Journal for Geometry and Graphics
Volume 27 (2023), No. 1, 59–67

Enforcing Surface Rigidity by Shadow-Line Constraints

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Abstract. A surface is under pressure and deforming. Is it bending without or with deforming the surface-metric? This is an important question in many applications. Mathematical concepts to deal with these kind of problems are differential geometry and infinitesimal bendings. Shadow-curves are an intuitive visualization tool. We prove in this paper that as long as the shadow-lines stay stationary during the deformation the surface is infinitesimal rigid.

Key Words: differential geometry, Computer Aided Design, computer graphics, computational geometry

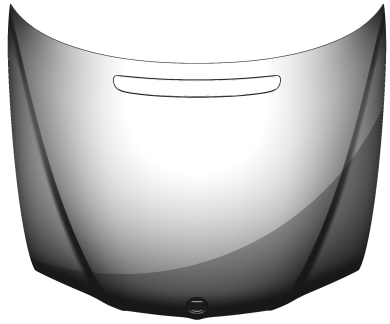
MSC 2020: 65D17 (primary), 65D18

1 Introduction

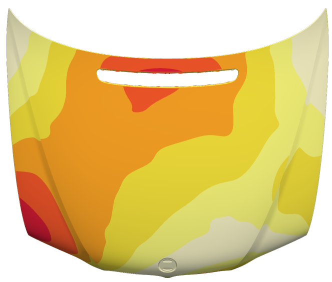
Closed convex surfaces (so called “Eiflächen”) are infinitesimal rigid. “Open” surfaces are not, but you can enforce rigidity by certain boundary conditions for the boundary (surface-) curves or by certain conditions for the deformation vector field. In many applications it is nearly impossible to impose conditions on the deformation vector field. Appropriate boundary conditions for the boundary curves are the “way to go”. We prove in this paper that as long as the shadow-lines stay stationary during the deformation the surface is infinitesimal rigid.

The shadow-line is the “boundary curve” between light and dark on the surface. Let e be the light direction of ”parallel lightning”, that means all light rays are parallel. We do not use spotlights. The shadow-line is them given by $\langle e, N\rangle = 0$. N is the normal vector of the surface and $\langle , \rangle$ the scalar product. $\langle e, N\rangle > 0$ descripes the “lighted part” of the surface.

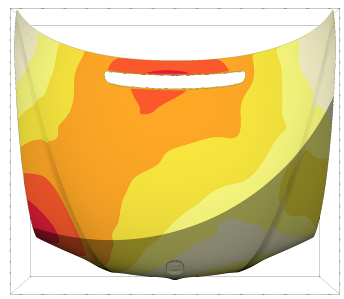
ISSN 1433-8157/\$ 2.50



(a) Shadow-lines on a car hood



(b) Pressure distribution on a car hood



(c) Shadow-lines are stationary under pressure, therefore the surface is infinitesimal rigid

2 Related Work

From the application point of view it makes no sense to impose conditions on the deformation vector field. The famous results of Grotemeyer [2] contain, at least to my knowledge the only conditions imposed to the boundary curves, which enforce rigidity by purely geometric constraints.

3 Shadow-Line Method

3.1 Basic Facts and Notations

We consider a parametrized c^2 surface $X : U \to E^3$.

$$X_u := \frac{\partial}{\partial u} X(u,w), \tag{1}$$

$$X_w := \frac{\partial}{\partial w} X(u,w), \tag{2}$$

$$X_{uw} := \frac{\partial}{\partial w} X_u(u,w), \ldots \tag{3}$$

T_uX is the tangend space of X, and N the normal vector. The matrix representation of the first fundamental form with respect to the basis $\{X_u, X_w\}$ of T_uX is given by

$$\{g_{ij}\}_{i,j=1}^{2} = \begin{pmatrix} \langle X_u, X_u \rangle & \langle X_u, X_w \rangle \\ \langle X_w, X_u \rangle & \langle X_w, X_w \rangle \end{pmatrix} \tag{4}$$

The Weingarten map $L = T_uX \to T_uX$ is defined by $L := -dN_u \circ (dX_u)^{-1}$. The matrix representation of the second fundamental form (the bilinear form definded by $\langle L(A), B\rangle$ for each $A, B \in T_uX$) is given by

$$\{h_{ij}\}_{i,j=1}^2 = \begin{pmatrix} \langle X_{uu}, N\rangle & \langle X_{uw}, N\rangle \\ \langle X_{wu}, N\rangle & \langle X_{ww}, N\rangle \end{pmatrix} \tag{5}$$

$k := \det(L)$ is the Gaussian curvature.

