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Three Collinear Points Generated by a Tetrahedron

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Abstract. Let $ABCD$ be a tetrahedron. For each point P inside of the tetrahedron $ABCD$, there is a unique set of points $\{E, F, G, H, I, J\}$ such that (1) E, F, G, H, I, and J are points on the edges DA, AB, BC, CD, AC, and BD, respectively, and (2) the segments EG, FH, and IJ concur at P. If the three planes FGJ, GHI, EHJ, intersect, say at A^*, then we will prove that the three points A, P, A^* are collinear. Let A' be the intersection of the line AP and the plane BCD. If the points B^*, C^*, D^* are defined similar to A^*, and if the points B', C', D' are defined similar to A', we will find the volume of the tetrahedra $A^*B^*C^*D^*$ and $A'B'C'D'$. We use barycentric coordinates to prove these results.

Key Words: tetrahedron, collinear points, barycentric coordinates
MSC 2020: 51M04 (primary), 51M25

1 Introduction

This note is motivated by Theorems 1 and 2 of [3], which we state here as Lemmas 1 and 2, respectively.

Lemma 1. *Suppose the points E, F, G, H, I, and J are points on the edges DA, AB, BC, CD, AC, and BD, respectively, of a tetrahedron $ABCD$ such that the segments EG, FH, and IJ intersect at P. Then the following are true:*

1. *The segments (BH, CJ and DG), (AH, CE and DI), (AJ, BE and DF), (AG, BI and CF) intersect, say at A', B', C', and D', respectively.*
2. *The segments AA', BB', CC', and DD' intersect at P. (See Figure 1.)*

Lemma 2. *Suppose P is a point inside of a tetrahedron $ABCD$. Suppose A', B', C', D' are intersections of (the line AP and the face BCD), (BP and ACD), (CP and ABD), and (DP and ABC), respectively. Then the following are true:*

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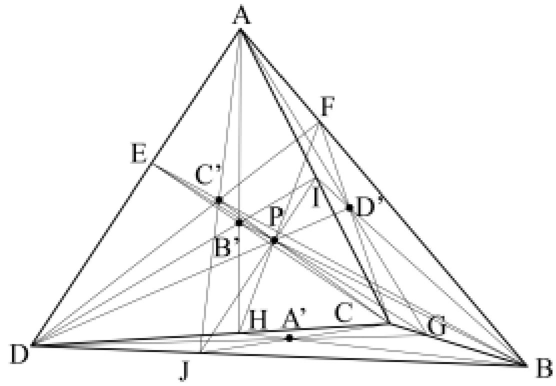

Figure 1: The labelings used in Lemmas 1 and 2 are explained.

1. *The segments (BC' and CB'), (DC' and CD'), (AD' and DA'), (AB' and BA'), (AC' and CA'), and (BD' and DB') intersect, say at E, F, G, H, I, and J, respectively. The points E, F, G, H, I, and J are points on the edges DA, AB, BC, CD, AC and BC respectively.*
2. *The segments EG, FH, and IJ intersect at P. (See Figure 1.)*

Definition 1. Throughout this note, let $ABCD$ be a tetrahedron in $\mathbb{R}^3$. And we denote a point inside of the tetrahedron $ABCD$ by P. Then Lemma 2 says that there is a unique set of points $\{E, F, G, H, I, J\}$ such that (1) E, F, G, H, I, and J are points on the edges DA, AB, BC, CD, AC, and BD, respectively, and (2) the segments EG, FH, and IJ concur at P. Let us call the six tuple of ordered point $[\![E, F, G, H, I, J]\!]$ to be the *edge-coordinates* of P *with respect to* the tetrahedron $ABCD$. Let Γ_A, Γ_B, Γ_C, Γ_D be the planes EFI, FGJ, GHI, EHJ, respectively.

We will prove the following theorem.

Theorem 1. *Let P be an interior point of a tetrahedron $ABCD$, and let $[\![E, F, G, H, I, J]\!]$ be the edge-coordinates of P with respect to the tetrahedron $ABCD$. If the three planes Γ_B, Γ_C, Γ_D intersect, say at A^*, then the three points A, P, and A^* are collinear. (See Figure 2.)*

Our proof of this theorem is computational using barycentric coordinates. We briefly introduce barycentric coordinates in the next section, and prove Theorem 1 in Section 3. If points B^*, C^*, and D^* are defined similarly to A^*, we will find the volumes of the tetrahedra $A'B'C'D'$ and $A^*B^*C^*D^*$ in Section 4.

2 Barycentric Coordinates

Since we are interested in the three-dimensional space, we only explain the barycentric coordinates in the space $\mathbb{R}^3$. There are several ways to introduce barycentric coordinates. Here, we use volumes, and this is sometimes called normalized barycentric coordinates. (See [2, 218–220] for two-dimensional case.)

