Journal for Geometry and Graphics
Volume 27 (2023), No. 1, 47–57

A General Trisectrix Curve and its Applications

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Abstract. We find a general trisectrix polar curve and then deduce special trisectrices including some of well known ones. The strategy is by trisecting angle and using some elementary properties of Euclidean geometry.

Key Words: geometric construction, angle trisection, trisectrix

MSC 2020: 51M04

1 Introduction

During the history of geometry, construction of a given angle by straightedge and compass has been a challenge problem. In particular, dividing an angle to n equal parts by this manner was of most importance. The special case $n = 3$ has occupied many scientists in the history.

As a matter of fact, it is impossible to trisect an arbitrary angle by straightedge and compass. However, sometimes, using a certain curve as an additional tool, called a trisectrix, helps it to become possible. Below is the history of some known trisectrices.

A historical trisectrix is the parabola which allows to trisect exactly an arbitrary angle with straightedge and compass. This trisection has been described by René Descartes in his book La Géométrie [2] (see also [6]).

The Tschirnhausen cubic is a plane curve defined by the polar equation $r = \ell \sec^3(\theta/3)$. This curve, sometimes known as de L'Hôpital's cubic or the trisectrix of Catalan, has been studied by von Tschirnhausen, de L'Hôpital, and Catalan [4, pp. 87-90] (see also [5]). Tschirnhausen trisectrix is the negative pedal of a parabola with respect to the focus of the parabola.

Between the years 1723 and 1728, the Italian mathematician, Guido Grandi, studied the "rhodonea" curves which translates to "rose" curves.[1] A rose is the set of points in polar coordinates specified by the polar equation $r = \ell \cos(k\theta)$ (see also [1]). When k is an integer there are k or $2k$ petals depending whether k is odd or even. If k is irrational then the number

[1]http://www-history.mcs.st-andrews.ac.uk/Biographies/Grandi.html

ISSN 1433-8157/$ 2.50

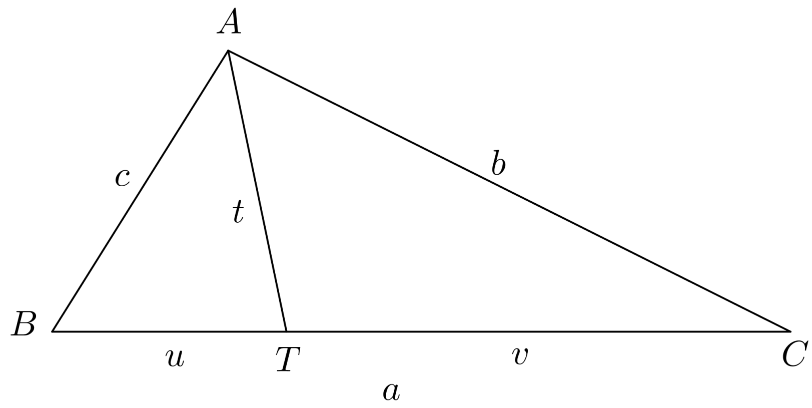

Figure 1: A triangle with a cevian t.

of petals is infinite. A rose with $k = 1/3$ is a Limacon trisectrix [4]. This rose has a single petal with two loops.

The trisectrix of Maclaurin [4] is a cubic plane curve defined as the locus of the intersection points between two lines, each in uniform rotation around a point, one of them going three times as fast as the other. The equation in polar coordinates is $r = \ell(4\cos(\theta) - \sec(\theta))$. This curve is named after Colin Maclaurin who investigated the curve in 1742. The Maclaurin trisectrix is an anallagmatic curve, and the origin is a crunode.

Freeth nephroid curve, defined in the polar system by $r = \ell(1 + 2\sin(\theta/2))$, is a strophoid of a circle with the pole at the center and a fixed point on the circumference of the circle. In a paper published by the London Mathematical Society in 1879, the English mathematician T. J. Freeth, described various strophoids, including the strophoid of a limacon trisectrix [3].

