fig. 3 shows an example of the curve $\mathcal{K}(X_1)$ for a certain triangle including the triangle centers on that curve.

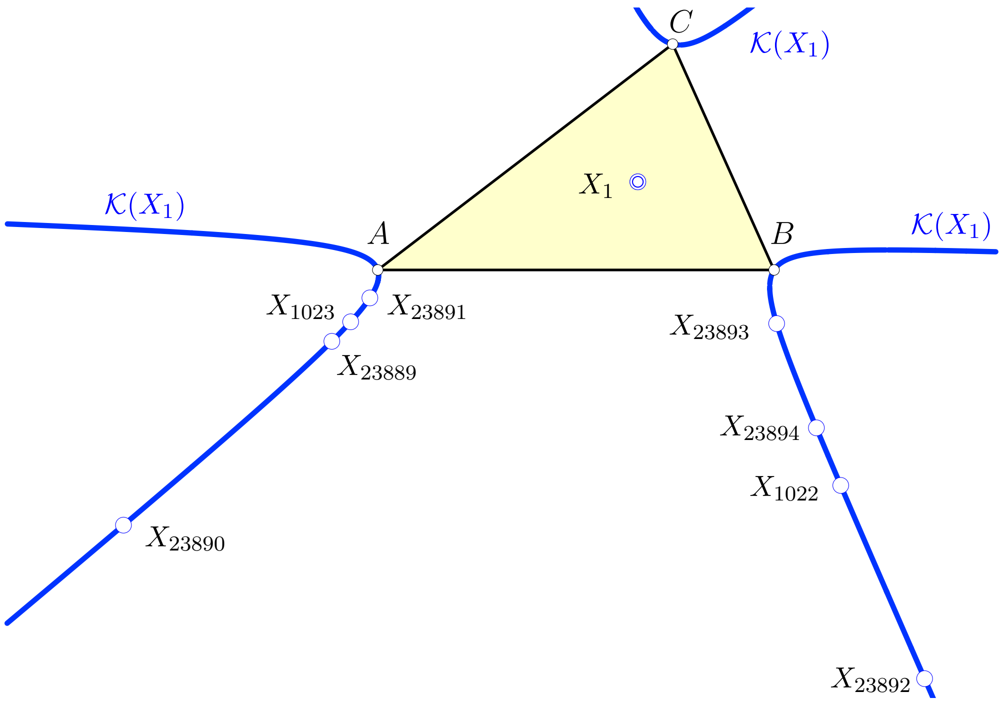

Figure 3: The rational cubic $\mathcal{K}(X_1)$ with the nine known triangle centers on it.

(5) The second cubic factor leads to an elliptic cubic $\mathcal{E}$ whose points do not lead to rational or degenerate permutation cubics.

(6) Finally, the sextic factor describes an elliptic curve $\mathcal{S}$ of degree 6. It has three tacnodes at the vertices of Δ and ordinary double points at

$$W_1 = [A, B] \cap \mathcal{L}_1 = 1 : -1 : 0, \ W_2 = [B, C] \cap \mathcal{L}_1 = 0 : 1 : -1, \ W_3 = [C, A] \cap \mathcal{L}_1 = 0 : 1 : -1. \quad (8)$$

Because of the ellipticity of the latter curve, its points cannot serve as pivot points for rational or even degenerate permutation cubics. The curves $\mathcal{E}$ and $\mathcal{S}$ are shown in Fig. 4. We shall summarize our results in:

Theorem 3.1. *The family* (3) *of permutation cubics contains one rational cubic, the cubic* $\mathcal{K}(X_1) = \mathcal{K}_{228}$ *with the pivot point* X_1.

In the family of permutation cubics, there exist exactly three degenerate cubics:

(i) the union of the Mandart circumellipse m *and the antiorthic axis* $\mathcal{L}_1$,

(ii) the union of the side lines of the excentral triangle Δ_e,

(iii) the union of the side lines of the base triangle Δ.

Each permutation cubic $K(V)$ touches the Mandart circumellipse m at the vertices of Δ along the exterior angle bisectors independent of the pivot point V. This holds also true for $\mathcal{E}$ and $\mathcal{S}$.