Let $\mathcal{V}$ be the volume of the tetrahedron $ABCD$. Let Q be any point in the space. Let $\mathcal{V}_{QBCD}$ be the *volume* of the tetrahedron $QBCD$ if Q and A are on the same side of the plane BCD, and $\mathcal{V}_{QBCD}$ to be the *negative* of the volume of the tetrahedron $QBCD$ if Q and A are

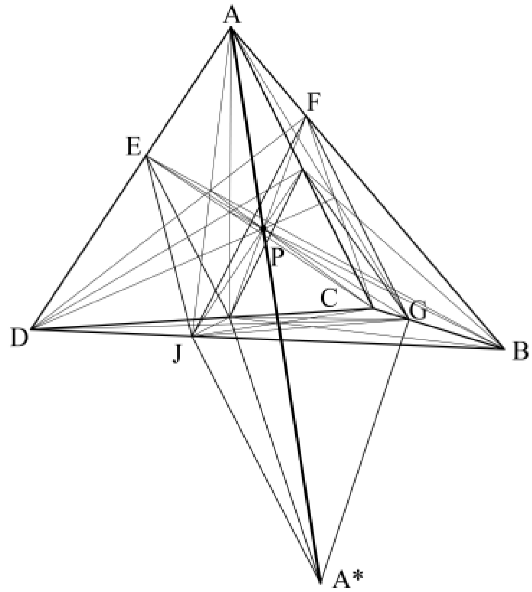

Figure 2: The planes Γ_B, Γ_C, and Γ_D, are represented by the quadrilaterals A^*GFJ, A^*GIH, and A^*HEJ, respectively. For the locations of the points H and I, please see Figure 1.

on the opposite sides of the plane BCD. Similarly, $\mathcal{V}_{QACD}$, $\mathcal{V}_{QABD}$, and $\mathcal{V}_{QABC}$ are defined. Then the *(normalized) barycentric coordinates* $[a, b, c, d]$ of Q are given by

$$a = \frac{\mathcal{V}_{QBCD}}{\mathcal{V}}, \quad b = \frac{\mathcal{V}_{QACD}}{\mathcal{V}}, \quad c = \frac{\mathcal{V}_{QABD}}{\mathcal{V}}, \quad d = \frac{\mathcal{V}_{QABC}}{\mathcal{V}}.$$

Note that every point in the space has unique barycentric coordinates. Let $[a, b, c, d]$ be the barycentric coordinates of a point Q with respect to the tetrahedron $ABCD$. Since $\mathcal{V}_{QBCD}+\mathcal{V}_{QACD}+\mathcal{V}_{QABD}+\mathcal{V}_{QABC} = \mathcal{V}$ for any choice of Q, we must always have $a+b+c+d = 1$.

The point Q is inside of the tetrahedron $ABCD$ if, and only if, a, b, c and d are all positive numbers.

Definition 2. Recall that P is a point inside of the tetrahedron $ABCD$. And we let $[a, b, c, d]$ be the barycentric coordinates of P. Hence, a, b, c and d are all positive numbers such that $a + b + c + d = 1$.

Next, we will give examples that are necessary for the proof of Theorem 1.

Example 1. The barycentric coordinates of the vertices of the tetrahedron $ABCD$ are given by $A = [1, 0, 0, 0]$, $B = [0, 1, 0, 0]$, $C = [0, 0, 1, 0]$, $D = [0, 0, 0, 1]$. The plane ABC is given by the set

$$\{[x, y, z, w] \colon w = 0\} = \{[x, y, z, 0] \colon x + y + z = 1\}.$$

Hence, the set of all points on the face ABC is given by the set

$$\{[x, y, z, 0] \colon x + y + z = 1 \text{ and } x, y, z \geq 0\}.$$

The line AB is given by $\{[x, y, 0, 0] \colon x + y = 1\}$, and the edge AB (the segment AB) is given by $\{[x, y, 0, 0] \colon x + y = 1 \text{ and } x, y \geq 0\}$.

Example 2. The line AP can be thought as the set of points $[1, 0, 0, 0]s + [a, b, c, d]t = [s + at, bt, ct, dt]$, where $s + t = 1$. This gives us

$$AP = \{[x, y, z, w] \colon x = s + at, y = bt, z = ct, w = dt, s + t = 1\}.$$

Similarly, we have

$$\begin{aligned} BP &= \{[x,y,z,w]\colon x = at, y = s + bt, z = ct, w = dt, s+t=1\}, \\ CP &= \{[x,y,z,w]\colon x = at, y = bt, z = s + ct, w = dt, s+t=1\}, \quad \text{and} \\ DP &= \{[x,y,z,w]\colon x = at, y = bt, z = ct, w = s + dt, s+t=1\}. \end{aligned}$$

Sometimes, we substitute s by $1-t$.