3.2 Deformations and Infinitesimal Bendings

Definition 1.

(a) $X(U)$ and $\tilde{X}(U)$ are two surfaces with a common parameter domain U. X and $\tilde{X}$ are called *isometric*, if $g_{ij} = \tilde{g}_{ij}$.

(b) A family of surfaces

$$X_t(u, w) := X(u, w) + t \cdot Z(u, w)$$

is called a *deformation* generated by a deformation vector field $Z(u, w)$.

(c) This deformation vector field $Z(u, w)$ is called an *infinitesimal bending* of the surface $X(u, w)$ if $L(\tilde{c}) = L(c) + o(\varepsilon)$ for each surface curve $c := X(u(s)), w(s))$ and $\tilde{c} := X_t(u(s), w(s))$.

Definition 2 (First Variation)**.** Let $\{X(U)\}_{t\in I}$ be a deformation generated by a deformation vector field $Z(u, w)$ and $f\colon U \times I \to \mathbb{R}$ a continuous mapping with continuous partial derivatives.

Then

$$\delta f := \frac{\partial f}{\partial t}|_{t=0}$$

is called the *first variation* of f.

Theorem 1 (Characterization of infinitesimal bendings)**.** *The following statements are equivalent:*

(I) $Z(u, w)$ *is an infinitesimal bending.*

(II) $\langle Z_u, X_u\rangle = 0$, $\langle Z_w, X_w\rangle = 0$ *and* $\langle Z_u, X_w\rangle + \langle Z_w, X_u\rangle = 0$

(III) $\delta g_{ij} = 0$

Proof. (I) $\iff$ *(II)*: $L(\tilde{c}) = L(c) + o(\varepsilon)$ if and only if $\tilde{g}_{rs}\dot{u}^r\dot{u}^s = g_{ij}\dot{u}^i\dot{u}^j + o(\varepsilon)$, but

$$\begin{aligned} &\langle X_u, X_u\rangle \dot{u}^2 + \langle X_w, X_w\rangle \dot{w}^2 + 2\langle X_u, X_w\rangle \dot{u}\dot{w} \\ &\qquad + 2\varepsilon\Big(\langle X_u, Z_u\rangle \dot{u}^2 + \langle X_w, Z_w\rangle \dot{w}^2 + (\langle Z_w, X_u\rangle + \langle Z_u, X_w\rangle)\dot{u}\dot{w}\Big) + o(\varepsilon) \\ &\qquad\qquad = g_{ij}\dot{u}^i\dot{u}^j + o(\varepsilon) \end{aligned}$$

if and only if $\langle Z_u, X_u\rangle = 0$, $\langle Z_w, X_w\rangle = 0$ and $\langle Z_u, X_w\rangle + \langle Z_w, X_u\rangle = 0$.

(II) $\Longrightarrow$ *(III)*:

$$\delta g_{ij} = \delta\langle X_i, X_j\rangle + \delta t(\langle X_i, Z_j\rangle + \langle X_j, Z_i\rangle) + \delta t^2(\sim),$$
$$X_1 = X_u,\ X_2 = X_w,\ Z_1 = Z_u,\ Z_2 = Z_w$$

Considering $\langle Z_u, X_u\rangle = 0$, $\langle Z_w, X_w\rangle = 0$ and $\langle Z_u, X_w\rangle + \langle Z_w, X_u\rangle = 0$, we get $\delta g_{ij} = 0$.

