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Family of Conics Having Double Contact with two Intersecting Ellipses

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Abstract. In this paper we prove using Projective Geometry, Analytic Geometry and Calculus the converse of the theorem, which is proved with Synthetic Projective Geometry in J. L. S. Hatton's book *The Principles of Projective Geometry Applied to the Straight Line and Conic,* Cambridge University Press, 1913, p. 287, case (b). This theorem, as well as its converse, refer to properties that exist when a conic C_3 contacts two other intersecting conics C_1 and C_2 and specifically concern the existing harmonic pencil between common chords of C_1, C_2 and the pair of their contact chords with C_3. With the proof of the converse theorem, which is achieved here in the case of two concentric ellipses, the problem of constructing a conic C_3 is also addressed. In addition we investigate the type of conic C_3, which is tangent to C_1, C_2, and the condition that is required for C_3 to be an ellipse, a hyperbola or a degenerate parabola, either inscribed or circumscribed to C_1, C_2. Finally, we refer to the existing involution between the common fixed chords and the changing contact chords.

Key Words: harmonic pencil, concentric ellipses, conjugate points, double contact conic, involution

MSC 2020: 51N15 (primary), 51N20, 68U05

1 Introduction

The notion of harmonic pencil of lines is a fundamental notion in Projective Geometry (s. [5, p. 24]): In the real projective plane a pencil of four concurring lines OA, OB, OC, OD, denoted by $O(A, B, C, D)$, is called *harmonic pencil* or *harmonic bundle*, if the cross ratio of the four lines (in that order), by times also denoted by $O(A, B, C, D)$, is equal to -1 (s. Figure 1).

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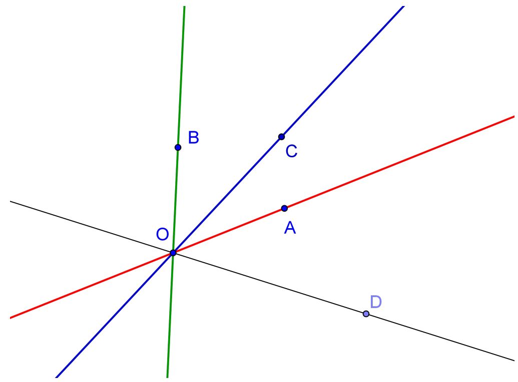

Figure 1: A harmonic pencil i.e. $O(A, B, C, D) = -1$

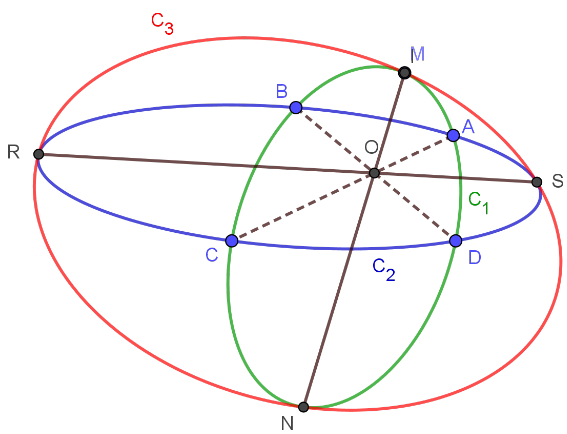

Figure 2: C_3 has double contact with C_1, C_2 and so $O(A, B, M, R) = -1$

In particular, four lines OA, OB, OC, OD through the origin O, with equations $y = \lambda_i x$, $i = 1, 2, 3, 4$, form a harmonic pencil $O(A, B, C, D)$, if their gradients λ_i, $i = 1, 2, 3, 4$ satisfy the following equation:

$$\frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_3} = -\frac{\lambda_4 - \lambda_1}{\lambda_2 - \lambda_4} \tag{1}$$

or equivalently

$$\lambda_4 = \frac{2\lambda_1\lambda_2 - \lambda_3(\lambda_1 + \lambda_2)}{\lambda_1 + \lambda_2 - 2\lambda_3}. \tag{2}$$

So, given two fixed lines through O, say OA, OB, determined by their gradients λ_1, λ_2, one can correspond to each line OC: $y = \lambda_3 x$, the unique line OD: $y = \lambda_4 x$, where λ_4 is given by (2), so that the four lines form a harmonic pencil. This transformation is called *harmonic conjugation* with respect to the two given lines.

J. L. S. Hatton in [2, p. 287], case (b) gives a property of two intersecting conics having double contact with a third conic, which concerns a harmonic pencil:

Let two conics C_1, C_2 intersect at four points A, B, C, D, that define a complete quadrangle[1]*. Let O be any of the three diagonal points of this quadrangle. If there exists a conic C_3, which has double contact with C_1 at M, N and double contact with C_2 at R, S, then MN, RS, AC, BD meet at O and it holds $O(A, B, M, R) = -1$, i.e. the chords of contact*

[1] S. [1, p. 388].

of C_1, C_2 with C_3 and two of the chords of intersection of C_1, C_2 are concurring and form a harmonic pencil (s. Figure 2).

Conversing the above theorem we will investigate the following question:

> Let two conics C_1, C_2 intersect at four points A, B, C, D with diagonal point O. Let MN, RS be chords of C_1, C_2 respectively passing through O and forming a harmonic pencil with the chords of intersection AC, BD of C_1, C_2, i.e. $O(A, B, M, R) = -1$. Is there a conic passing through M, N, R, S and having double contact with C_1 and C_2 at M, N and R, S respectively?

In this initial question O can be any of the three diagonal points of the complete quadrangle defined by A, B, C, D, i.e. O can be any vertex of the common polar triangle of C_1, C_2 (s. [1, p. 278, 294]). In what follows we will investigate this question especially in the case of two ellipses C_1, C_2 [2] having common centre O in relation to existence, number and type of conics that pass through M, N, R, S and have double contact with C_1 and C_2 at M, N and R, S respectively. Although we will study in depth the case that the diagonal point O lies inside C_1, C_2, our investigation method can also be applied in case O lies outside C_1, C_2. In our study we will use methods of Projective Geometry, Analytic Geometry and Calculus.

After the proof of the converse theorem, the two theorems are unified in the case of two concentric ellipses as follows:

Theorem 1. *For there to be a conic C_3 having double contact with two intersecting ellipses C_1 and C_2 with common centre O sufficient and necessary condition is the common chords of C_1, C_2 and the pair of contact chords of C_3 with C_1 and C_2 to form a harmonic pencil with centre O.*

Based on this Theorem we can construct any of the infinite number of conics C_3, which passes through the four points of contact and tangents to the corresponding tangent lines to these four points.