Example 3. Let A', B', C', D' be intersections of (the line AP and the face BCD), (BP and ACD), (CP and ABD), and (DP and ABC), respectively. Since the plane $BCD = \{[x,y,z,w]\colon x = 0\}$, the 1^{st} barycentric coordinate of A' is $x = 0$. So by letting $x = s + at = 1-(1-a)t = 0$ in the line AP in Example 2, we have $t = \frac{1}{1-a} = \frac{1}{b+c+d}$. This gives us $A' = \left[0, \frac{b}{b+c+d}, \frac{c}{b+c+d}, \frac{d}{b+c+d}\right]$. As a summary, we have

$$\begin{aligned} A' &= \left[0, \frac{b}{b+c+d}, \frac{c}{b+c+d}, \frac{d}{b+c+d}\right], \quad & B' &= \left[\frac{a}{a+c+d}, 0, \frac{c}{a+c+d}, \frac{d}{a+c+d}\right], \\ C' &= \left[\frac{a}{a+b+d}, \frac{b}{a+b+d}, 0, \frac{d}{a+b+d}\right], \quad & D' &= \left[\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}, 0\right]. \end{aligned}$$

From A', we have

$$BA' = \left\{[x,y,z,w]\colon x = 0, y = s + \tfrac{b}{b+c+d}\cdot t, z = \tfrac{c}{b+c+d}\cdot t, w = \tfrac{d}{b+c+d}\cdot t, s+t=1\right\}.$$

Let H be the intersection of BA' and the edge CD. Since $CD = \{[0,0,z,w]\colon z+w=1\}$, the 2^{nd} barycentric coordinate of H must be 0. From this, we have $y = s + \frac{b}{b+c+d}\cdot t = 0$. Since $s+t=1$ and $a+b+c+d=1$, we have $t = \frac{b+c+d}{c+d}$. Hence, $H = \left[0, 0, \frac{c}{c+d}, \frac{d}{c+d}\right]$. As a summary, we have

$$\begin{aligned} H &= \left[0,0,\frac{c}{c+d},\frac{d}{c+d}\right], \quad & J &= \left[0,\frac{b}{b+d},0,\frac{d}{b+d}\right], \quad & G &= \left[0,\frac{b}{b+c},\frac{c}{b+c},0\right], \\ E &= \left[\frac{a}{a+d},0,0,\frac{d}{a+d}\right], \quad & I &= \left[\frac{a}{a+c},0,\frac{c}{a+c},0\right], \quad \text{and} & F &= \left[\frac{a}{a+b},\frac{b}{a+b},0,0\right]. \end{aligned}$$

By Definition 1, $[\![E,F,G,H,I,J]\!]$ is the edge-coordinate of the point P.

Example 4. The line GJ is given by

$$GJ = \left\{[x,y,z,w]\colon x = 0, y = \frac{b}{b+c}\cdot s + \frac{b}{b+d}\cdot t, z = \frac{c}{b+c}\cdot s, w = \frac{d}{b+d}\cdot t, s+t=1\right\}.$$

Let $Q = \left[0, \frac{b}{b+c}\cdot s + \frac{b}{b+d}\cdot t, \frac{c}{b+c}\cdot s, \frac{d}{b+d}\cdot t\right] \in GJ$ for some $s+t=1$. Then the line FQ is given by

$$\begin{aligned} FQ = \Bigg\{[x,y,z,w]\colon x = \frac{a}{a+b}\cdot l, y = \frac{b}{a+b}\cdot l + \left(\frac{b}{b+c}\cdot s + \frac{b}{b+d}\cdot t\right)\cdot r, \\ z = \left(\frac{c}{b+c}\cdot s\right)\cdot r, w = \left(\frac{d}{b+d}\cdot t\right)\cdot r, l+r=1\Bigg\}. \end{aligned}$$

The collection of the lines FQ for all possible s and t such that $s+t=1$ is the plane FGJ. Let $s \cdot r = m$ and $t \cdot r = n$. Then $l+m+n = l+sr+tr = l+(s+t)r = l+r = 1$, and the plane $FGJ = \Gamma_B$ is given by the set

$$\Gamma_B = \left\{[x,y,z,w] : x = \tfrac{a}{a+b}l, y = \tfrac{b}{a+b}l + \tfrac{b}{b+c}m + \tfrac{b}{b+d}n, z = \tfrac{c}{b+c}m, w = \tfrac{d}{b+d}n, l+m+n=1\right\}.$$