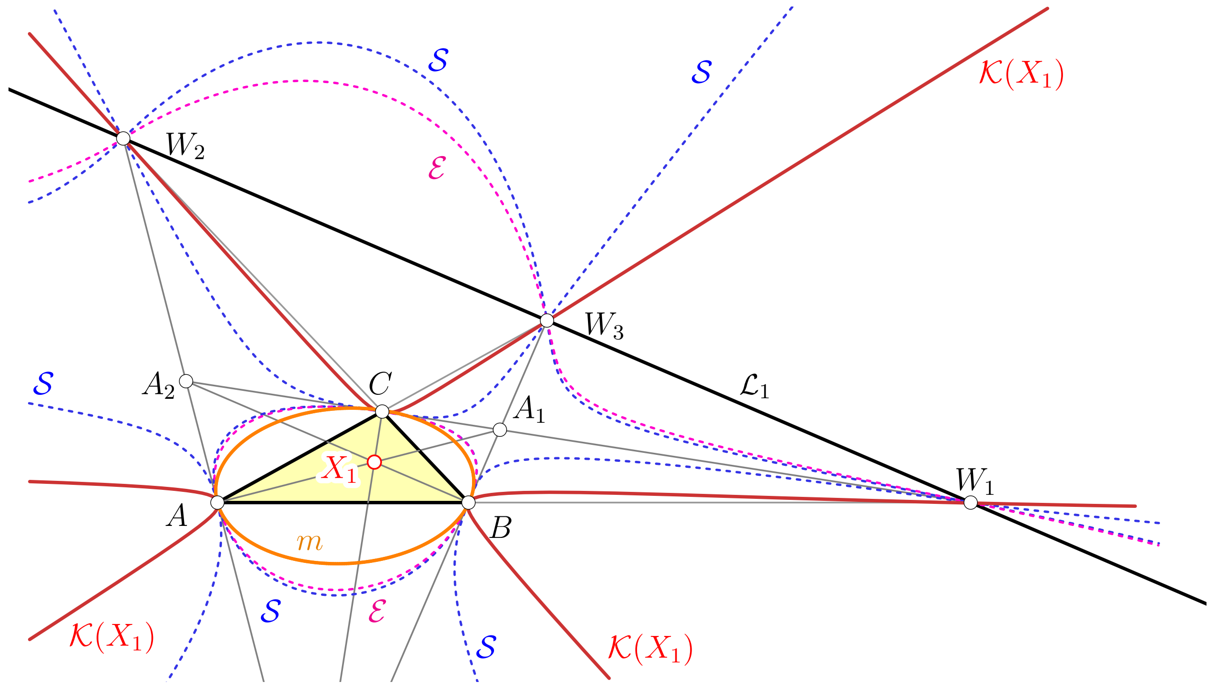

Figure 4: The side lines of Δ and Δ_e are degenerate permutation cubics. The cubic $\mathcal{K}(X_1)$ is the only rational permutation cubic. The elliptic sextic $\mathcal{S}$ and the elliptic cubic $\mathcal{E}$ are artifacts of the computation.

3.2 The duals of the permutation cubics defined by the isogonal conjugation

We compute the dual curves of (3) by eliminating the homogeneous point coordinates ξ, η, ζ from the following system of equations:

$$\partial_\xi \mathcal{K} = \rho u_0, \quad \partial_\eta \mathcal{K} = \rho u_0, \quad \partial_\zeta \mathcal{K} = \rho u_0, \quad \mathcal{K} = 0.$$

Consequently, we find

$$\begin{aligned}\mathcal{K}^\star(V)\colon\ & Q^4 \sum_{\text{cyc}} u_0^6 + 2Q^3(P+Q)\sum_{\text{cyc}} u_0^5(u_1+u_2) \\ & + Q^2(P^2-4PQ-13Q^2)\sum_{\text{cyc}} u_0^4(u_1^2+u_2^2) + 2Q^2(2P^2+10PQ+15Q^2)u_0u_1u_2\sum_{\text{cyc}} u_0^3 \\ & - 2Q^2(P^2-2PQ-10Q^2)\sum_{\text{cyc}} u_0^3u_1^3 + (P^4-6P^2Q^2+36PQ^3+90Q^4)u_0^2u_1^2u_2^2 \\ & + 2Q(P^3+4P^2Q+PQ^2-8Q^3)u_0u_1u_2\sum_{\text{cyc}} u_0^2(u_1+u_2) = 0. \quad (9)\end{aligned}$$

In general, the dual curves $\mathcal{K}^\star$ are of degree six since the permutation cubics $\mathcal{K}$ are elliptic. Except in the special case $V = X_1 = 1 : 1 : 1$, *i.e.*, $P = 6$ and $Q = 1$, we arrive at

$$\mathcal{K}^\star(X_1)\colon\ \sum_{\text{cyc}} u_0^2\Big(u_0^2 + 12u_0(u_1+u_2) - 26u_1^2 + 244u_1u_2\Big) = 0,$$

where the factor $u_0 + u_1 + u_2$ (the dual of $\mathcal{L}_1$) has split off with multiplicity two. $\mathcal{K}^\star(X_1)$ has three real cusps corresponding to the three real inflection points of $\mathcal{K}(X_1)$ and one real double tangent corresponding to the acnode X_1.

3.3 Hessian pencil and inflection points

The Hessian curve H$\mathcal{C}$ of a cubic $\mathcal{C}$ is again a cubic independent of the genus and class of $\mathcal{C}$. The curves $\mathcal{C}$ and H$\mathcal{C}$ span the *syzygetic pencil* of cubics (cf. [5]) and share their inflection points. The equation of the Hessian curve H$\mathcal{C}$ of an algebraic curve $\mathcal{C}$ can be computed via the determinant of the Hessian matrix $(\partial_{ij}\mathcal{C})_{i,j=\xi,\eta,\zeta}$ and, applied to (3), we find

$$\mathrm{H}\mathcal{K}(V)\colon 8Q^2(P+2Q)\sum_{\mathrm{cyc}}\xi^3+2(P^3-12PQ^2-24Q^3)\xi\eta\zeta-2P^2Q\sum_{\mathrm{cyc}}\xi^2(\eta+\zeta)=0. \quad (10)$$

The Hessian curve of a permutation cubic can only be a permutation cubic if, the coefficient of $\sum_{\mathrm{cyc}}\xi^3$ vanishes, *i.e.*, either $P+2Q=0$ or $Q=0$. Inserting $P=-2Q$ in (3), we obtain $(\xi+\eta)(\eta+\zeta)(\zeta+\xi)=0$, *i.e.*, the side lines of the excentral triangle Δ_e which is one of the degenerate permutation cubics mentioned in Thm. 3.1. On the other hand, both (3) and (10) become $\xi\eta\zeta=0$ if $Q=0$ which describes another degenerate permutation cubic (cf. Thm. 3.1). For pivot points on $\mathcal{L}_1$, the Hessian curve becomes the union of $\mathcal{L}_1$ and the Mandart circumellipse m.

Now, we can show the following:

Theorem 3.2. *The rational and elliptic permutation cubics share their three real inflection points which are the intersections of $\mathcal{L}_1$ with the side lines of Δ.*

Proof. The three points given in (8) satisfy all requirements on inflection points: They are located on $\mathcal{K}$ and H$\mathcal{K}$, they are regular points on $\mathcal{K}$. Further, it is easy to see that they are the intersections of Δ's sides with $\mathcal{L}_1$. □

For all regular permutation cubics, the triangles Δ_W built by the inflection tangents is perspective to Δ and its excentral triangle Δ_e with perspector X_1. In both cases, $\mathcal{L}_1$ serves as the perspectrix. This is illustrated in Fig. 5. It is worth mentioning that the permutation cubics form a two-parameter family of triangle cubics independent of the chosen quadratic transformation. All members of these families share their three real inflection points like the one-parameter family of distance product cubics from [11]. However, the latter are defined by means of metric properties, while the permutation cubics can be defined in a purely projective way.