Remark 1. The results of the above Theorem hold true for any two regular conics (not just ellipses) C_1 and C_2 having four intersection points and for any vertex O of its common polar triangle. If the configuration of the two conics C_1 and C_2 and the point O is not projectively equivalent to that of the above Theorem, minor modifications of its proof are necessary. We will occasionally hint at this possibility.

2 Two Concentric Intersecting Ellipses

In the real projective plane we consider two conics C_1, C_2 having four intersection points A, B, C, D. The three diagonal points of A, B, C, D form the common polar triangle of both conics. Let O be the diagonal point lying in the interior of C_1. In what follows, we assume that O is also in the interior of C_2 and both, C_1 and C_2 are ellipses. Using a homology we can always map two intersecting ellipses to two concentric ellipses. Therefore we assume that O is the common centre of C_1, C_2. With no loss of generality we consider C_2 as a circle, since there is always a projectivity mapping an ellipse on a circle. We choose a coordinate system

[2] The case of two concentric intersecting ellipses is of a special interest investigating the Four Ellipses Problem (s. [3, 4, 6]).

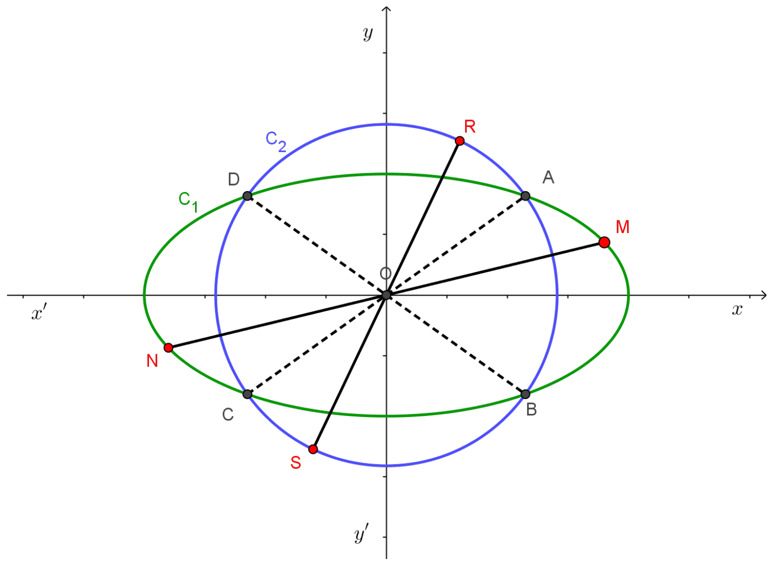

Figure 3: Two concentric ellipses with $O(A, B, M, R) = -1$

so that C_1, C_2 have the following equations:

$$C_1\colon \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{3}$$

$$C_2\colon x^2 + y^2 = r^2, \qquad b < r < a \tag{4}$$

(s. Figure 3). In this case lines OA, OB are symmetric with respect to $x'x$ axis. If $y = \lambda_i x$, $i = 1, 2$ are the equations of OA, OB respectively, then it holds

$$\lambda_1 = -\lambda_2. \tag{5}$$

Let $MN\colon y = \lambda_3 x$ be the line of an arbitrary diameter of C_1 and $RS\colon y = \lambda_4 x$ the harmonic conjugate line of MN with respect to the given lines OA, OB, intersecting C_2 at R, S. According to (2) and (5) it holds

$$\lambda_4 = -\frac{\lambda_1 \lambda_2}{\lambda_3} = \frac{\lambda_1^2}{\lambda_3}. \tag{6}$$

Since $A(x_A, y_A)$ is an intersection point of C_1, C_2, the following hold

$$\frac{x_A^2}{a^2} + \frac{y_A^2}{b^2} = 1, \qquad y_A = \lambda_1 x_A, \qquad x_A^2 + y_A^2 = r^2. \tag{7}$$

Eliminating x_A, y_A we obtain

$$r^2 = \frac{a^2 b^2}{b^2 + \lambda_1^2 a^2}(1 + \lambda_1^2). \tag{8}$$

For $M(x_M, y_M)$ and $R(x_R, y_R)$ it holds respectively

$$\frac{x_M^2}{a^2} + \frac{y_M^2}{b^2} = 1, \qquad y_M = \lambda_3 x_M \tag{9}$$

and

$$x_R^2 + y_R^2 = r^2, \qquad y_R = \lambda_4 x_R. \tag{10}$$

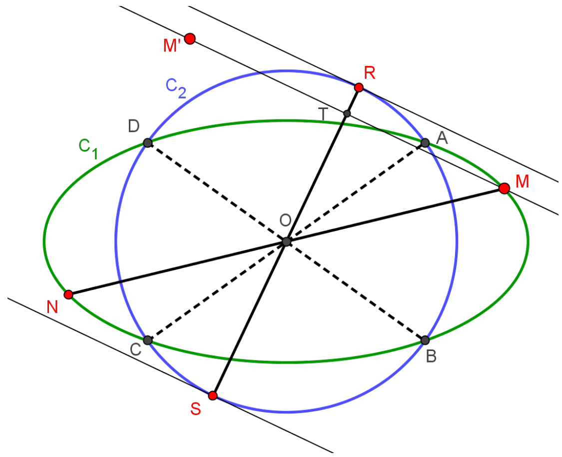

Figure 4: M' is the conjugate point of M with respect to RS and C_2

So, according to (6) and (8) we have

$$x_M^2 = \frac{a^2b^2}{b^2+\lambda_3^2a^2}, \qquad y_M = \lambda_3 x_M \tag{11}$$

and

$$x_R^2 = \frac{\lambda_3^2(1+\lambda_1^2)(b^2+\lambda_3^2a^2)}{(\lambda_3^2+\lambda_1^4)(b^2+\lambda_1^2a^2)}x_M^2, \qquad y_R = \frac{\lambda_1^2}{\lambda_3}x_R. \tag{12}$$