Similarly, we have

$$\Gamma_C = \left\{[x,y,z,w] : x = \tfrac{a}{a+c}l, y = \tfrac{b}{b+c}m, z = \tfrac{c}{a+c}l + \tfrac{c}{b+c}m + \tfrac{c}{c+d}n, w = \tfrac{d}{c+d}n, l+m+n=1\right\},$$
$$\Gamma_D = \left\{[x,y,z,w] : x = \tfrac{a}{a+d}l, y = \tfrac{b}{b+d}m, z = \tfrac{c}{c+d}n, w = \tfrac{d}{a+d}l + \tfrac{d}{b+d}m + \tfrac{d}{c+d}n, l+m+n=1\right\},$$

and

$$\Gamma_A = \left\{[x,y,z,w] : x = \tfrac{a}{a+b}l + \tfrac{a}{a+c}m + \tfrac{a}{a+d}n, y = \tfrac{b}{a+b}l, z = \tfrac{c}{a+c}m, w = \tfrac{d}{a+d}n, l+m+n=1\right\}.$$

3 Proof of Theorem 1

Proof of Theorem 1. Suppose that $P=[a,b,c,d]$ is a point inside of the tetrahedron $ABCD$. So, a, b, c and d are all positive numbers such that $a+b+c+d=1$.

From Example 3, we have

$$H = \left[0,0,\frac{c}{c+d},\frac{d}{c+d}\right], \quad J = \left[0,\frac{b}{b+d},0,\frac{d}{b+d}\right], \qquad G = \left[0,\frac{b}{b+c},\frac{c}{b+c},0\right],$$
$$E = \left[\frac{a}{a+d},0,0,\frac{d}{a+d}\right], \quad I = \left[\frac{a}{a+c},0,\frac{c}{a+c},0\right], \quad \text{and} \quad F = \left[\frac{a}{a+b},\frac{b}{a+b},0,0\right].$$

And from Example 4, we have equations of the planes Γ_B, Γ_C, and Γ_D.

Assume that $1-2a \neq 0$.

Case 1: We will prove that $A^* = \left[\frac{-a}{1-2a}, \frac{b}{1-2a}, \frac{c}{1-2a}, \frac{d}{1-2a}\right]$ is a point on the plane Γ_B. From Example 4, we have that

$$\Gamma_B = \left\{[x,y,z,w] \colon x = \tfrac{a}{a+b}l, y = \tfrac{b}{a+b}l + \tfrac{b}{b+c}m + \tfrac{b}{b+d}n, z = \tfrac{c}{b+c}m, w = \tfrac{d}{b+d}n, l+m+n=1\right\}.$$

Let $l = -\frac{a+b}{1-2a}$, $m = \frac{b+c}{1-2a}$, $n = \frac{b+d}{1-2a}$. Then,

$$l+m+n = -\frac{a+b}{1-2a} + \frac{b+c}{1-2a} + \frac{b+d}{1-2a} = \frac{-a+b+c+d}{1-2a} = 1,$$

and we have

$$\begin{aligned}
x &= \tfrac{a}{a+b}l = \tfrac{a}{a+b}\left(-\tfrac{a+b}{1-2a}\right) = \tfrac{-a}{1-2a},\\
y &= \tfrac{b}{a+b}l + \tfrac{b}{b+c}m + \tfrac{b}{b+d}n - \tfrac{b}{a+b}\left(-\tfrac{a+b}{1-2a}\right) + \tfrac{b}{b+c}\cdot\tfrac{b+c}{1-2a} + \tfrac{b}{b+d}\cdot\tfrac{b+d}{1-2a} = \tfrac{b}{1-2a},\\
z &= \tfrac{c}{b+c}m = \tfrac{c}{b+c}\cdot\tfrac{b+c}{1-2a} = \tfrac{c}{1-2a}, \quad \text{and} \quad w = \tfrac{d}{b+d}n = \tfrac{d}{b+d}\cdot\tfrac{b+c}{1-2a} = \tfrac{d}{1-2a}.
\end{aligned}$$

This shows that $A^* \in \Gamma_B$.

Case 2: From Example 4, we have

$$\Gamma_C = \left\{[x,y,z,w] : x = \tfrac{a}{a+c}l, y = \tfrac{b}{b+c}m, z = \tfrac{c}{a+c}l + \tfrac{c}{b+c}m + \tfrac{c}{c+d}n, w = \tfrac{d}{c+d}n, l+m+n=1\right\},$$

If we let $l = \frac{-(a+c)}{1-2a}$, $m = \frac{b+c}{1-2a}$, $n = \frac{c+d}{1-2a}$ in Γ_C, then $l+m+n=1$ and

$$x = \frac{a}{a+c}l = \frac{-a}{1-2a}, \quad y = \frac{b}{b+c}m = \frac{b}{1-2a},$$
$$z = \frac{c}{a+c}l + \frac{c}{b+c}m + \frac{c}{c+d}n = \frac{-c+c+c}{1-2a} = \frac{c}{1-2a}, \quad \text{and} \quad w = \frac{d}{c+d}n = \frac{d}{1-2a}.$$

This shows that $A^* \in \Gamma_C$.