The main purpose of this paper is to find the general trisectrix curve

$$r = \frac{2pc}{1-p}\cos\frac{\theta}{3}, \tag{1}$$

where $0 < p < 1$. Then, particular trisectrices, including some of the above well known ones, are deduced for certain values of p. Our strategy is to trisect the angle $\angle BAC$ in $\triangle ABC$ (Figure 2), take $a_3 = pa$ for $0 < p < 1$, and establish in Theorem 5 the following relation.

$$(1-p)^3b^4 - p^2(3-2p)b^2c^2 + p^3a^2c^2 - p^3c^4 = 0.$$

This, arranging the vertices of $\triangle ABC$ in polar coordinate system as $A = (0, 0)$, $B = (c, 0)$ and $C = (r, \theta)$ turns to the polar equation

$$r^3 - \frac{3p^2c^2}{(1-p)^2}r - \frac{2p^3c^3}{(1-p)^3}\cos\theta = 0, \tag{2}$$

the solution of which is our trisectrix (1).

2 Preliminaries

In a triangle $\triangle ABC$, a cevian of the vertex A, by definition, is any line segment AT from the vertex A intersecting the opposite edge BC at T (Figure 1). The word “cevian” is in honor of the Italian mathematician Giovanni Ceva, who is the first studied cevians and proved the well known useful Ceva’s Theorem about cevians. We follow the standard labeling shown in Figure 1 and recall the following two well known theorems of geometry.

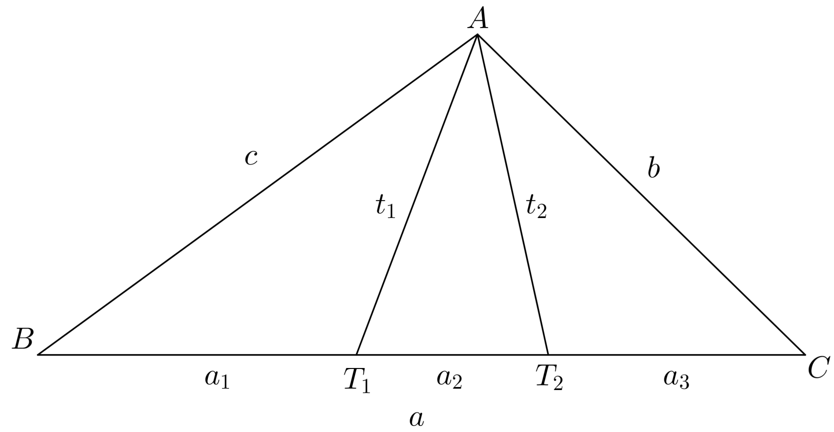

Figure 2: A triangle with two trisector cevians t_1 and t_2.

Theorem 1 (Angle Bisector Theorem)**.** *Suppose AT is the angle bisector of $\angle BAC$. Then,*

$$\frac{c}{b} = \frac{u}{v}.$$

Theorem 2 (Stewart Theorem)**.** *The length t of the cevian AT is given by*

$$b^2u + c^2v = a(t^2 + uv).$$

We follow the labeling scheme in $\triangle ABC$ of Figure 2 and prove some properties of trisector cevians.

Proposition 3. *Let AT_1, AT_2 be trisector cevians with lengths t_1, t_2, respectively. Then,*

$$a_1 = \frac{act_1}{bt_2 + ct_1 + t_1t_2}, \tag{3}$$
$$a_2 = \frac{at_1t_2}{bt_2 + ct_1 + t_1t_2},$$
$$a_3 = \frac{abt_2}{bt_2 + ct_1 + t_1t_2}. \tag{4}$$