Theorem 3.3. *The harmonic polars of the three inflection points are the interior angle bisectors of Δ.*

Proof. The harmonic polar h of a point Q on a cubic $\mathcal{C}$ is defined as the set of all harmonic conjugates H of Q with respect to the intersections $S_1, S_2 \neq Q$ of all lines in the pencil about Q (cf. [5]). If $W=\xi_W:\eta_W:\zeta_W$ is an inflection point of $\mathcal{C}$, then h is a straight line which is a part of the singular polar $p(W)\colon \partial_\xi\mathcal{C}\,\xi_W+\partial_\eta\mathcal{C}\,\eta_W+\partial_\zeta\mathcal{C}\,\zeta_W=0$. For example, $p(W_1)\colon (\eta-\xi)(Q\xi+Q\eta-(P+2Q)\zeta)=0$, where the second factor equals $\mathcal{K}$'s tangent at W_1. Therefore, the first factor equals the harmonic polar h_1 of W_1. It is easily verified that h_1 contains the triangle vertex C, X_1, and the third excenter A_3, and is thus Δ's interior angle bisector at C. Further, it houses the points of contact of the tangents from W_1 to $\mathcal{K}$. In the same way we argue for the (harmonic) polars of W_2 and W_3. □

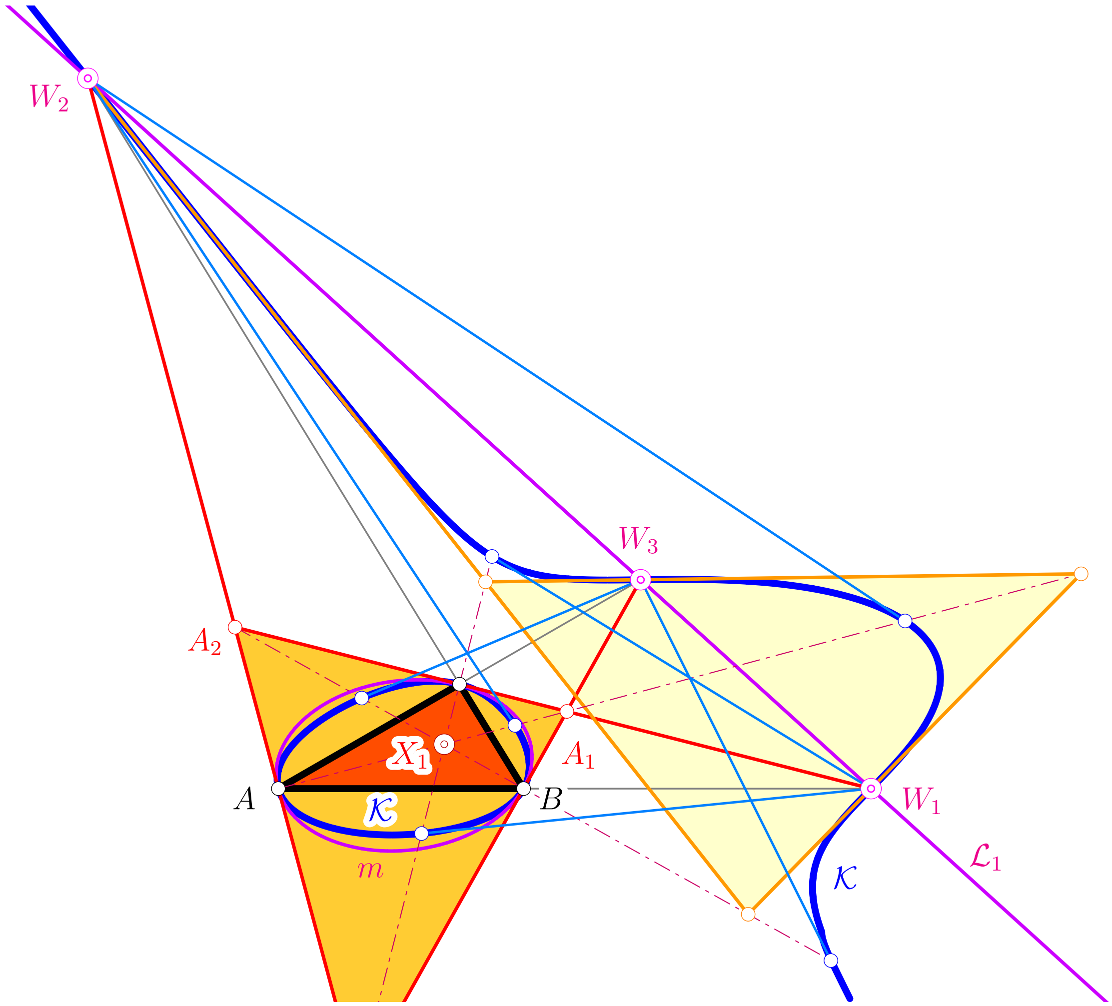

Figure 5: The permutation cubics share the three real inflection points W_1, W_2, W_3 on $\mathcal{L}_1$. The harmonic polars of the inflection points are the interior angle bisectors of Δ and carry the contact points of the tangents from W_i to $\mathcal{K}$.

3.4 Triangle centers on permutation cubics

Each triangle center X_i determines a permutation cubic $\mathcal{K}(X_i)$. Since the permutation cubics (3) are invariant under the isogonal conjugation ι (according to Thm. 2.1), each cubic $\mathcal{K}(X_i)$ also contains $\iota(X_i)$, and thus, $\mathcal{K}(X_i) = \mathcal{K}(\iota(X_i))$.