(III) $\Longrightarrow$ *(II)*: $\delta g_{ij} = 0$ leads to

$$\langle X_i, Z_j\rangle + \langle X_j, Z_i\rangle = 0 \tag{6}$$

for all $i, j = 1, 2$ which is equivalent to

$$\langle Z_u, X_u\rangle = 0, \tag{7}$$
$$\langle Z_w, X_w\rangle = 0, \tag{8}$$
$$\langle Z_u, X_w\rangle + \langle Z_w, X_u\rangle = 0. \tag{9}$$

□

Definition and Theorem 2 (Rotation vector field Y)**.** *For an infinitesimal bending $Z(u, w)$ there is a unique so called rotation vector field $Y(u, w)$ with*

$$[Y, X_u] = Z_u \quad \textit{and} \quad [Y, X_w] = Z_w.$$

Proof. First, we prove the existence:

$$\langle X_u, Z_u\rangle = 0 \quad \text{and} \quad \langle X_w, Z_w\rangle = 0$$

means that there are vector fields $Y_1(u, w)$ and $Y_2(u, w)$ with $Z_u = [Y_1, X_u]$ and $Z_w = [Y_2, X_w]$. Further is $\langle [Y_1, X_u], X_w\rangle + \langle [Y_2, X_w], X_u\rangle = 0$ equivalent to $\det|X_u, X_w, Y_1 - Y_2| = 0$, which leads to

$$Y_1 - Y_2 = \lambda_1 X_u + \lambda_2 X_w.$$

Define $Y := Y_1 - \lambda_1 X_u = Y_2 + \lambda_2 X_w$.

To show uniqueness we assume there exists $Y(u, w)$ and $\tilde{Y}(u, w)$ with $[Y, X_u] = [\tilde{Y}, X_u]$ and $[Y, X_w] = [\tilde{Y}, X_w]$. This implies $[Y - \tilde{Y}, X_u] = 0 = [Y - \tilde{Y}, X_w]$ which means $Y = \tilde{Y}$. □

Remarks.

(1) The vector field $Z(u, w)$ can be visualized as the velocity field of the deformation.

(2) The vector field $Y(u, w)$ can be visualized as the "momentary rotation vector field" of a rigid body attached to the surface.

Definition 3 (infinitesimal movement and infinitesimal rigidity)**.**

(a) An infinitesimal bending $Z(u, w)$ is called trivial or an infinitesimal movement if $Z(u, w) = [c, X(u, w)] + d$ with constant vectors c and d.

(b) A surface which allows only trivial infinitesimal bendings is called infinitesimal rigid.

Lemma 3 (Rigidity)**.** *A surface $X(U)$ is infinitesimal rigid if and only if the first variation of the second fundamental form is zero ($\delta h_{ij} = 0$) for all infinitesimal bendings ($\delta g_{ij} = 0$).*

Proof. Let $X(u)$ be inifintesimal rigid

$$\begin{aligned}\tilde{X}(u, w) = X_t(u, w) &= X(u, w) + tZ(u, w)\\ &= X(u, w) + t([c, X(u, w)] + d),\end{aligned}$$

which means

$$\tilde{X}_u = X_u + t[c, X_u],$$
$$\tilde{X}_w = X_w + t[c, X_w]$$

and

$$\begin{aligned}[\tilde{X}_1, \tilde{X}_2] &= [X_1, X_2] + t[X_1, [c, X_2]] + t[[c, X_1], X_2] + o(t) \\ &= [X_1, X_2] + t\left([X_1, [c, X_2]] - [X_2, [c, X_1]]\right) + o(t) \\ &= [X_1, X_2] + t\left(\langle X_1, X_2\rangle c - \langle X_1, c\rangle X_2 - \langle X_2, X_1\rangle c + \langle X_2, c\rangle X_1\right) + o(t) \\ &= [X_1, X_2] + t[c, [X_1, X_2]] + o(t).\end{aligned}$$