Proof. Applying Angle Bisector Theorem 1 for $\triangle ABT_2$, we get $\frac{c}{t_2} = \frac{a_1}{a_2}$, and another apply for $\triangle AT_1C$ gives $\frac{t_1}{b} = \frac{a_2}{a_3}$. It follows that

$$a_1 = \frac{ct_1}{bt_2}a_3,\ \ a_2 = \frac{t_1}{b}a_3,$$

and hence,

$$a_3 = \frac{abt_2}{bt_2 + ct_1 + t_1t_2}.$$

The relations for a_1 and a_2 are obtained similarly. □

Proposition 4. *For the trisector cevians AT_1, AT_2 with lengths t_1, t_2, respectively, we have*

$$\frac{a_1}{a_2 + a_3} \cdot \frac{a_1 + a_2}{a_3} = \frac{c^2}{b^2}. \tag{5}$$

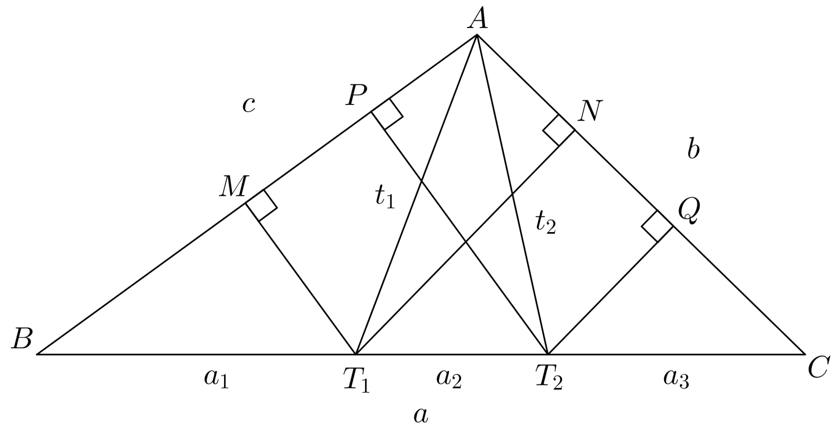

Figure 3: A triangle with two trisector cevians t_1 and t_2 (extended).

Proof. By triangle similarities $\triangle AMT_1 \sim \triangle AQT_2$ as well as $\triangle APT_2 \sim \triangle ANT_1$ in Figure 3 we have

$$\frac{T_1M}{T_2Q} = \frac{t_1}{t_2} = \frac{T_1N}{T_2P}. \tag{6}$$

Computing the areas in two ways and using (6), we approach to the result:

$$\frac{a_1}{a_2+a_3} \cdot \frac{a_1+a_2}{a_3} = \frac{S_{ABT_1}}{S_{AT_1C}} \cdot \frac{S_{ABT_2}}{S_{AT_2C}} = \frac{cT_1M}{bT_1N} \cdot \frac{cT_2P}{bT_2Q} = \frac{c^2}{b^2}.$$

Here, by S_{ABC} we mean the area of $\triangle ABC$. □

Note that there is nothing special about being trisector cevians in the proof of Theorem 4. In fact, this theorem holds for any isogonal conjugate cevians. However, we will not effort with this concept.

3 Finding a general trisectrix curve

Now, we attempt to find our general trisectrix curve (1). Suppose $a_3 = pa$, $0 < p < 1$, where $p = p(a, b, c)$ is a function of side lengths a, b and c. Then, $a_1 + a_2 = (1-p)a$ and from (5) we have

$$a_1 = \frac{apc^2}{(1-p)b^2 + pc^2}. \tag{7}$$

Now, from (4) we get

$$p = \frac{bt_2}{bt_2 + ct_1 + t_1t_2}$$

or, $bt_2 = \frac{p}{1-p}(ct_1 + t_1t_2)$. Putting this in (3) and equating the right hand sides of the resulted equation and (7) we obtain

$$\frac{apc^2}{(1-p)b^2 + pc^2} = \frac{ac(1-p)}{c+t_2}.$$

This leads to

$$t_2 = \frac{(1-p)^2b^2 - p^2c^2}{pc}. \tag{8}$$

On the other hand, Stewart Theorem 2 implies

$$t_2^2 = (1-p)b^2 + pc^2 - p(1-p)a^2. \tag{9}$$

Equate the right hand sides of (9) and the square of (8) and sum up to the following basic result.