The cubic $\mathcal{K}(X_1)$ is a very special case. Here, the pivot point $X_1 = \iota(X_1)$ equals its isogonal conjugate which causes its singularity on $\mathcal{K}(X_1)$ and also the rationality of $\mathcal{K}(X_1)$. Furthermore, $\mathcal{K}(X_1)$ contains the following triangle centers (different from X_1)

$$(X_{1022}, X_{1023}), \quad (X_{23889}, X_{23894}), \quad (X_{23890}, X_{23893}), \quad (X_{23891}, X_{23892}),$$

where each pair is a pair of isogonal conjugate centers.

Among the approximately 52000 triangle centers listed in [13] (as to September 2022), there are only a few that share their permutation cubics with more than one center.

The centroid X_2 and the Symmedian point X_6 are each others isogonal conjugates, and thus, $\mathcal{K}(X_2) = \mathcal{K}(X_6)$. The permutation cubic $\mathcal{K}(X_2)$ contains the pairs

$$(X_2, X_6), \quad (X_{3570}, X_{3572})$$

of isogonal conjugate triangle centers. The tangents to $\mathcal{K}(X_2)$ at X_2 and X_6 meet $\mathcal{K}(X_2)$ in the center X_{3572}, while the line $\mathcal{L}_{2,6} := [X_2, X_6]$ carries also X_{3570}. Further, the tangents to $\mathcal{K}(X_2)$ at X_{3570} and X_{3572} meet $\mathcal{K}(X_2)$ at X_{52222} (see Fig. 6, left).

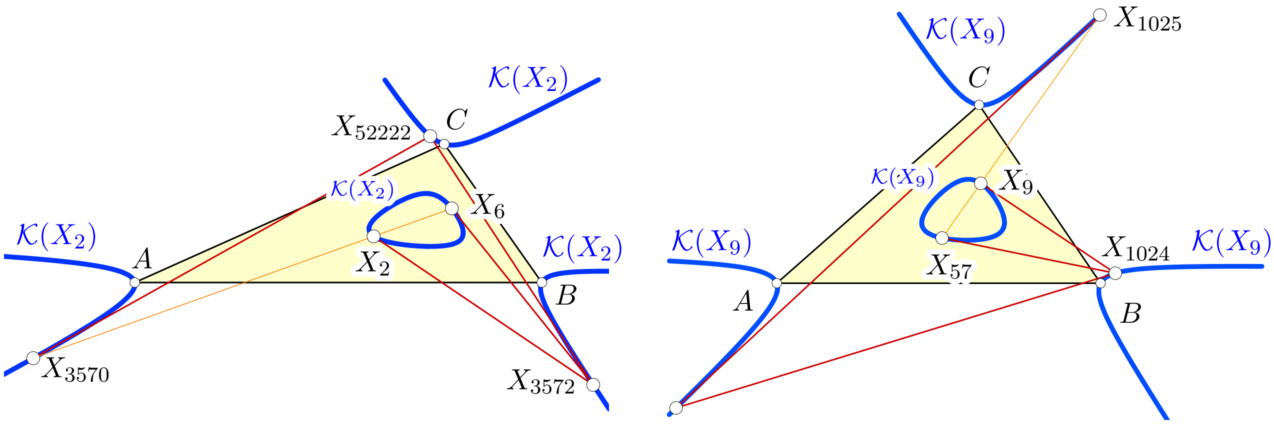

Figure 6: Two permutation cubics: $\mathcal{K}(X_2)$ (left) and $\mathcal{K}(X_9)$ (right) with some centers on them.

The circumcenter X_3 and the orthocenter X_4 are each others isogonal conjugates, and thus, they both lie on $\mathcal{K}(X_3) = \mathcal{K}(X_4)$. On $\mathcal{K}(X_3)$, we can also find X_{1981} and its isogonal conjugate which is a yet unnamed center with the comparably short trilinear center function

$$\alpha = a(b-c)(a^2-b^2-c^2)\Big(a^3c + a^2(b^2-2c^2) - ac(b^2-c^2) - b^2(b^2-c^2)\Big) \cdot \Big(a^3b - a^2(2b^2-c^2) + ab(b^2-c^2) + c^2(b^2-c^2)\Big).$$

The permutation cubic $\mathcal{K}(X_9)$ determined by the Mittenpunkt contains two pairs of isogonal conjugate (and known) centers:

$$(X_9, X_{57}), \quad (X_{1024}, X_{1025})$$

(see Fig. 6, right). Comparable to the case of the centroid's permutation cubic $\mathcal{K}(X_2)$, we observe that the curve's tangent at X_9 and X_{57} intersect the cubic in the same point: X_{1024}. The tangents at X_{1024} and X_{1025} meet $\mathcal{K}(X_9)$ at the same, yet unnamed center with the rather lengthy trilinear center function

$$\begin{aligned}\alpha = {} & (b-c)(a-b-c)(ab+ac-b^2-c^2)\Big(a^4-(b+c)a^3+a^2bc+bc(b-c)^2\Big)\\ & \cdot\Big(ca^7-2c(b+c)a^6+(b+c)(b^2+bc+c2)a^5-b^3(3b+c)a^4+(3b^5-b^4c+4b^3c^2-6b^2c^3+c^5)a^3\\ & \qquad -(b-c)(b^5+4b^3c^2-2c^5)a^2+c^2(b^3+b^2c+c^3)(b-c)^2a-b^3c^2(b-c)^3\Big)\\ & \cdot\Big(ba^7-2b(b+c)a^6+(b+c)(b^2+bc+c^2)a^5-c^3(3c+b)a^4+(b^5-6b^3c^2+4b^2c^3-bc^4+3c^5)a^3\\ & \qquad -(b-c)(2b^5-4b^2c^3-c^5)a^2+b^2(b^3+bc^2+c^3)(b-c)^2a+b^2c^3(b-c)^3\Big).\end{aligned}$$

Further, X_9, X_{57}, and X_{1025} are three collinear points on $\mathcal{K}(X_9)$.