Case 3: From Example 4, we have

$$\Gamma_D = \left\{[x,y,z,w] : x = \tfrac{a}{a+d}l, y = \tfrac{b}{b+d}m, z = \tfrac{c}{c+d}n, w = \tfrac{d}{a+d}l + \tfrac{d}{b+d}m + \tfrac{d}{c+d}n, l+m+n=1\right\}.$$

If we let $l = \frac{-(a+d)}{1-2a}$, $m = \frac{b+d}{1-2a}$, $n = \frac{c+d}{1-2a}$ in Γ_C, then $l+m+n=1$ and

$$x = \frac{a}{a+d}l = \frac{-a}{1-2a}, \quad y = \frac{b}{b+d}m = \frac{b}{1-2a},$$
$$z = \frac{c}{c+d}n = \frac{c}{1-2a}, \quad \text{and} \quad w = \frac{d}{a+d}l + \frac{d}{b+d}m + \frac{d}{c+d}n = \frac{-d+d+d}{1-2a} = \frac{d}{1-2a}.$$

This shows that $A^* \in \Gamma_D$.

Hence, from Cases 1–3, we have shown that $A^* = \left[\frac{-a}{1-2a}, \frac{b}{1-2a}, \frac{c}{1-2a}, \frac{d}{1-2a}\right]$ is the intersection of the planes Γ_B, Γ_C, and Γ_D. Moreover, this shows that three planes Γ_B, Γ_C and Γ_D intersect if, and only if, $a \neq \frac{1}{2}$.

Finally, we will show that $A^* = \left[\frac{-a}{1-2a}, \frac{b}{1-2a}, \frac{c}{1-2a}, \frac{d}{1-2a}\right]$ is a point on the line AP. From Example 2, the line AP is given by the set

$$AP = \{[x,y,z,w] : x = s+at, y = bt, z = ct, w = dt, s+t=1\}.$$

Let $s = \frac{-2a}{1-2a}$ and $t = \frac{1}{1-2a}$. Then $s+t=1$ and

$$x = s+at = \frac{-2a}{1-2a} + \frac{a}{1-2a} = \frac{-a}{1-2a}, \quad y = bt = \frac{b}{1-2a},$$
$$z = ct = \frac{c}{1-2a}, \quad \text{and} \quad w = dt = \frac{d}{1-2a}.$$

This shows that $A^* = \left[\frac{-a}{1-2a}, \frac{b}{1-2a}, \frac{c}{1-2a}, \frac{d}{1-2a}\right]$ is a point on the line AP. Therefore, the three points A, P, A^* are collinear. □

Corollary 1. *The three planes Γ_B, Γ_C, Γ_D intersect if, and only if, $a \neq \frac{1}{2}$. If $a \neq \frac{1}{2}$, then the barycentric coordinates of the intersecting point A^* of the three planes Γ_B, Γ_C, Γ_D is given by $A^* = \left[\frac{-a}{1-2a}, \frac{b}{1-2a}, \frac{c}{1-2a}, \frac{d}{1-2a}\right]$.*

Proof. Suppose the three planes Γ_B, Γ_C, Γ_D intersect at A^*. From the proof of Theorem 1, A^* has the barycentric coordinate $\left[\frac{-a}{1-2a}, \frac{b}{1-2a}, \frac{c}{1-2a}, \frac{d}{1-2a}\right]$. This is only possible when $a \neq \frac{1}{2}$.

On the other hand, suppose $a \neq \frac{1}{2}$. Then $\left[\frac{-a}{1-2a}, \frac{b}{1-2a}, \frac{c}{1-2a}, \frac{d}{1-2a}\right]$ is a point. Again, from the proof of Theorem 1, it is the intersection of the three planes Γ_B, Γ_C, Γ_D. □

Remark 1. If $a = \frac{1}{2}$, then the only way the planes Γ_B, Γ_C, Γ_D do not intersect is for these planes to form a prism-like tunnel. If $a \neq \frac{1}{2}$, then the three planes Γ_B, Γ_C, Γ_D intersect at A^*. If $a < \frac{1}{2}$, then A^* and A are on the same side of the plane BCD. If $a > \frac{1}{2}$, then A^* and A are on the opposite sides of the plane BCD.

Definition 3. Suppose $(1-2a)(1-2b)(1-2c)(1-2d) \neq 0$. Then the planes ($\Gamma_B$, Γ_C, and Γ_D), (Γ_A, Γ_C, and Γ_D), (Γ_A, Γ_B, and Γ_D), and (Γ_A, Γ_B, and Γ_C) intersect by Corollary 1, and we denote their intersections by A^*, B^*, C^*, D^*, respectively. See Figure 3.