3 Construction of Conic C_3

Let P be the pole of RS with respect to circle C_2 and T be the intersection point of PM and RS. We consider point M' so that M, M' are harmonic conjugate to P, T. In what follows we will call M' *the conjugate point of* M *with respect to* RS *and* C_2. Since conics C_1, C_2 are concentric and so RS is a diameter of C_2, the tangent lines of C_2 at R, S are parallel and point P lies at infinity. Consequently, PM is a line through M parallel to the tangent lines of C_2 at R, S and M' is the symmetric point of M with respect to T (s. Figure 4). The tangent line of C_2 at R or S is perpendicular to RS. So, its gradient and also the gradient of MM' is equal to (s. (6))

$$-\frac{1}{\lambda_4} = -\frac{\lambda_3}{\lambda_1^2}. \tag{13}$$

So, point $M'(x_{M'}, y_{M'})$ satisfies the following equations:

$$y_{M'} - y_M = -\frac{1}{\lambda_4}(x_{M'} - x_M), \qquad \frac{y_M + y_{M'}}{2} = \lambda_4\frac{x_{M'}+x_M}{2}. \tag{14}$$

According to (11) and (13) it holds:

$$x_{M'} = \frac{\lambda_3^2 - \lambda_1^4 + 2\lambda_1^2\lambda_3^2}{\lambda_1^4+\lambda_3^2}x_M, \qquad y_{M'} = \frac{(2\lambda_1^2 - \lambda_3^2 + \lambda_1^4)\lambda_3}{\lambda_1^4+\lambda_3^2}x_M. \tag{15}$$

Let C_3 be the conic through M, N, R, S, M'. We will prove that C_3 has double contact with C_1 at M, N and double contact with C_2 at R, S.

Remark 2. For more constructions of conics from five points or four points and a tangent line in one of them we refer to [1, p. 162] and [5, p. 254].

3.1 The Equation of Conic C_3

The equation of C_3 is given by

$$C_3\colon \begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_M^2 & x_M y_M & y_M^2 & x_M & y_M & 1 \\ x_N^2 & x_N y_N & y_N^2 & x_N & y_N & 1 \\ x_R^2 & x_R y_R & y_R^2 & x_R & y_R & 1 \\ x_S^2 & x_S y_S & y_S^2 & x_S & y_S & 1 \\ x_{M'}^2 & x_{M'} y_{M'} & y_{M'}^2 & x_{M'} & y_{M'} & 1 \end{vmatrix} = 0. \tag{16}$$

Substituting $x_N = -x_M$, $y_N = -y_M$, $x_S = -x_R$, $y_S = -y_R$ in (16) and using determinant properties we get

$$C_3\colon \begin{vmatrix} x^2 & xy & y^2 & 1 \\ x_M^2 & x_M y_M & y_M^2 & 1 \\ x_R^2 & x_R y_R & y_R^2 & 1 \\ x_{M'}^2 & x_{M'} y_{M'} & y_{M'}^2 & 1 \end{vmatrix} = 0 \tag{17}$$

considering that in general x_M, $x_R \neq 0$ and $\lambda_3 \neq \lambda_4$.

Remark 3. We notice that the conic C_3 is concentric with C_1, C_2 (s. (17)).

Substituting x_M, y_M, x_R, y_R, $x_{M'}$, $y_{M'}$ in (17) through (11), (12), (15) and considering that in general $\lambda_3 \neq \lambda_1$, (17) turns to

$$F(x,y) := \alpha x^2 + 2\beta xy + \gamma y^2 + \delta = 0 \tag{18}$$

where

$$\alpha = (\lambda_1^2 - \lambda_3^2)(b^2 + a^2\lambda_1^2) - \lambda_1^4(a^2 - b^2), \tag{19}$$
$$\beta = \lambda_1^2\lambda_3(a^2 - b^2), \tag{20}$$
$$\gamma = (1 + \lambda_1^2)(b^2\lambda_1^2 - a^2\lambda_3^2) + \lambda_1^4(a^2 - b^2), \tag{21}$$
$$\delta = (1 + \lambda_1^2)(\lambda_3^2 - \lambda_1^2)a^2b^2. \tag{22}$$

So, the equation of conic C_3 is the following:

$$[(\lambda_1^2 - \lambda_3^2)(b^2 + a^2\lambda_1^2) - \lambda_1^4(a^2 - b^2)]x^2 + 2\lambda_1^2\lambda_3(a^2 - b^2)xy + [(1 + \lambda_1^2)(b^2\lambda_1^2 - a^2\lambda_3^2) + \lambda_1^4(a^2 - b^2)]y^2 + (1 + \lambda_1^2)(\lambda_3^2 - \lambda_1^2)a^2b^2 = 0. \tag{23}$$

Figure 5 shows the conic C_3 passing through M, N, R, S, M'.

3.2 Proof of the Double Contact of C_3 with C_1

The tangent line of C_3 at M has the following equation:

$$\left(\frac{\partial F}{\partial x}\right)_M (x - x_M) + \left(\frac{\partial F}{\partial y}\right)_M (y - y_M) = 0 \tag{24}$$

i.e.

$$(\alpha x_M + \beta y_M)(x - x_M) + (\beta x_M + \gamma y_M)(y - y_M) = 0. \tag{25}$$

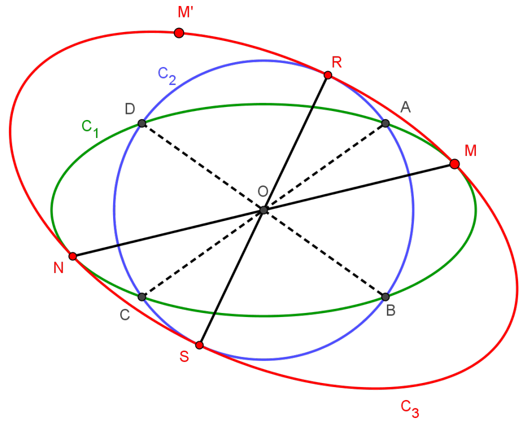

Figure 5: The Double Contact Conic C_3

On the other hand the tangent line of C_1 at M has the following equation:

$$\frac{x_M x}{a^2} + \frac{y_M y}{b^2} = 1. \tag{26}$$

In order for the two lines to coincide we have to show that

$$\begin{vmatrix} \alpha x_M + \beta y_M & \beta x_M + \gamma y_M \\ b^2 x_M & a^2 y_M \end{vmatrix} = 0. \tag{27}$$

According to (11) we must prove equivalently

$$\begin{vmatrix} \alpha + \beta\lambda_3 & \beta + \gamma\lambda_3 \\ b^2 & a^2\lambda_3 \end{vmatrix} = 0 \tag{28}$$

which can be easily verified by substituting α,β, γ through (19), (20), (21). Since (28) depends only on λ_3 and not on point M, it is obvious that C_3 and C_1 have a double contact at M, N.