The pairs of isogonal triangle centers (X_{101}, X_{514}) and (X_{34905}, X_{34906}) determine the same permutation cubic and $(X_{101}, X_{514}, X_{34906})$ form a collinear triple on $\mathcal{K}(X_{101})$. The cubic $\mathcal{K}(X_{101})$ equals the cubic $\mathcal{K}_{721}$ in B. Gibert's list (see [12]).

There are also some triangle centers that determine, and therefore, lie on the degenerate permutation cubic $\mathcal{K}(X_{44}) = \mathcal{L}_1 \cup m$. Obviously, X_{44} is the triangle center with the smallest

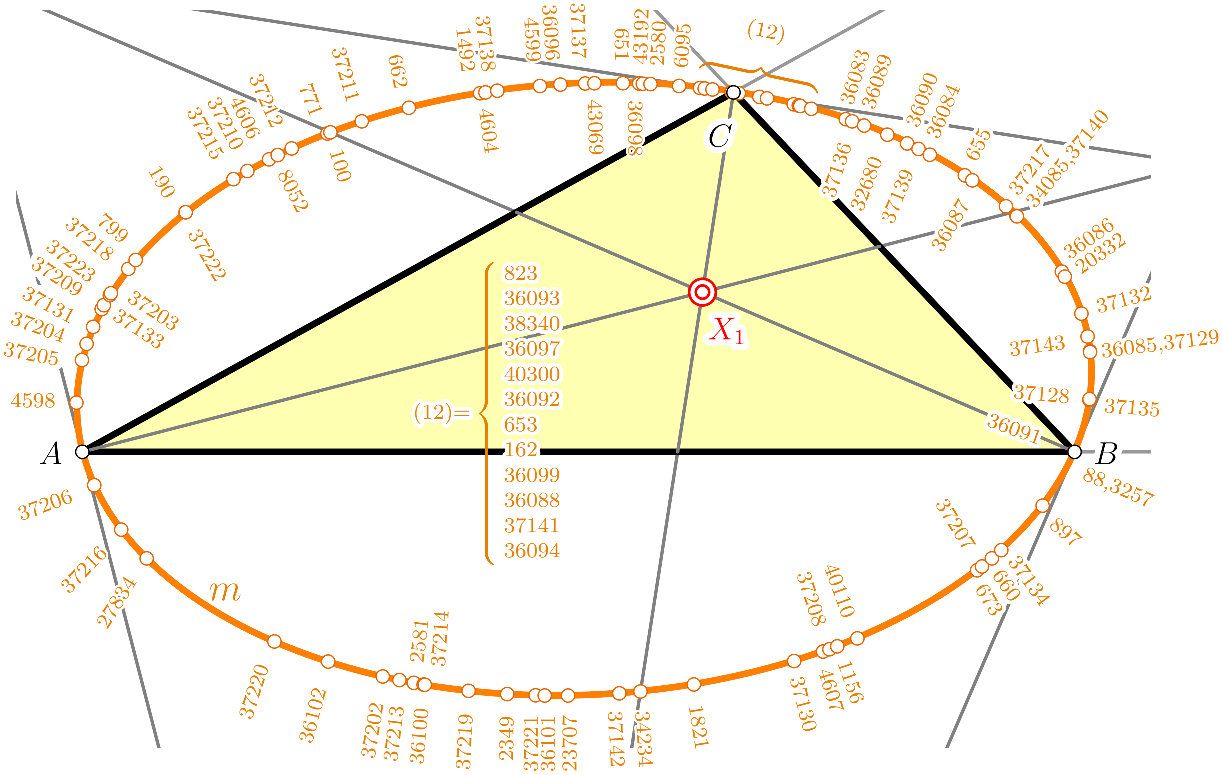

Figure 7: Triangle centers on Mandart's circumellipse m sharing their degenerate permutation cubic.

Kimberling number that determines this particular curve. There are two disjoint sets of triangle centers on $\mathcal{K}(X_{44})$: On the Mandart circumellipse m, we find 114 triangle centers (shown in Fig. 7) whose Kimberling numbers (≤ 50000) are

$$\begin{aligned}&88, 100, 162, 190, 651, 653, 655, 658, 660, 662, 673, 771, 799, 823, 897, 1156, 1492, 1821,\\ &2349, 2580, 2581, 3257, 4598, 4599, 4604, 4606, 4607, 8052, 20332, 23707, 24624, 27834,\\ &32680, 34085, 34234, 36083-36102, 37128-37143, 37202-37223, 38340, 40300, 40110,\\ &43069, 43192, 43757-43764, 45875, 46116-46122.\end{aligned} \tag{11}$$

The antiorthic axis $\mathcal{L}_1$ carries 184 triangle centers (shown in Fig. 8) with Kimberling indices (≤ 50000):

$$\begin{aligned}&44, 513, 649, 650, 652, 654, 656, 657, 659, 661, 672, 770, 798, 822, 851, 896, 899, 910, 1155, 1491, 1575,\\ &1635, 1755, 2173, 2182, 2183, 2225, 2227-2240, 2243-2247, 2252-2254, 2265, 2272, 2290,\\ &2312-2315, 2348, 2483, 2484, 2503, 2509, 2511, 2515, 2516, 2522, 2526, 2578, 2579, 2590, 2591,\\ &2600, 2610, 2624, 2630, 2631, 2635, 2637, 2641, 2642, 3000, 3013, 3287, 3330, 3768, 4394, 4724,\\ &4782, 4784, 4790, 4813, 4893, 4979, 7655, 7659, 8043, 8061, 9356, 9360, 9393, 9404, 9508, 9511,\\ &10495, 13401, 14298-14300, 15586, 17410, 17418, 17420, 18116, 20331, 20979, 21127, 21894,\\ &22108, 22443, 23503, 24533, 24749, 25142, 29357, 29361, 30600, 38472, 39690, 40109, 40137,\\ &40338, 44151, 44319, 45877, 45881-45886, 46380-46393, 47777, 47810, 47811, 47826-47828,\\ &47842, 48019-48033, 48160, 48162, 48193, 48194, 48213, 48226, 48244, 48544, 48572.\end{aligned}$$

The permutation cubics $\mathcal{K}(X_i)$ for $i = \{1, 2, 3, 9, 101, 238, 239, 1822, 1823\}$ are on at least 4 centers from [13]. Similarly, the cubics $\mathcal{K}(X_i)$ with $i = \{240, 241, 1983, 3960, 23343\}$ are on 3 triangle centers.