Theorem 5. *In $\triangle ABC$ of Figure 2, let the trisector cevian AT_2 meet the side a such that $a_3 = ap$, where $0 < p < 1$. Then the sides of triangle satisfy the following.*

$$(1-p)^3 b^4 - p^2(3-2p)b^2c^2 + p^3a^2c^2 - p^3c^4 = 0. \tag{10}$$

Writing (10) in terms of p, leads to the following corollary.

Corollary 6. *The angle $\angle BAC$ can be trisected by straightedge and compass if and only if roots of the cubic*

$$(a^2c^2 - (b^2-c^2)^2)p^3 + 3b^2(b^2-c^2)p^2 - 3pb^4 + b^4 = 0$$

are constructable.

Suppose in $\triangle ABC$ of Figure 2 that the vertices have polar coordinates $A = (0,0)$, $B = (c,0)$ and $C = (r,\theta)$. Then

$$b = r,\ a = \sqrt{r^2 + c^2 - 2cr\cos\theta}. \tag{11}$$

Substituting these values in (10), gives

$$(1-p)^3r^4 - p^2(3-2p)r^2c^2 + p^3c^2(r^2 + c^2 - 2cr\cos\theta - c^2) = 0,$$

which simplifies

$$r^3 - \frac{3p^2c^2}{(1-p)^2}r - \frac{2p^3c^3}{(1-p)^3}\cos\theta = 0,$$

with a root

$$r = \frac{2pc}{1-p}\cos\frac{\theta}{3}.$$

We summarize the above discussion and get our general trisectrix curve.

Theorem 7. *Suppose $c > 0$ is a constant and $p = p(r,\theta)$ is a constructable function with $0 < p < 1$. Put $A = (0,0)$, $B = (c,0)$ and let $C = (r,\theta)$ be a point on the polar curve*

$$r = \frac{2pc}{1-p}\cos\frac{\theta}{3}. \tag{12}$$

Then $\angle BAC$ is trisectable. In other words, (12) *is a trisectrix.*

Remark 8. When we refer to Equation (12), we typically use the core equations (11) without mentioning them.

Remark 9. The trisectrix (12) depends on the choice $a_3 = pa$ for some $0 < p < 1$. If, instead, we had taken $a_1 = pa$ for $0 < p < 1$ then we would have gotten the trisectrix curve

$$r = \frac{c(1-p)}{2p\cos(\theta/3)}, \tag{13}$$

which is the inversion of (12) with respect to the circle $r = c$.

4 Applications

In this section we deduce some special trisectrices as examples, by taking certain values of p in trisectrix (12) or, its inversion (13). In particular, we obtain some well known trisectrices mentioned in the introductory section.

Example 10 (Maclaurin trisectrix). Let $0 < p < 1$ be a constant and consider the trisectrix (13),

$$r = \frac{c(1-p)}{2p\cos(\theta/3)}.$$

In the special case $p = 1/3$, the trisectrix $r = c\sec(\theta/3)$ is the Maclaurin trisectrix.

Example 10 is the only application of trisectrix (13). All of the oncoming examples concerns the application of trisectrix (12).

Example 11 (Limacon trisectrix). Suppose $0 < p < 1$ is constant. Then the trisectrix (12) is the limacon

$$r = \frac{2cp}{1-p}\cos\frac{\theta}{3}.$$

In particular, if $p = 1/2$ (i.e, the median of side BC coincides with T_2), then the limacon turns to

$$r = 2c\cos\frac{\theta}{3}$$

which is the limacon trisectrix $r = c(1 + 2\cos\theta)$ translated c units to the right on the polar axis.