3.3 Proof of the Double Contact of C_3 with C_2

The tangent line of C_3 at R has the following equation:

$$\left(\frac{\partial F}{\partial x}\right)_R (x - x_R) + \left(\frac{\partial F}{\partial y}\right)_R (y - y_R) = 0 \tag{29}$$

i.e.

$$(\alpha x_R + \beta y_R)(x - x_R) + (\beta x_R + \gamma y_R)(y - y_R) = 0. \tag{30}$$

On the other hand the tangent line of C_2 at R has the following equation:

$$x_R x + y_R y = r^2. \tag{31}$$

In order for the two lines to coincide we have to show that

$$\begin{vmatrix} \alpha x_R + \beta y_R & \beta x_R + \gamma y_R \\ x_R & y_R \end{vmatrix} = 0. \tag{32}$$

According to (12) the above equation is equivalent to

$$\begin{vmatrix} \alpha\lambda_3 + \beta\lambda_1^2 & \beta\lambda_3 + \gamma\lambda_1^2 \\ \lambda_3 & \lambda_1^2 \end{vmatrix} = 0. \tag{33}$$

Dividing by λ_3 all elements of the determinant and using (6), equation (33) turns equivalently to

$$\begin{vmatrix} \alpha + \beta\lambda_4 & \beta + \gamma\lambda_4 \\ 1 & \lambda_4 \end{vmatrix} = 0, \tag{34}$$

which can be easily verified by substituting α, β, γ through (19), (20), (21). Since (34) depends only on λ_4 and not on point R, it is obvious that C_3 and C_2 have a double contact at R, S. Furthermore, in general case that no three of the points M, N, R, S, M' are collinear, C_3 is the unique conic that has double contact with C_1 at M, N and double contact with C_2 at R, S. So, we have proved the following:

Proposition 1. *Let C_1, C_2 be two ellipses with common centre O intersecting at four points A, B, C, D. Let MN, RS be chords of C_1, C_2 respectively passing through O and forming a harmonic pencil with two of the chords of intersection, i.e. $O(A, B, M, R) = -1$. Then, there is a unique conic C_3 passing through M, N, R, S and having double contact with C_1 and C_2 at M, N and R, S, respectively.*

In what follows the above constructed conic C_3 will be called *the double contact conic of C_1, C_2 with respect to MN* or simply *the double contact conic of C_1, C_2*.

Remark 4. Each diameter MN of C_1 corresponds to a unique double contact conic C_3 of C_1, C_2. So, the family of conics having double contact with two intersecting ellipses is a one-parameter family of conics. The parameter of the family is exactly the gradient of diameter MN.

Remark 5. It is known from the Theory of Involution in Projective Geometry that if there exists a pencil of rays with two fixed rays, say OA, OB, as well as a variable pair of corresponding rays, say OM, OR, such that $O(A, B, M, R) = -1$, then there exists a hyperbolic involution with double rays OA, OB in which the variable corresponds. Therefore, when conic C_3 runs through the one-parameter family of the double contact conics of the concentric intersecting ellipses C_1, C_2, then a corresponding hyperbolic involution with centre O is created with double lines the intersection lines AC, BD of C_1, C_2.

4 Characteristic Points of C_3

Constructing conic C_3 we considered point M' as the fifth point of the conic passing through M, N, R, S. Let us now consider point N', the conjugate point of N with respect to RS and C_2. It can be easily verified in analytical way, that conic C_3 passes also through N', since $x_N = -x_M$, $y_N = -y_M$ (s. (15) and (17)). It can also be verified in analytical way, that C_3 passes through R', S', the conjugate points of R, S respectively with respect to MN and C_1 (s. Figure 6).

So, we have proved the following:

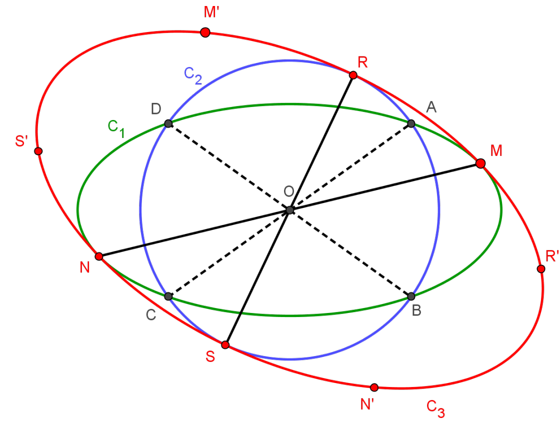

Figure 6: The constructed conic C_3 passes also through N', R', S'

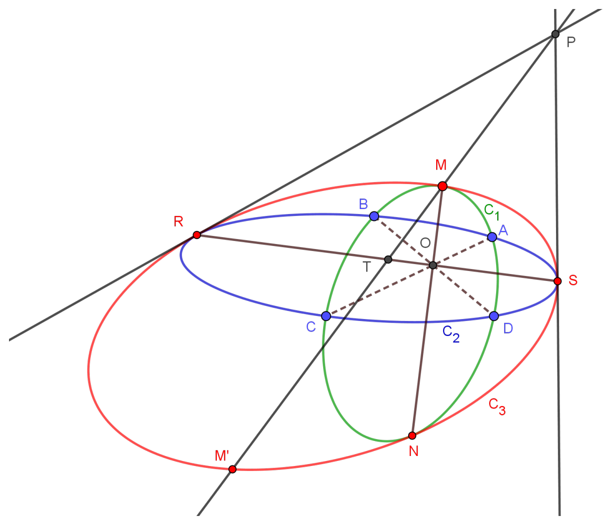

Figure 7: The double contact conic C_3 passes through M'

Proposition 2. *Let C_1, C_2 be two ellipses with common centre O intersecting at four points A, B, C, D. Let MN, RS be chords of C_1, C_2 respectively passing through O and forming a harmonic pencil with two of the chords of intersection, i.e. $O(A,B,M,R) = -1$. Then, the double contact conic C_3 of C_1, C_2 passes through the conjugate points of M, N with respect to RS and C_2 and through the conjugate points of R, S with respect to MN and C_1.*