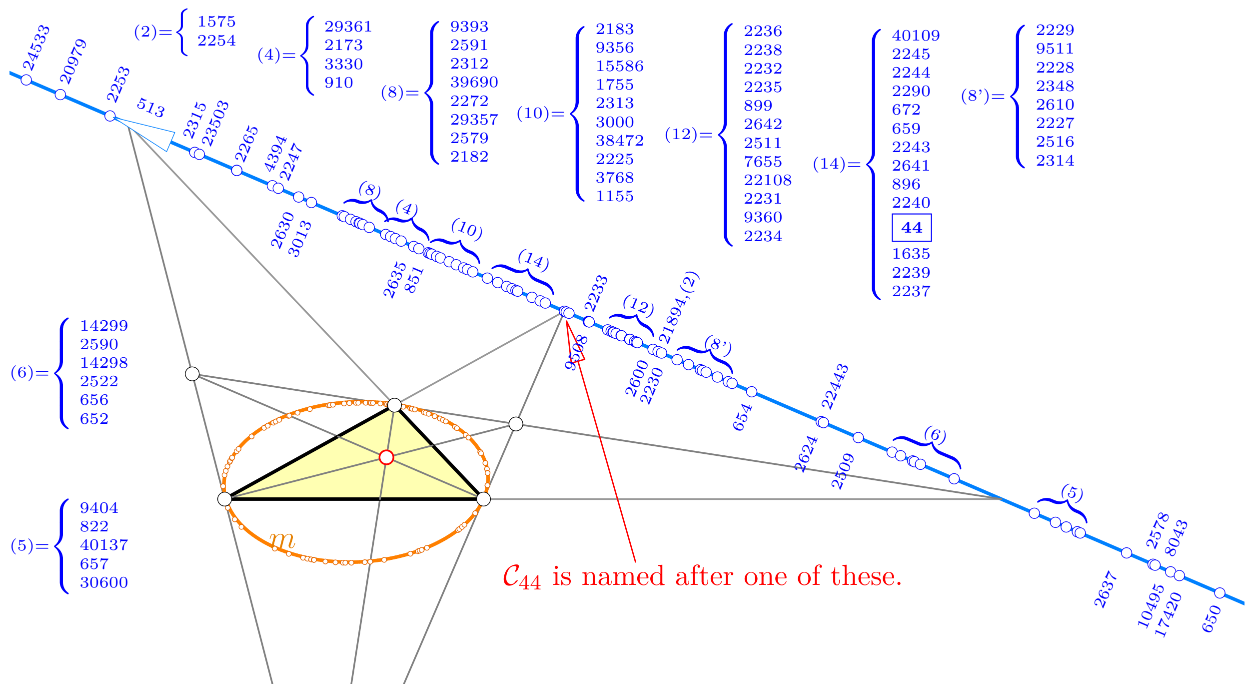

Figure 8: The straight part $\mathcal{L}_1$ of $\mathcal{K}(X_{44})$ with some triangle centers on it.

4 Permutation cubics defined by the isotomic transformation

The two-parameter family of cubics described by (3) is determined by a pivot point $V = p : q : r$ and defined by the isogonal conjugation. Now, we shall replace the isogonal conjugation by the *isotomic conjugation* τ. Following (2), the isotomic conjugation sends the pivot point $V = p : q : r$ to

$$\tau(V) = b^2c^2qr : c^2a^2rp : a^2b^2pq$$

which has also six permutation points (including the point V). Now, we can state and prove:

Theorem 4.1. *The six permutation points of a point V and the six permutation points of its isotomic conjugate $\tau(V)$ lie on a single cubic $\mathcal{T}(V)$ with the trilinear equation*

$$\begin{aligned}\mathcal{T}(V)\colon\ & a^2b^2c^2p^2q^2r^2\Big(\sum_{cyc} a^2b^2pq(a^2p+b^2q) - a^2b^2c^2\sum_{cyc} pq(p+q)\Big)\sum_{cyc}\xi^3 \\ & - pqr\Big(\sum_{cyc} a^6b^6p^3q^3 - a^4b^4c^4\sum_{cyc} p^3qr\Big)\sum_{cyc}\xi\eta(\xi+\eta) - \Big(a^2b^2c^2pqr\sum_{cyc} a^4p^5(b^2q+c^2r) \\ & \quad - \sum_{cyc} a^6b^6p^4q^4(p+q) - pqr\sum_{cyc} a^4p^4(b^6(a^2-c^2)q^2 + c^6(a^2-b^2)r^2) \\ & \qquad - p^2q^2r^2\sum_{cyc} a^2b^2pq\Big(b^4(a^4-c^4)p + a^4(b^4-c^4)q\Big)\Big)\xi\eta\zeta = 0. \qquad (12)\end{aligned}$$

Proof. We use the same techniques as in the proof of Thm. 2.1. □

The equation (12) of the permutation cubics with respect to the isotomic conjugation does not become substantially simpler if we use barycentric coordinates.

The cubics (12) do not pass through the vertices of the base triangle Δ. Therefore, they can neither be self-isogonal nor self-isotomic. However, the cubics (12) are permutation cubics, and by virtue of Lem. 1.1 and comparable to Thm. 2.2, we have:

Theorem 4.2. *Each of the permutation cubics* (12) *with respect to the isotomic conjugation is transformed into itself under each element of the discrete group* G_6.