Remark 2. Suppose $(1-2a)(1-2b)(1-2c)(1-2d) \neq 0$, or equivalently, A^*, B^*, C^*, D^*, exists. Then $E \in B^*C^*$, $F \in C^*D^*$, $G \in A^*D^*$, $H \in A^*B^*$, $I \in B^*D^*$, $J \in A^*C^*$. Hence, $[\![E, F, G, H, I, J]\!]$ be the edge-coordinates of P with respect to the tetrahedron $A^*B^*C^*D^*$. By Theorem 1, the planes EHI, EFJ, and FGI intersect. But planes $EHI = ACD$, $EFJ = ABD$, $FGI = ABC$. Hence, the intersection of these three planes is A. Lemma 1 applies to the tetrahedron $A^*B^*C^*D^*$. The three segments B^*F, C^*I, D^*E intersect, say at $A^{*\prime}$. Then $A^{*\prime}$ is the intersection of the line A^*P and the plane $B^*C^*D^*$. In other words, the points A, P, A', A^*, $A^{*\prime}$ are all collinear.

4 The Volumes of the Tetrahedra $A'B'C'D'$ and $A^*B^*C^*D^*$

We will find the volume of the tetrahedra $A'B'C'D'$ and $A^*B^*C^*D^*$. See Figure 3 for the tetrahedron $A^*B^*C^*D^*$.

Notation: The determinant of the matrix $\begin{bmatrix} s_1 & \cdots & s_n \\ \vdots & \ddots & \vdots \\ t_1 & \cdots & t_n \end{bmatrix}$ is denoted by $\begin{vmatrix} s_1 & \cdots & s_n \\ \vdots & \ddots & \vdots \\ t_1 & \cdots & t_n \end{vmatrix}$.

The next lemma may be known, but since we could not find a reference to it, we will prove it. A related result for a two-dimensional case can be found at the bottom of page 295 in [1] without a proof.

Lemma 3. *Let $S = [s_1, s_2, s_3, s_4]$, $T = [t_1, t_2, t_3, t_4]$, $U = [u_1, u_2, u_3, u_4]$, $V = [v_1, v_2, v_3, v_4]$ be points given in barycentric coordinates of points in $\mathbb{R}^3$ with respect to the tetrahedron $ABCD$. Let $\mathcal{V}$ be the volume of the tetrahedron $ABCD$, and let*

$$\delta = \begin{vmatrix} s_1, & s_2, & s_3, & s_4 \\ t_1, & t_2, & t_3, & t_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \end{vmatrix}.$$

Then the volume $\mathcal{V}'$ of the tetrahedron $STUV$ is given by $|\delta|\mathcal{V}$, i.e., $\mathcal{V}' = |\delta|\mathcal{V}$.

Proof. Note that determinants have the property that

$$\begin{vmatrix} s_1 & \dots, & a_i + b_i, & \dots & s_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_1 & \dots, & c_i + d_i, & \dots & t_n \end{vmatrix} = \begin{vmatrix} s_1 & \dots, & a_i, & \dots & s_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_1 & \dots, & c_i, & \dots & t_n \end{vmatrix} + \begin{vmatrix} s_1 & \dots, & b_i, & \dots & s_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_1 & \dots, & d_i, & \dots & t_n \end{vmatrix}. \tag{a}$$

Applying (a) multiple times, and since $s_1+s_2+s_3+s_4 = 1$, $t_1+t_2+t_3+t_4 = 1$, $u_1+u_2+u_3+u_4 = 1$, and $v_1 + v_2 + v_3 + v_4 = 1$, we can show that

$$\begin{vmatrix} s_1, & s_2, & s_3, & s_4 \\ t_1, & t_2, & t_3, & t_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \end{vmatrix} = \delta = -\begin{vmatrix} t_1 - s_1, & t_2 - s_2, & t_3 - s_3 \\ u_1 - s_1, & u_2 - s_2, & u_3 - s_3 \\ v_1 - s_1, & v_2 - s_2, & v_3 - s_3 \end{vmatrix}. \tag{b}$$

Let Σ' be the volume of the parallelepiped defined by $\overrightarrow{ST}$, $\overrightarrow{SU}$, and $\overrightarrow{SV}$ in the affine coordinate system having lines AB, AC, and AD as its coordinate axes with unit lengths being the segments AB, AC, and AD, respectively. Hence, the volume of the parallelepiped defined by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ is the unit. Let Σ be the volume of the parallelepiped defined by vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$. Let $\dot{S}=(s_1,s_2,s_3)$, $\dot{T}=(t_1,t_2,t_3)$, $\dot{U}=(u_1,u_2,u_3)$, $\dot{V}=(v_1,v_2,v_3)$ be points in $\mathbb{R}^3$ with the usual rectangular coordinates. Then $|\delta|$ is the volume of the parallelepiped defined by vectors $\overrightarrow{\dot{S}\dot{T}}=\langle t_1-s_1,t_2-s_2,t_3-s_3\rangle$, $\overrightarrow{\dot{S}\dot{U}}=\langle u_1-s_1,u_2-s_2,u_3-s_3\rangle$, and $\overrightarrow{\dot{S}\dot{V}}=\langle v_1-s_1,v_2-s_2,v_3-s_3\rangle$ by the Equation b. Hence, we have that $\frac{\Sigma'}{\Sigma}=|\delta|$. (This idea is similar to [2, p. 218].) Therefore, $\mathcal{V}'=\frac{1}{6}\Sigma'=\frac{1}{6}|\delta|\Sigma=|\delta|\mathcal{V}$. □