Remark 6. In the general case, let two conics C_1, C_2 intersect at four points A, B, C, D with diagonal point O being the intersection point of AC,BD, which is not necessarily their centre. Let MN, RS be chords of C_1, C_2 respectively passing through O and forming a harmonic pencil with the chords of intersection AC, BD of C_1, C_2, i.e. $O(A,B,M,R) = -1$ (s. Figure 7). Let P be the pole of RS with respect to C_2 and T be the intersection point of PM and RS. We consider point M' so that M, M' are harmonic conjugate to P, T, i.e. M' is the conjugate point of M with respect to RS and C_2. If there is a conic C_3 passing through M, N, R, S and having double contact with C_2 at R, S, then RS is the polar of P with respect to C_2, but also the polar of P with respect to C_3. Considering PM as an intersecting line through P, its point of intersection T with RS is the conjugate point of P with respect to the intersection points M, M' of line PM with C_3. So, C_3 passes through M'. Similarly, C_3 passes through N', which is the conjugate point of N with respect to RS

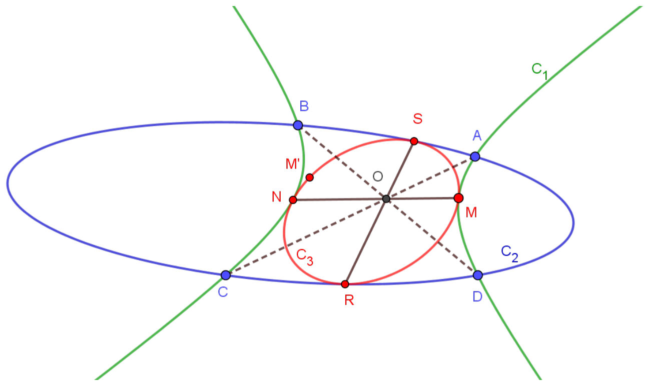

Figure 8: Conic C_3 in case C_1 is a hyperbola and C_2 an ellipse

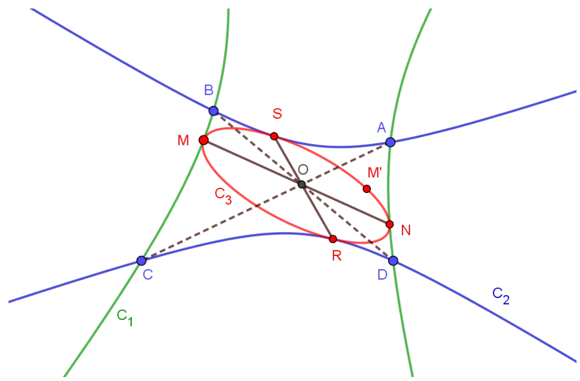

Figure 9: Conic C_3 in case C_1, C_2 are both hyperbolas

and C_2, and also through R', S', which are the conjugate points of R, S with respect to MN and C_1 respectively.

Figures 7, 8 and 9 show the double contact conic C_3 in the following cases respectively: C_1, C_2 are both ellipses, C_1 is a hyperbola and C_2 is an ellipse and C_1, C_2 are both hyperbolas.

Remark 7. Let now O be another diagonal point of the complete quadrangle defined by A, B, C, D, say the intersection point of AB, CD (s. Figure 10). Let MN, RS be chords of C_1, C_2 respectively and lines MN, RS pass through O and form a harmonic pencil with the lines of the intersection chords AB, CD of C_1, C_2, i.e. $O(A, C, M, R) = -1$. Using the same method we can construct the double contact conic C_3 passing through M, N, R, S and having double contact with C_1 and C_2 at M, N and R, S respectively. C_3 passes through M', N', the conjugate points of M, N with respect to RS and C_2, and also through R', S', the conjugate points of R, S with respect to MN and C_1 respectively.

Regarding this case, i.e. O being the intersecting point of AB, CD, Figures 10, 11 and 12 show the double contact conic C_3 in the following cases respectively: C_1, C_2 are both ellipses, C_1 is a hyperbola and C_2 is an ellipse and C_1, C_2 are both hyperbolas.

5 Type of Conic C_3

Let C_1, C_2 be the conics with equations (3), (4) respectively. We consider now the common tangent lines ε_1, ε_2, ε_3, ε_4 of C_1, C_2 (s. Figure 13). Let E_i^1, E_i^2, $i = 1, 2, 3, 4$ be the contact

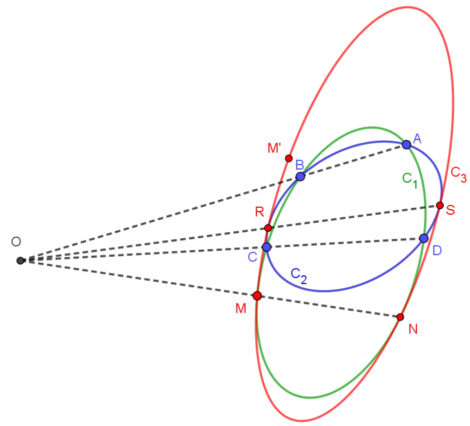

Figure 10: Conic C_3 in case C_1, C_2 are both ellipses and O is the intersection point of AB, CD

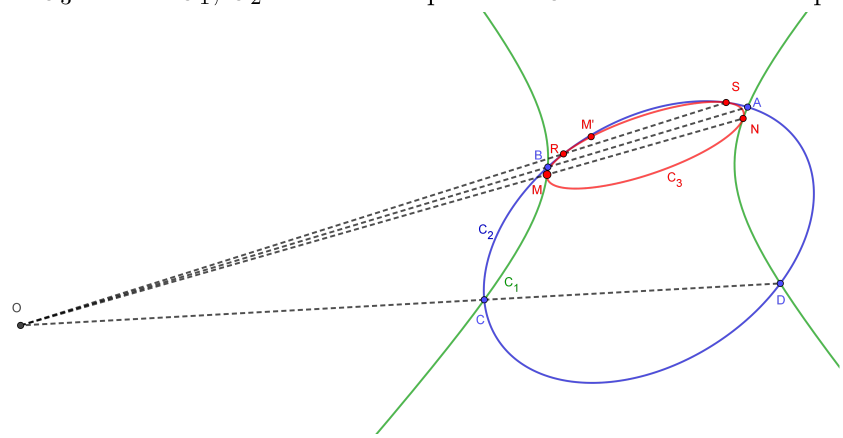

Figure 11: Conic C_3 in case C_1 is a hyperbola, C_2 is an ellipse and O is the intersection point of AB, CD