The permutation cubics with respect to the isotomic conjugation contain, besides the six permutation points of the pivot V and the six permutation points of the isotomic conjugate $\tau(V)$, also all isotomic conjugates of the six permutation points of V. This is not clear from the very beginning since permutation and isotomic conjugation do not commute. In the case of the isogonal conjugation, the commutation is rather obvious.

We can also show:

Theorem 4.3. *All permutation cubics* (12) *with respect to the isotomic conjugation share the three real inflection points which lie on the line* $\mathcal{L}_1$ *and agree with* (8).

(i) The permutation cubic $\mathcal{T}(X_2)$ *degenerates completely,* i.e., *its equation is the zero form.*

(ii) The permutation cubic $\mathcal{T}(X_{661}) = \mathcal{T}(X_{799})$ *equals* $\mathcal{K}(X_{44})$.

(iii) The cubics (12) *whose pivot points are the triangle centers with Kimberling numbers*

$(44, 20568), (513, 668), (649, 1978), (650, 4554), (652, 46404), (654, 46405), (656, 811), (657, 46406), (659, 4583), (661, 799), (672, 18031), 770, (789, 1491), (798, 4602), 822, 851, (896, 46277), (899, 31002), 910, 1155, (1575, 32020), 1635, (1755, 46273), (2173, 33905), 2182, 2183, 2225, 2227 - 2237, (2238, 40017), 2239, 2240, 2243 - 2247, 2252 - 2253, (2254, 51560), 2265, 2272, 2290, 2312, 2313 - 2315, 2348, 2483, 2484, 2503, 2509, 2511, 2515, 2516, 2522, 2526, 2578, 2579, 2590, 2591, 2600, 2610, 2624, 2630, 2631, 2635, 2637, 2641, 2642, 3000, 3013, 3287, 3330, 3768, 4394, (4593, 8061), 4724, 4782, 4784, 4790, 4813, 4893, 4979, 7655, 7659, 8043, 9356, 9360, 9393, 9404, 9508, 9511, 10495, 13401, 14298 - 14300, 15586, 17410, 17418, 17420, 18031, 18116, 20331, 20979, 21127, 21894, 22108, 22443, 23503, 24533, 29357, 29361, 30600, 33805, 36036, 38472, 39690, (40338, 40339), 40109, 40137, 44151, 44319, 45877, 45881 - 45886, 46380 - 46393,$

split into the line $\mathcal{L}_1$ *and a further conic different from* m*, where the pairs of isotomic conjugate pivots (enclosed in brackets) determine the same quadratic part.*

It should be noted that the quadratic component of the degenerate permutation cubic $\mathcal{T}(X_i)$ is not always the same conic and it depends on V.

Fig. 9 shows some of the cubics $\mathcal{T}_i$ defined by triangle centers with low Kimberling indices $i \in \{1, \ldots, 12\}$. Note that $\mathcal{T}_7 = \mathcal{T}_8$ since $X_8 = \tau(X_7)$.

Comparable to Thm. 3.3, we have:

Theorem 4.4. *The harmonic polars of the three real inflection points of the permutation cubics* $\mathcal{T}(V)$ (12) *are the three interior angle bisectors of* Δ.

The latter theorem holds true for all cubics of the form

$$A\sum_{\text{cyc}}\xi^3 + B\sum_{\text{cyc}}\xi(\eta^2+\zeta^2) + C\xi\eta\zeta = 0$$

with A, B, $C \in \mathbb{R}$, since their Hessian curves are

$$\begin{aligned} &2(2B-C)(6AB+3AC-4B^2)\sum_{\text{cyc}}\xi^3 \\ &\quad + 2B(6A-C)^2\sum_{\text{cyc}}\xi(\eta^2+\zeta^2) + (216A^3-72AB^2+48B^3-24B^2C+2C^3)\xi\eta\zeta = 0 \end{aligned}$$

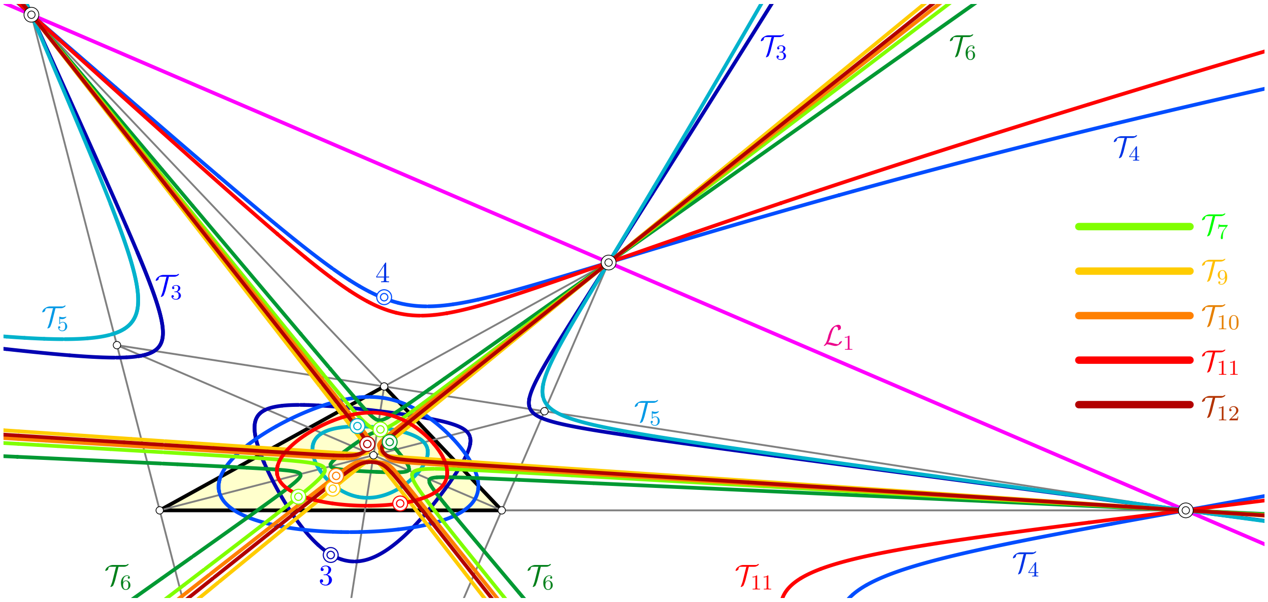

Figure 9: Some permutation cubics (12) with respect to the isotomic conjugation and with low-indexed centers for their pivot points. For the sake of simplicity, we have set $\mathcal{T}(X_i) = \mathcal{T}_i$.