So we get for the normal vectors

$$\begin{aligned}\tilde{N} &= (N + t[c, N])\frac{\|[X_u, X_w]\|}{\|[\tilde{X}_u, \tilde{X}_w]\|}, \\ \tilde{X}_{ij} &= X_{ij} + t[c, X_{ij}]\end{aligned}$$

and for the second fundamental form

$$\begin{aligned}\tilde{h}_{ij} &= \left(\langle X_{ij}, N\rangle + t\langle X_{ij}, [c, N]\rangle + t\langle [c, X_{ij}], N\rangle\right) \cdot \frac{\|[X_u, X_w]\|}{\|[\tilde{X}_u, \tilde{X}_w]\|} + o(t) \\ &= \left(\langle X_{ij}, N\rangle + t(\det|X_{ij}, c, N| + \det|N, c, X_{ij}|)\right) \cdot \frac{\|[X_u, X_w]\|}{\|[\tilde{X}_u, \tilde{X}_w]\|} + o(t) \\ &= \langle X_{ij}, N\rangle \cdot \frac{\|[X_u, X_w]\|}{\|[\tilde{X}_u, \tilde{X}_w]\|} + o(t).\end{aligned}$$

This leads to

$$h_{ij}(t) = h_{ij}(\cdot) \cdot \frac{\|[X_u, X_w]\|}{\|[\tilde{X}_u, \tilde{X}_w]\|} + o(t)$$

and $\delta h_{ij} = 0$. This completes the first part of the proof.

The first variation of the second fundamental form of an infinitesimal bending ($\delta g_{ij} = 0$) vanishes ($\delta h_{ij} = 0$).

$$\delta h_{ij} = 0 \iff h_{ij}(t) = h_{ij}(0) + o(t)$$

$$\tilde{h}_{ij} = \frac{\|[X_u, X_w]\|}{\|[\tilde{X}_u, \tilde{X}_w]\|} h_{ij} + t\langle Z_{ij}, N\rangle + t\langle X_{ij}, [X_u, Z_w] + [Z_u, X_w]\rangle + o(t).$$

This leads to

$$\begin{aligned}0 &= \langle Z_{ij}, N\rangle + \langle X_{ij}, [X_u, [Y, X_w]] + [[Y, X_u], X_w]\rangle \\ &= \langle Z_{ij}, N\rangle + \langle X_{ij}, \langle X_1, X_2\rangle Y - \langle X_1, Y\rangle X_2 - \langle X_2, X_1\rangle Y + \langle X_2, Y\rangle X_1\rangle \\ &= \langle [Y_j, X_i] + [Y, X_{ij}], N\rangle + \langle X_{ij}, \langle X_2, Y\rangle X_1 - \langle X_1, Y\rangle X_2\rangle \\ &= \langle [Y_j, X_i] + \Gamma^l_{ij}[Y, X_l], N\rangle + \Gamma^l_{ij}\langle X_l, \langle X_2, Y\rangle X_1 - \langle X_1, Y\rangle X_2\rangle.\end{aligned}$$

This implies

$$0 = \langle [Y_j, X_i] + \Gamma^l_{ij}[Y, X_l], N\rangle \tag{10}$$

and

$$\begin{aligned}0 &= \Gamma^l_{ij}\langle X_l, \langle X_2, Y\rangle X_1 - \langle X_1, Y\rangle X_2\rangle \\ &= \Gamma^l_{ij}\left(\langle X_2, Y\rangle\langle X_l, X_1\rangle - \langle X_1, Y\rangle\langle X_2, X_l\rangle\right) \\ &= \Gamma^l_{ij}\langle [Y, X_l], [X_2, X_1]\rangle\end{aligned} \tag{11}$$

Using Eq. (11) in Eq. (10) yields

$$0 = \langle [Y_j, X_i], N\rangle = \langle Y_j, [X_i, N]\rangle,$$

which implies

$$\langle Y_j, [X_i, N]\rangle = 0. \tag{12}$$

Consider now $Z_{uw} = Z_{wu} \implies [Y_w, X_u] + [Y, X_{uw}] = [Y_u, X_w] + [Y, X_{wu}]$. This implies $[Y_w, X_u] - [Y_u, X_w] = 0$ and further

$$\begin{aligned} \implies \langle X_w, [Y_w, X_u]\rangle &= 0 \\ \langle X_u, [Y_u, X_w]\rangle &= 0 \\ \iff \det|X_w, Y_w, X_u| &= 0 = \det|Y_w, X_u, X_w|\,, \end{aligned}$$

analogous for Y_u.