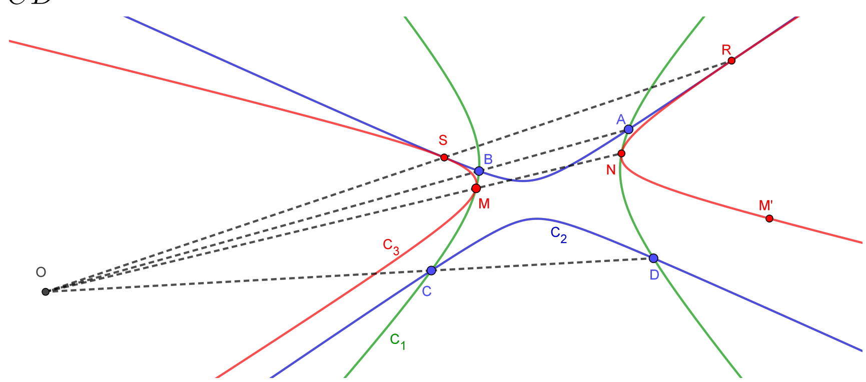

Figure 12: Conic C_3 in case C_1, C_2 are both hyperbolas and O is the intersection point of AB, CD

points of line ε_i and C_1, C_2 respectively. Since ε_1 is the tangent line of C_1 at E_1^1 it holds

$$\varepsilon_1\colon \frac{xx_{E_1^1}}{a^2} + \frac{yy_{E_1^1}}{b^2} = 1 \tag{35}$$

and

$$b^2x_{E_1^1}^2 + a^2y_{E_1^1}^2 = a^2b^2. \tag{36}$$

But ε_1 is also a tangent line of C_2. So, the distance between O and ε_1 is equal to r. Therefore,

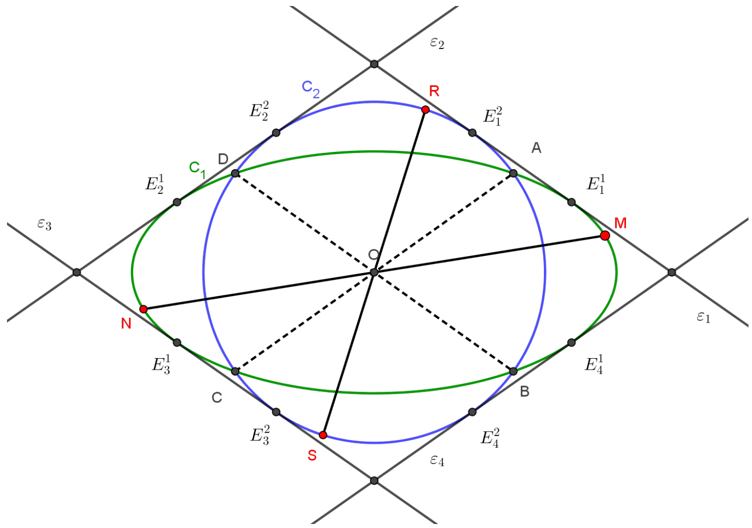

Figure 13: The common tangent lines of conics C_1, C_2

it holds

$$\frac{a^2b^2}{\sqrt{b^4x^2_{E_1^1}+a^4y^2_{E_1^1}}}=r. \tag{37}$$

According to (8), (36) and (37) we get

$$x_{E_1^1}=\frac{a}{\sqrt{1+\lambda_1^2}}, \qquad y_{E_1^1}=\frac{b\lambda_1}{\sqrt{1+\lambda_1^2}}. \tag{38}$$

Then, the gradient of line OE_1^1 is equal to $\frac{y_{E_1^1}}{x_{E_1^1}}=\frac{b\lambda_1}{a}$.

Since MN is an arbitrary diameter of C_1, it is expected that the choice of MN effects the type of the double contact conic C_3 of C_1, C_2. We will prove the following (s. Figure 14):

- If M is a point inside the elliptic arc $E_4^1E_1^1$ or $E_2^1E_3^1$, i.e. $|\lambda_3|<\frac{b|\lambda_1|}{a}$, then conic C_3 is an ellipse.
- If M is a point outside circle C_2 and outside the elliptic arcs $E_4^1E_1^1$ and $E_2^1E_3^1$, i.e. $\frac{b|\lambda_1|}{a}<|\lambda_3|<|\lambda_1|$, then conic C_3 is a hyperbola.
- If M is a point inside the circle C_2, i.e. $|\lambda_1|<|\lambda_3|$, then conic C_3 is an ellipse.
- If M coincides to E_1^1 or E_3^1, i.e. $\lambda_3=\frac{b\lambda_1}{a}$, then conic C_3 degenerates to two parallel lines ε_1, ε_3.
- If M coincides to E_2^1 or E_4^1, i.e. $\lambda_3=-\frac{b\lambda_1}{a}$, then conic C_3 degenerates to two parallel lines ε_2, ε_4.
- If M coincides to A, C (resp. B, D), i.e. $\lambda_3=\lambda_1$ (resp. $\lambda_3=-\lambda_1$), then segments MN, RS coincide with AC (resp. BD), and conic C_3 degenerates to the double line AC (resp. BD).

So, the following holds:

Proposition 3. *Let C_1, C_2 be two ellipses with common centre O intersecting at four points A, B, C, D. Let MN, RS be chords of C_1, C_2 respectively passing through O and forming a harmonic pencil with two of the chords of intersection, i.e. $O(A,B,M,R)=-1$. The choice of the chord MN determines the type of the double contact conic C_3 of C_1, C_2 in the following way:*