Theorem 2. *Let A', B', C', D' be intersections of (the line AP and the face BCD), (BP and ACD), (CP and ABD), and (DP and ABC), respectively. Then we have the following:*

(1) $A'=\left[0,\frac{b}{b+c+d},\frac{c}{b+c+d},\frac{d}{b+c+d}\right]$, $B'=\left[\frac{a}{a+c+d},0,\frac{c}{a+c+d},\frac{d}{a+c+d}\right]$, $C'=\left[\frac{a}{a+b+d},\frac{b}{a+b+d},0,\frac{d}{a+b+d}\right]$, $D'=\left[\frac{a}{a+b+c},\frac{b}{a+b+c},\frac{c}{a+b+c},0\right]$.

(2) If $\mathcal{V}$ is the volume of the tetrahedron $ABCD$, the volume $\mathcal{V}'$ of the tetrahedron $A'B'C'D'$ is given by $\frac{abcd}{(b+c+d)(a+c+d)(a+b+d)(a+b+c)}\mathcal{V}$.

(3) $\mathcal{V}'\le\frac{1}{81}\mathcal{V}$. The equality holds only when P is the centroid of the tetrahedron $ABCD$.

Proof. (1) is given in Example 2. (2) is an application of Lemma 3. So, we will prove (3). Let $f(a,b,c,d)=\frac{abcd}{(b+c+d)(a+c+d)(a+b+d)(a+b+c)}$. Then we want to maximize $f(a,b,c,d)$ subject to $a+b+c+d=1$; $a,b,c,d>0$. We will use Lagrange's multiplier method. Let $g(a,b,c,d)=a+b+c+d$. Then the critical points are given by $\nabla f=\lambda\nabla g$ for some λ. Since $\nabla g=\langle 1,1,1,1\rangle$, we must have $\lambda=\frac{\partial f}{\partial a}=\frac{\partial f}{\partial b}=\frac{\partial f}{\partial c}=\frac{\partial f}{\partial d}$. From $\frac{\partial f}{\partial a}=\frac{\partial f}{\partial b}$, we have $(b-a)(bc+bd+ca+c^2+cd+ad+dc+d^2)=0$ after simplification. Since $a,b,c,d>0$, this implies $a=b$. Similarly, we have $a=b=c=d$. Since $a+b+c+d=1$, $(a,b,c,d)=(\frac14,\frac14,\frac14,\frac14)$ is the only critical point. Since $\lim_{a\to 0}f(a,b,c,d)=0$, we can see that $f(\frac14,\frac14,\frac14,\frac14)=\frac{1}{81}$ is the maximum value of f. Again, the barycentric coordinates $[\frac14,\frac14,\frac14,\frac14]$ is the centroid of the tetrahedron $ABCD$. □

Theorem 3. *Suppose* $(1-2a)(1-2b)(1-2c)(1-2d)\neq 0$. *Then, we have the following:*

(1) The barycentric coordinates of the points A^*, B^*, C^*, D^* are

$$A^*=\left[\frac{-a}{1-2a},\frac{b}{1-2a},\frac{c}{1-2a},\frac{d}{1-2a}\right],\quad B^*=\left[\frac{a}{1-2b},\frac{-b}{1-2b},\frac{c}{1-2b},\frac{d}{1-2b}\right],$$
$$C^*=\left[\frac{a}{1-2c},\frac{b}{1-2c},\frac{-c}{1-2c},\frac{d}{1-2c}\right],\quad D^*=\left[\frac{a}{1-2d},\frac{b}{1-2d},\frac{c}{1-2d},\frac{-d}{1-2d}\right].$$

(2) If $\mathcal{V}$ is the volume of the tetrahedron $ABCD$, then the volume $\mathcal{V}^*$ of the tetrahedron $A^*B^*C^*D^*$ is given by $\mathcal{V}^*=\frac{16abcd}{|(1-2a)(1-2b)(1-2c)(1-2d)|}\mathcal{V}$.