and the inflection tangents

$$(3A - B)\xi + (3A - B)\eta + (2B - C)\zeta = 0,$$
$$(3A - B)\xi + (2B - C)\eta + (3A - B)\zeta = 0,$$
$$(2B - C)\xi + (3A - B)\eta + (3A - B)\zeta = 0$$

split off from the polars of the inflection points.

5 Final remarks

It is not at all clear whether all quadratic Cremona transformations can be used to construct permutation cubics. Although the isogonal and isotomic conjugation are the most prominent representatives of the wide class of these mappings, there are some others which are in a special relation to the triangle. One could also consider inversions in special triangle circles, *e.g.*, the inversion in the circumcircle or the incircle. Unfortunately, two base points of these two quadratic Cremona transformations coincide with the absolute points of Euclidean geometry and the third base point is the center of the respective circle. This makes the coordinate representation of these inversions (with respect to the standard frame) more complicated and the equations of the corresponding permutation cubics may be hard to handle even with a CAS.

The inversion β in the Mandart ellipse m with inversion center X_1 (which is not the center of m) has a relatively simple coordinate representation in terms of trilinear coordinates:

$$\beta\colon \xi : \eta : \zeta \mapsto \xi^2 - \eta\zeta : \eta^2 - \zeta\xi : \zeta^2 - \xi\eta.$$

Sometimes, $\beta(X)$ is also referred to as the X_1-Hirst inverse of $X = \xi : \eta : \zeta$. It can easily be shown that the following holds

Theorem 5.1. *The six permutation points of $V = p : q : r$ and the six permutation points of $\beta(V)$, where β is the inversion in the Mandart ellipse m with center X_1 lie on a single cubic with the equation*

$$\begin{aligned} \mathcal{B}(V)\colon \Big(2\sum_{cyc} p^2qr(p-q)(p^2+q^2)-p^3q^3(p+q)\Big)\sum_{cyc}\xi^3 \\ +\Big(\sum_{cyc} p\Big)\cdot\Big(\sum_{cyc} p^2-qr\Big)\cdot\Big(\sum_{cyc} pq(pq-r^2)\Big)\cdot\sum_{cyc}\xi\eta(\xi+\eta) \\ -\Big(\sum_{cyc} 2p^7-p^6(q+r)-2p^3q^3(p+q-2r)\Big)\xi\eta\zeta = 0. \end{aligned} \tag{13}$$

All cubics $\mathcal{B}(V)$ share the three real inflection points which are equal to (8)*.*

It is obvious that (13) yields the zero form if $V = X_1$. There seem to be no rational curves among the cubics (13).

Degenerate permutation cubics among the cubics (13) can be also be detected:

- For pivot points V on the lines $[X_1, W_i]$ (with $i \in \{1,2,3\}$), the cubics $\mathcal{B}(V)$ split into the triple of lines $(2\xi-\eta-\zeta)(\xi-2\eta+z\eta)(\xi+\eta-2\zeta)=0$ passing through X_1.
- For pivot points V on either line $\mathcal{L}_1$, c, $\overline{c}$ (the complex conjugate pair (4) mentioned prior to Thm. 2.1), the cubics $\mathcal{B}(V)$ split into the triple of lines $(\mathcal{L}_1, c, \overline{c})$.

Again, the harmonic polars of the real inflection points are the interior angle bisectors of Δ (cf. Thm. 3.3 and Thm. 4.4).

It is clear that β-conjugate triangle centers X_i like the pairs $(3,1936)$, $(4,243)$, $(5,2596)$, $(7,14189)$, $(8,5205)$, $(10,17763)$, $(19,240)$, $(21,2651)$, $(29,2659)$, … share their permutation cubic $\mathcal{B}(X_i)$.

$\mathcal{B}(X_2)$ carries the two pairs $(2,239)$, $(869,\beta(869))$, of β-conjugate triangle centers with

$$\alpha_{\beta(869)} = a^3(b+c)(b^2+c^2)-a^2bc(b^2+bc+c^2)-b^3c^3$$

and so is the case with $\mathcal{B}(X_6)$ carrying $(6,238)$, $(984,\beta(984))$, where

$$\alpha_{\beta(984)} = a^3+a(b^2+bc+c^2)-(b+c)(b^2+c^2),$$

and further $\mathcal{B}(X_9)$ houses the pairs $(9,518)$, $(7290,\beta(7290))$, wehre

$$\alpha_{\beta(7290)} = 4a^3-3a^2(b+c)+2a(2b^2-bc+2c^2)-(b+c)(b^2+c^2),$$

and, moreover, $X_{100} = \beta(X_{100})$ and the pair $(244,1054)$ share their β-permutation cubic.

In particular all triangle centers on m are fixed under β, *i.e.*, the centers listed in (11).

Finally, we shall give some triples of triangle centers that share the cubic $\mathcal{B}(X_i)$ with the following Kimberling indices

$$\begin{gathered}(2,239,869),(6,238,984),(9,518,7290),(31,1580,51836),(55,37772,37773),\\(202,19551,39151),(203,7126,39150),(259,8076,8083),(291,292,3802),\\(1094,2153,51805),(1095,2154,51806).\end{gathered}$$

Acknowledgements

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