Example 12 (Freeth nephroid). In Figure 3, suppose $CQ = qb$ for some $0 < q < 1$. Then it is easy to see that $t_2^2 - a_3^2 = (1 - 2q)b^2$. Moreover, putting $p^2a^2 = a_3^2$ in (9), we get $t_2^2 - a_3^2 = (1 - p)b^2 + pc^2 - pa^2$. Equating the right hand sides of the above two equations implies the following.

$$p = \frac{qb}{b - c\cos\theta}. \tag{14}$$

Let q be a constant. Then, the corresponding trisectrix (12) with the above p is

$$r = \frac{c}{1-q}(\cos\theta + 2q\cos\frac{\theta}{3}).$$

In particular, when $q = 1/2$ (T_2Q is the perpendicular bisector of AC) then the trisectrix is the Freeth nephroid

$$r = 2c(\cos\theta + \cos\frac{\theta}{3}).$$

Reflection of this curve about line $x = c/2$ gives the curve $r = c(1 + 2\sin\frac{\theta}{2})$ which is used to construct regular heptagon (see [5, pp.135]).

Another interesting case occurs opposing to Freeth's case, i.e, when the perpendicular bisector of side AC intersects the side BC at T_1. Therefore, $t_1 = a_2 + a_3$. On the other hand, by Theorems 1 and 2,

$$t_1 = \frac{ab^2}{a^2 + b^2 - c^2}, \quad \frac{a_3}{a_2} = \frac{b}{t_1}. \tag{15}$$

The latter equation of (15) implies

$$\frac{b}{b + t_1} = \frac{a_3}{a_2 + a_3} = \frac{pa}{t_1},$$

from which and the former equation of (15) we obtain the following.

$$p = \frac{b}{a + 2b - 2c\cos\theta}.$$

With this value of p, by a tedious but straightforward calculations, the corresponding trisectrix (12) turns to

$$r = \frac{c\sin\frac{5\theta}{3}}{\sin\frac{2\theta}{3}}. \tag{16}$$

Example 13 (Tschirnhausen cubic and Cayley sextic). Inspired by equation (14) in Example 12, suppose

$$p = \frac{kb^2}{a^2 + mb^2 + nc^2},$$

where m, n, k are real constants establishing the inequality $0 < p < 1$. With this p, the equation of trisectrix (12) turns to

$$(m - k + 1)r^2 - 2cr(\cos\theta + k\cos\frac{\theta}{3}) + (n + 1)c^2 = 0.$$

If $m = k - 1$ and $n \neq -1$, then

$$r = \frac{(n + 1)c}{2(\cos\theta + k\cos\theta/3)},$$

where the spacial case $k = 3$, $n = 7$ is the Tschirnhausen cubic $r = c\sec^3\frac{\theta}{3}$, also known as Catalan trisectrix.

If $n = -1$ and $m \neq k - 1$ then, $r = 2c(\cos\theta + k\cos(\theta/3))/(m - k + 1)$, where the spacial case $k = 3$ and $m = 10$ gives rise to the curve $r = c\cos^3(\theta/3)$, known as Cayley sextic [5, pp. 155] which is the inversion of Tschirnhausen cubic with respect to the circle $r = c$. This curve is also the pedal curve (or roulette) of a cardioid with respect to its cusp.

Example 14 (Freeth supertrisectrix). Consider $t_2 = bq$ for some $q > 0$. By the law of cosines in $\triangle AT_2C$,

$$2p^2 = b^2 + b^2q^2 - 2b^2q\cos\frac{\theta}{3}.$$

and by Stewart Theorem, $b^2q^2 = c^2p + b^2(1 - p) - a^2p + a^2p^2$. Therefore,

$$p = \frac{2b^2(1 - q\cos(\theta/3))}{a^2 + b^2 - c^2}.$$

For constant q, the trisectrix (12) becomes

$$r = \frac{c}{q}(1 - 2q\cos\frac{\theta}{3} + 2\cos\frac{2\theta}{3}).$$

The special case $q = 1$, is called Freeth supertrisectrix (see [5, pp. 136]) and is strophoid of limacon trisectrix.