If λ_1,λ_3 are the gradients of AB, MN respectively, then

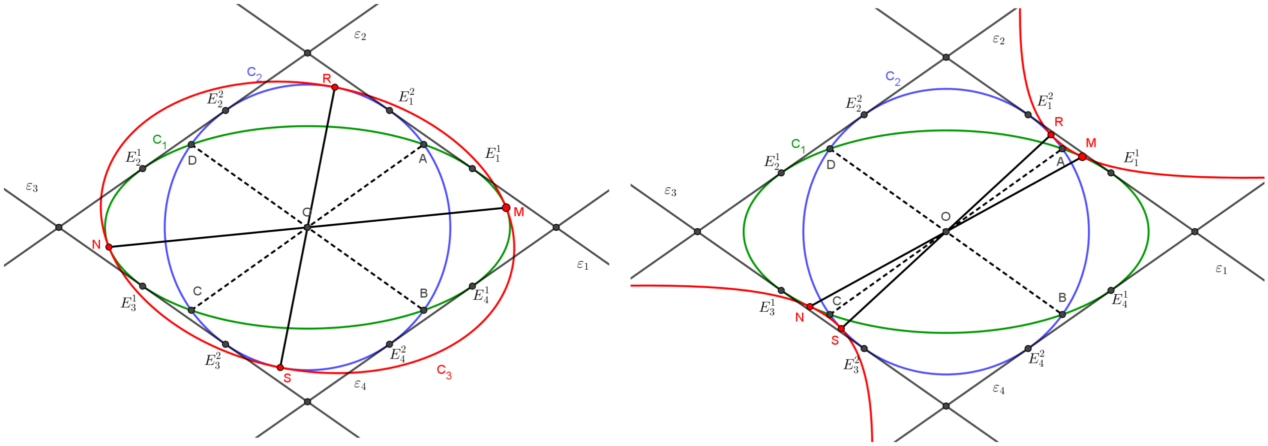

Figure 14: The choice of MN determines the type of C_3

- *C_3 is an ellipse, if $|\lambda_3| < \frac{b|\lambda_1|}{a}$ or $|\lambda_3| > |\lambda_1|$,*
- *C_3 is a hyperbola, if $\frac{b|\lambda_1|}{a} < |\lambda_3| < |\lambda_1|$ and*
- *C_3 is a degenerate parabola (i.e. a pair of parallel lines or a double line) in all other cases.*

Proof. The equation (18) of the conic C_3 can be written in matrix notation as

$$C_3\colon \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \delta = 0 \tag{39}$$

or in homogeneous form as

$$C_3\colon \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0. \tag{40}$$

It is known that the type of the conic (40) is determined by the invariants $I = \alpha + \gamma$, $J = \alpha\gamma - \beta^2$ and $\Delta = \delta J$ (s. [1, p. 362]).[3] According to (19), (20), (21), (22) we get

$$I = (\lambda_1^2 - \lambda_3^2)(b^2 + a^2\lambda_1^2) + (1 + \lambda_1^2)(b^2\lambda_1^2 - a^2\lambda_3^2) \tag{41}$$
$$J = (1 + \lambda_1^2)(b^2 + a^2\lambda_1^2)(\lambda_1^2 - \lambda_3^2)(b^2\lambda_1^2 - a^2\lambda_3^2) \tag{42}$$
$$\Delta = -(1 + \lambda_1^2)(\lambda_1^2 - \lambda_3^2)a^2b^2J \tag{43}$$

So, it holds (s. Figure 14):

- $J > 0$ if and only if $|\lambda_3| < \frac{b|\lambda_1|}{a}$ or $|\lambda_3| > |\lambda_1|$. In each case it holds $I\Delta < 0$ and so C_3 *is an ellipse.*
- $J < 0$ if and only if $(\lambda_1^2 - \lambda_3^2)(b^2\lambda_1^2 - a^2\lambda_3^2) < 0$, i.e. if $\frac{b|\lambda_1|}{a} < |\lambda_3| < |\lambda_1|$. In this case it holds $\Delta > 0$ and so C_3 *is a hyperbola.*
- $J = 0$ if and only if $|\lambda_3| = \frac{b|\lambda_1|}{a}$ or $|\lambda_3| = |\lambda_1|$. In this case it holds $\Delta = 0$ and so C_3 is never a parabola. Now C_3 is a degerate conic and the invariant δI will determine the type of C_3:

[3] I and J are the trace and the determinant of matrix $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ and Δ is the determinant of matrix $\begin{pmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & \delta \end{pmatrix}$. I, J, Δ are invariants under arbitrary rotations and translations of the coordinate axes.

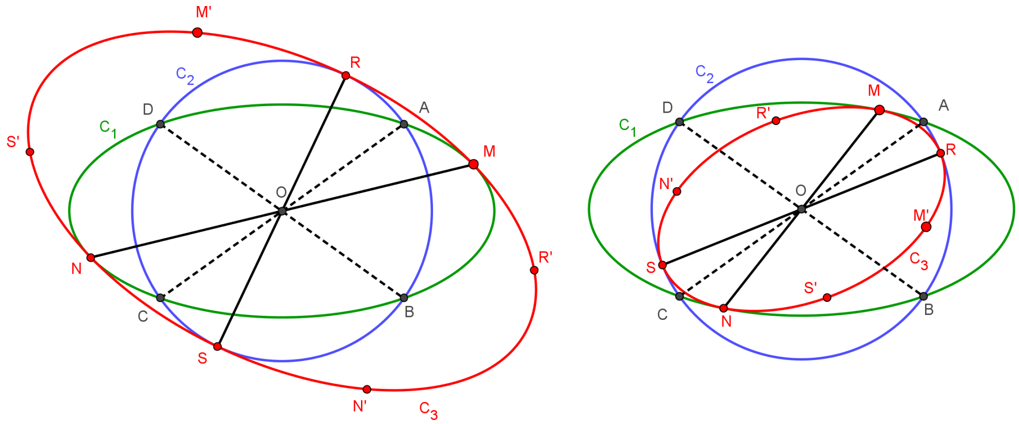

Figure 15: The constructed conic C_3 is either circumscribed or inscribed to C_1, C_2

- If $|\lambda_3| = \frac{b|\lambda_1|}{a}$, then $\delta I < 0$, so C_3 *degenerates to a pair of parallel lines.* In particular:
  - If $\lambda_3 = \frac{b\lambda_1}{a}$ the equation (23) of C_3 turns to

$$C_3\colon bx + a\lambda_1 y = \pm ab\sqrt{1+\lambda_1^2}. \tag{44}$$

    i.e. C_3 degenerates to the common tangent lines ε_1, ε_3 of C_1, C_2.
  - If $\lambda_3 = -\frac{b\lambda_1}{a}$ the equation (23) of C_3 turns to

$$C_3 : bx - a\lambda_1 y = \pm ab\sqrt{1+\lambda_1^2} \tag{45}$$

    i.e. C_3 degenerates to the common tangent lines ε_2, ε_4 of C_1, C_2 (s. Figure 13).
- If $|\lambda_3| = |\lambda_1|$, then $\delta I = 0$, so C_3 *is a double line.* In particular:
  - If $\lambda_3 = \lambda_1$ the equation (23) of C_3 turns to $C_3\colon y = \lambda_1 x$ i.e. C_3 degenerates to the double line AC.
  - If $\lambda_3 = -\lambda_1$ the equation (23) of C_3 turns to $C_3 : y = -\lambda_1 x$ i.e. C_3 degenerates to the double line BD (s. Figure 13).