This implies

$$\begin{aligned} \langle Y_w, [X_u, X_w]\rangle = 0 \quad &\text{and} \quad \langle Y_u, [X_u, X_w]\rangle = 0 \\ \langle Y_j, [X_i, X_l]\rangle = 0 \quad &\text{and} \quad \langle Y_j, [X_i, N]\rangle = 0 \qquad i, j, l = 1, 2 \end{aligned}$$

It follows that Y is constant and therefore $Z = [c, X] + d$. This concludes the proof. □

3.3 Enforcing Rigidity – the Principle

Surfaces are in general not infinitesimal rigid, besides closed convex surfaces. But you can enforce rigidity by "certain geometric boundary conditions" for the boundary (surface-) curves. You can enforce rigidity also through conditions for the deformation vectors, but in many applications it is nearly impossible to impose these kind of conditions. Appropriate boundary conditions for the boundary curves are the "way to go". For more interesting details see Efimov [1].

Results based on constraints on the boundary curves are in the Grotemeyer [2] paper: An open convex surface is rigid under infinitesimal bendings which do not change along the boundary of the boundary curve:

- the normal section curvature of the surface
- the geodesic torsion of the surface
- the curvature of the boundary curve
- the torsion of the boundary curve

It is somewhat hard to visualize these curvature functions along the boundaries in an intuitive way. Shadow-lines are an intuitive visualization tool. We prove in this paper that the surface is infinitesimal rigid as long as the shadow-lines are stationary during the deformation. The proofs of these or similar results are based on the "Integralformelmethode". This fundamental technique is nicely presented in all details in Huck et al [3]. A specific form of Stokes theorem is used

$$\int\!\!\int_U \mathcal{E}^{ij} T_{i\|j} \mathrm{d}F = \int_{\partial U} T_j \frac{\mathrm{d}u^j}{\mathrm{d}t} \mathrm{d}t\,,$$

where T_i is a C^1 vector field,

$$T_i := \mathcal{E}^{rs} \delta h_{ir} A_s,$$

$$\mathcal{E}^{11} = 0, \tag{13}$$

$$\mathcal{E}^{22} = 0, \tag{14}$$

$$\mathcal{E}^{12} = \frac{1}{\sqrt{g}}, \tag{15}$$

$$\mathcal{E}^{21} = -\frac{1}{\sqrt{g}}, \tag{16}$$

$$g := \det(g_{ij}). \tag{17}$$

$T_{i\|j}$ is the covariant differentiation of the vector field T_i.

An appropriate tensor A_s then "delivers" the result. In more details:

Lemma 4. *A_{ir} is a symmetric tensor and there is a positive definite, symmetric tensor S_{ks} with $\mathcal{E}^{ik}\mathcal{E}^{rs}A_{ir}S_{ks} = 0$. Then*

$$\det(A_{ir}) \le 0$$

and for $\det(A_{ir}) = 0$ we get $A_{ir} = 0$.

Proof. Diagonalize $\mathcal{E}^{ik}\mathcal{E}^{rs}A_{ir}$ with respect to S_{ks}. $\mathcal{E}^{ik}\mathcal{E}^{rs}A_{ir}S_{ks}$ is then the trace, the sum of the two real eigenvalues. If $\mathcal{E}^{ik}\mathcal{E}^{rs}A_{ir}S_{ks} = 0$ then that means that $\det(\mathcal{E}^{ik}\mathcal{E}^{rs}A_{ir}) \le 0$ and since S_{ks} is positive definite $\det(A_{ir}) \le 0$.