Proof. Proof of (1) is a repeated application of the proof of Theorem 1. As for (2), by our assumption, we have $(1-2a)(1-2b)(1-2c)(1-2d)\neq 0$ by Corollary 1. Hence, (2) is an application of Lemma 2. □

Remark 3. Unlike the inequality relation in Theorem 2(3) between $\mathcal{V}$ and $\mathcal{V}'$, there is no inequality relation between $\mathcal{V}$ and $\mathcal{V}^*$ in Theorem 3. In order to see this, we consider the segment AA'. The segment AA' is given by $AA'=\{[1-y,\frac{t}{3},\frac{t}{3},\frac{t}{3}]\colon 0\le t\le 1\}$. So, let $P(t)=[1-t,\frac{t}{3},\frac{t}{3},\frac{t}{3}]$, $0<t<1$. Then $P(0)=A$, $P(\frac12)=[\frac12,\frac16,\frac16,\frac16]$, $P(\frac34)=[\frac14,\frac14,\frac14,\frac14]$, and $P(1)=A'$. So, there is no tetrahedron $A^*B^*C^*D^*$ that corresponds to the point $P(\frac12)$, and $P(\frac34)$ is the centroid of the tetrahedron $ABCD$.

Let $f(a,b,c,d)=\frac{16abcd}{|(1-2a)(1-2b)(1-2c)(1-2d)|}$. Then $f(1-t,\frac{t}{3},\frac{t}{3},\frac{t}{3})=\frac{16(1-t)t^3}{|(1-2t)(3-2t)^3|}$. Hence, $\lim_{t\to 0}f(1-t,\frac{t}{3},\frac{t}{3},\frac{t}{3})=0$, $\lim_{t\to\frac12}f(1-t,\frac{t}{3},\frac{t}{3},\frac{t}{3})=\infty$, and $\lim_{t\to 1}f(1-t,\frac{t}{3},\frac{t}{3},\frac{t}{3})=0$. From

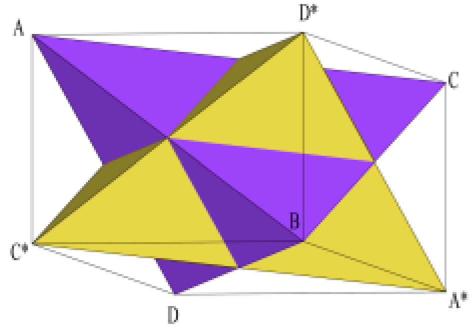

Figure 3: The tetrahedron $A^*B^*C^*D^*$ is exhibited when the point P is the centroid of the tetrahedron $ABCD$.

Theorem 3, the volume $\mathcal{V}^*$ of the tetrahedron $A^*B^*C^*D^*$ can be made as large as and as close to zero as you wish depending on the choice of P.

Remark 4. It is not difficult to see that the following three statements (i)–(iii) are equivalent:

(i) P be the centroid of a tetrahedron $ABCD$.

(ii) A', B', C', D' are the centroid of the triangular faces BCD, ACD, ABD, and BCD, respectively.

(iii) E, F, G, H, I, and J are the mid-points of the edges DA, AB, BC, CD, AC, and BD, respectively.

Let $P = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$, the centroid of the tetrahedron $ABCD$. We will investigate the tetrahedron $A^*B^*C^*D^*$ that corresponds to the centroid P. If we let $t = \frac{3}{4}$, we have $P = P(\frac{3}{4})$ and $f(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = 1$ from Remark 3. So, Theorem 3 shows that the volumes of the tetrahedra $ABCD$ and $A^*B^*C^*D^*$ are the same.

Moreover, the barycentric coordinates of A^* and B^* are given by $[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$ and $[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$, respectively. Note that $H = [0, 0, \frac{1}{2}, \frac{1}{2}]$ from Example 2. This shows that H is the midpoint of the segment A^*B^*. Similarly, we can see that E, F, G, I, and J are also the midpoints of the edges B^*C^*, C^*D^*, D^*A^*, B^*D^*, and A^*C^*, respectively. Then the six quadrilaterals AB^*CD^*, A^*BC^*D, AC^*BD^*, A^*CB^*D, AB^*DC^*, and A^*BD^*C are all parallelograms since all the diagonals of faces bisect each other. Hence, the hexahedron $AB^*CD^*A^*BC^*D$ is a parallelepiped. See Figure 3. In addition, for example, the parallelogram AB^*CD^* contains the edge AC, and is on the plane parallel to the lines AC and BD. The parallelepiped $AB^*CD^*A^*BC^*D$ inscribes both tetrahedra $ABCD$ and $A^*B^*C^*D^*$. Hence, the tetrahedra $ABCD$ and $A^*B^*C^*D^*$ are not only having the same volume, but they are congruent. As a matter of fact, the tetrahedra $ABCD$ and $A^*B^*C^*D^*$ are mirror images of each other, but are not identical unless the tetrahedron $ABCD$ is regular.

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