Example 15 (Tomahawk trisector tool). In Example 14 let $q = \frac{c}{b}$. Then $t_2 = c$ and AT_1 is the altitude (Figure 4). We have $a_1 = a_2 = T_2D$, i.e., the circle with center T_2 passing through T_1 is tangent to side AC. The bold segments together with the semicircle in Figure 4 is looking

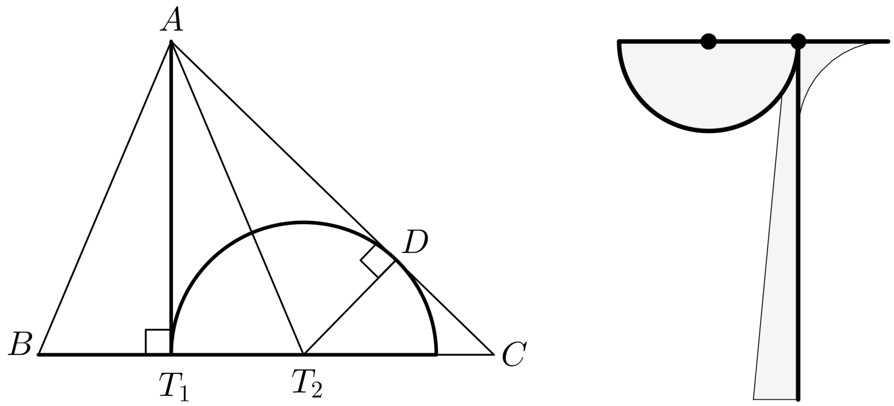

Figure 4: Tomahawk trisector tool and its trisecting method.

like a tomahawk, a native American axe [7], which is a trisector tool called by the same name. This tool is shown in the left side of Figure 4.

A simple calculation shows that

$$p = \frac{2b(b - c\cos(\theta/3))}{a^2 + b^2 - c^2} = \frac{b^2 - c^2}{a^2}.$$

Substituting this in trisectrix (12), we get

$$r = \frac{2r\cos\theta/3 - 2c\cos^2\theta/3}{\cos\theta/3 - \cos\theta}.$$

Solving this relation for r leads to

$$r = c\cos\frac{\theta}{3}\sec\frac{2\theta}{3}.$$

Example 16. Suppose in Figure 2 that D is a point on side AC with $CD = qb$ for some $0 < q < 1$ and T_2 is the intersection of angle bisector of $\angle BDC$ with side BC. Then by Angle Bisector Theorem 1 and the fact $a_3 = pa$ we have

$$p = \frac{bq}{BD + bq}.$$

With this p, the trisectrix (12) leads to the equation $BD = 2cq\cos\frac{\theta}{3}$. Now, if BD is the angle bisector of $\angle ABC$, then $q = \frac{a}{a+c}$. It follows that

$$BD^2 = \frac{4c^2a^2}{(a+c)^2}\cos^2\frac{\theta}{3}. \tag{17}$$

On the other hand, by the law of cosines in $\triangle ABD$,

$$BD^2 = c^2 + \frac{b^2c^2}{(a+c)^2} - \frac{2bc^2}{a+c}\cos\theta, \tag{18}$$

since $AD = bc/(a+c)$. Equating the right hand sides of (17) and (18), by a tedious but straightforward calculation we get

$$r = \frac{2c\cos\frac{\theta}{3}\sin(\frac{2\theta}{3} \pm \theta)}{-\sin\frac{5\theta}{3}}$$

which is either the limacon $r = -2c\cos(\theta/3)$ or the curve

$$r = \frac{c\sin\frac{2\theta}{3}}{\sin\frac{5\theta}{3}}.$$

Compare this trisectrix with (16).