In the special case $\det(A_{ir}) = 0$ we get:
A_{ir} symmetric $\implies A_{ir} = a_i \cdot a_r \implies \mathcal{E}^{ik}\mathcal{E}^{rs}a_i a_r = 0$. As S_{ks} is positive definite, we get $a_i = 0$ and finally $A_{ir} = 0$. □

3.4 Enforcing Rigidity by Shadow-Line Constraints

Surfaces in CAD / CAM technology are often "inspected" by parallel lighting, that means all light rays are parallel. We do not use spotlights. So called "shadow-lines" are of special interest.

We prove now that a surface is rigid under inifinitesimal bendings which keep the shadow-lines stationary.

e is the light direction of the shadow-line ("Eigenschattengrenze") with $\langle e, N\rangle = 0$. e is a constant vector with $\langle e, N\rangle > 0$ in the lighted part of the surface.

Theorem 5. *An open convex surface with boundary curves which are shadow-lines ("Eigenschattengrenze") by parallel lighting e is infinitesimally rigid under infinitesimal bending which keep $\langle e, X_i\rangle$, $i = 1, 2$ stationary.*

Proof. Along the boundary it holds that $\delta\langle e, X_i\rangle = 0$ for $i = 1, 2$.

$$\int_U\!\!\int \mathcal{E}^{ij} T_{i\|j} \mathrm{d}F = \int_{\partial U} T_j \frac{\mathrm{d}u^j}{\mathrm{d}t}\mathrm{d}t,$$

$$T_j := \mathcal{E}^{rs}\delta h_{jr}\delta\langle e_i, X_s\rangle.$$

This leads to

$$\int_{\partial U} (\mathcal{E}^{rs}\delta h_{jr}\delta\langle e, X_s\rangle)\frac{\mathrm{d}u^j}{\mathrm{d}t}\mathrm{d}t = 0$$

and

$$\int\int_U \mathcal{E}^{ij}(\mathcal{E}^{rs}\delta h_{jr}\delta\langle e, X_s\rangle)_{\|i}\mathrm{d}F = 0,$$
$$\int\int_U \mathcal{E}^{ij}\mathcal{E}^{rs}\delta h_{jr}\delta\langle e, X_s\rangle_{\|i}\mathrm{d}F = 0.$$

Considering

$$\langle e, X_s\rangle_{\|i} = \langle e, N\rangle h_{si},$$
$$\delta\langle e, X_s\rangle_{\|i} = \delta\langle e, N\rangle h_{si} + \langle e, N\rangle\delta h_{si}$$

so we get

$$\int\int_U \mathcal{E}^{ij}\mathcal{E}^{rs}\delta h_{jr}\delta h_{si}\langle e, N\rangle\mathrm{d}F = 0,$$
$$\int\int_U \langle e, N\rangle\frac{\det(\delta h_{ir})}{\det(g_{ir})}\mathrm{d}F = 0.$$

Here, δh_{ir} is a symmetric tensor, h_{ks} is a symmetric, positive definite tensor and

$$\mathcal{E}^{ih}\mathcal{E}^{rs}\delta h_{ir}h_{ks} = 0$$

implies $\det(\delta h_{ir}) \leq 0$.

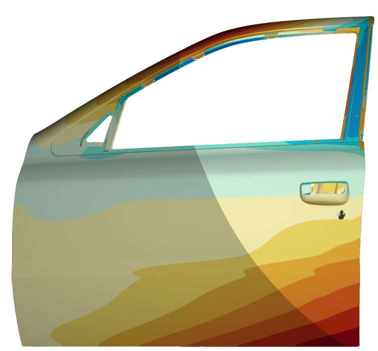
In this special situation $\det(\delta h_{ir})$ is even zero, which implies $\delta h_{ir} = 0$.

This means the surface is rigid under these kind of infinitesimal bendings. □

4 Results

Deformations which do not change the metric are not a problem in design and manufacturing. To guarantee the rigidity by simulation is an important goal. The shadow-lines are stationary under pressure. That means we have the situation we want around the door handle.

(a) Pressure distribution on a car door



(b) Shadow-lines are stationary under pressure, therefore the surface is infinitesimal rigid

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Received March 24, 2023; final form July 14, 2023.