Finally, let BD be the altitude to the side AC. Then

$$q = \frac{a^2 + b^2 - c^2}{2b^2}.$$

In this case we have

$$c\sin\theta = BD = \frac{2c(r^2 - cr\cos\theta)}{r^2}\cos\frac{\theta}{3}$$

which simplifies to the following trisectrix

$$r = \frac{2c\cos\theta\cos\frac{\theta}{3}}{2\cos\frac{\theta}{3} - \sin\theta}.$$

5 Conclusion

We found the general trisectrix curve

$$r = \frac{2pc}{1-p}\cos\frac{\theta}{3}, \quad 0 < p < 1, \tag{19}$$

and its inversion with respect to the circle $r = c$ which is

$$r = \frac{c(1-p)}{2p\cos(\theta/3)}, \quad 0 < p < 1. \tag{20}$$

In the first row of Table 1, for a particular value of p in the curve (20) the Maclaurin trisectrix is obtained. The remaining rows are special trisectrices concerned from the curve (19), the well known trisectrices of them are specified with their names in the table.

The strategy in Section 3 may be used to find other general trisectrices. Furthermore, the methodology in Section 4 may be applied to create more special trisectrices from either (19) or (20).

Acknowledgment

The authors would like to thank the reviewer(s) for all useful and helpful comments which definitely improved the paper.

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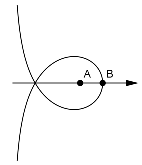
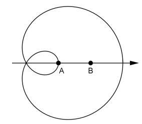
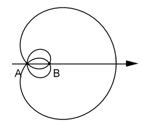
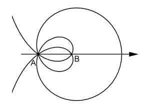
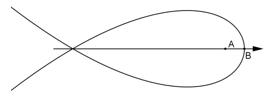
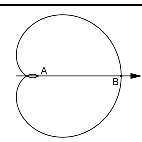
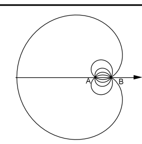
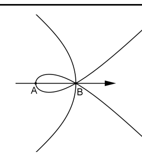
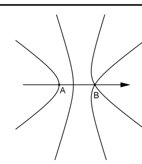
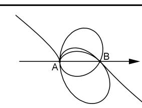
Table 1: Special trisectrices concluded from the curves (20) (the first line) and (19) (the remaining lines) for various values of p.

p	polar equation	graph	resulted trisectrix
$\frac{1}{3}$	$r = c\sec\frac{\theta}{3}$		Maclaurin trisectrix
$\frac{1}{2}$	$r = 2c\cos\frac{\theta}{3}$		Limacon trisectrix
$\frac{b}{2b-2c\cos\theta}$	$r = 2c(\cos\theta + \cos\frac{\theta}{3})$		Freeth nephroid
$\frac{b}{a+2b-2c\cos\theta}$	$r = c\sin\frac{5\theta}{3}\csc\frac{2\theta}{3}$		
$\frac{3b^2}{a^2+2b^2+7c^2}$	$r = c\sec^3\frac{\theta}{3}$		Tschirnhausen cubic
$\frac{3b^2}{a^2+10b^2-c^2}$	$r = c\cos^3\frac{\theta}{3}$		Cayley sextic
$\frac{2b^2(1-\cos\frac{\theta}{3})}{a^2+b^2-c^2}$	$r = c(1 - 2\cos\frac{\theta}{3} + 2\cos\frac{2\theta}{3})$		Freeth supertrisectrix
$\frac{b^2-c^2}{a^2}$	$r = c\cos\frac{\theta}{3}\sec\frac{2\theta}{3}$		
$\frac{b}{b+2c\cos(\frac{\theta}{3})}$	$r = c\sin\frac{2\theta}{3}\csc\frac{5\theta}{3}$		
$\frac{b-c\cos\theta}{b-c(\cos\theta-\sin\theta)}$	$r = \frac{2c\cos\theta\cos(\frac{\theta}{3})}{2\cos(\frac{\theta}{3})-\sin\theta}$		

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Received June 7, 2023; final form July 3, 2023.