We notice that in case $J = 0$, C_3 degenerates to a pair of parallel lines or to a double line. So, C_3 is a degenerate parabola. □

We can also verify the following:

- If $|\lambda_3| < |\lambda_1|$, then point M lies outside C_2 and the above constructed conic C_3 is circumscribed to C_1, C_2.
- If $|\lambda_3| > |\lambda_1|$, then point M lies inside C_2 and the above constructed conic C_3 is inscribed to C_1, C_2 (s. Figure 15).
- If $|\lambda_3| = |\lambda_1|$, then conic C_3 degenerates to a double line.

So, the following holds:

Proposition 4. *Let C_1, C_2 be two ellipses with common centre O intersecting at four points A, B, C, D. Let MN, RS be chords of C_1, C_2 respectively passing through O and forming a harmonic pencil with two of the chords of intersection, i.e. $O(A,B,M,R) = -1$. Then, the double contact conic C_3 of C_1, C_2 is circumscribed (resp. inscribed) to C_1, C_2 in case M lies outside (resp. inside) C_2 and degenerates to a double line in case M lies on C_2.*

Figure 16 shows the one-parameter family of conics having double contact with two concentric intersecting ellipses.

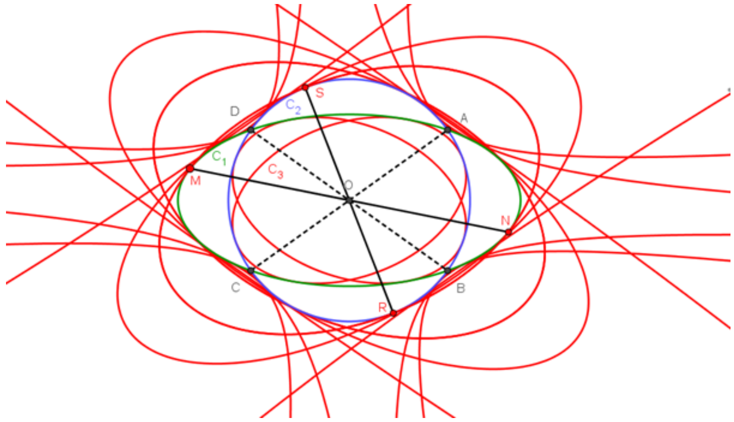

Figure 16: The family of conics having double contact with two intersecting ellipses

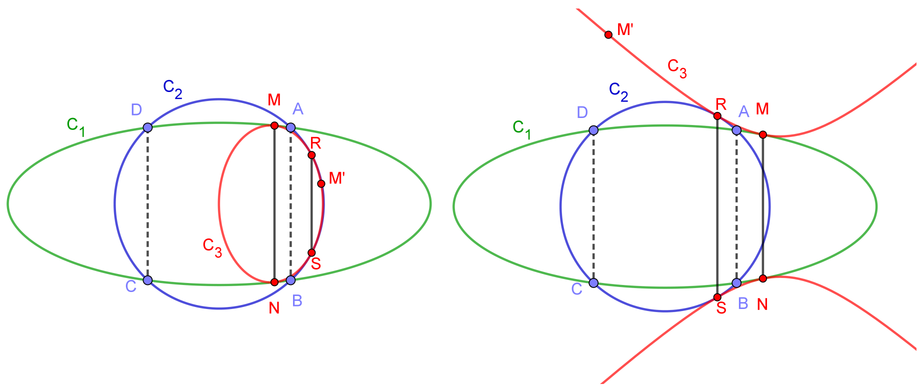

Figure 17: Conic C_3 in case O is the intersection point of AB, CD

Remark 8. Let now O be the intersection point of AB, CD, so O lies at infinity (s. Figure 17). Let MN, RS be chords of C_1, C_2 respectively and lines MN, RS pass through O and form a harmonic pencil with the lines of the intersection chords AB, CD of C_1, C_2, i.e. $O(A, C, M, R) = -1$. Using the conjugate point of M' of M with respect to RS and C_2, we can construct the double contact conic C_3 passing through M, N, R, S and having double contact with C_1 and C_2 at M, N and R, S respectively. The choice of the chord MN determines again the type of the double contact conic C_3 of C_1, C_2. Figure 17 shows the double contact conic C_3 inscribed or circumscribed to C_1, C_2 in case O lies at infinity.

6 Canonical Form of the Equation of C_3

We consider matrix $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ of the equation (39) of conic C_3. Let e_1, e_2 be the eigenvalues of the matrix. It is known that the invariants I and J satisfy the equations $I = e_1 + e_2$ and $J = e_1 e_2$. Then, according to (41), (42) it can be easily verified that the eigenvalues of the matrix are

$$e_1 = (\lambda_1^2 - \lambda_3^2)(b^2 + a^2\lambda_1^2), \qquad e_2 = (1 + \lambda_1^2)(b^2\lambda_1^2 - a^2\lambda_3^2) \tag{46}$$

with corresponding eigenvectors

$$v_1 = \begin{pmatrix} \lambda_3 \\ \lambda_1^2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -\lambda_1^2 \\ \lambda_3 \end{pmatrix} \tag{47}$$

i.e.

$$\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\begin{pmatrix}\lambda_3\\ \lambda_1^2\end{pmatrix} = e_1\begin{pmatrix}\lambda_3\\ \lambda_1^2\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\begin{pmatrix}-\lambda_1^2\\ \lambda_3\end{pmatrix} = e_2\begin{pmatrix}-\lambda_1^2\\ \lambda_3\end{pmatrix}.$$

Remark 9. Eigenvector v_1 is parallel to line RS, since its gradient is equal to λ_4 (s. (6)). Eigenvector v_2 is vertical to v_1, since the above matrix is symmetric.

Hence, rotating the given coordinate axes through the origin O so that the new $\tilde{x}'\tilde{x}$ axis is parallel to eigenvector v_1 i.e. parallel to RS, the equation (39) of C_3 (s. (39)) turns to

$$e_1\tilde{x}^2 + e_2\tilde{y}^2 = -\delta \tag{48}$$

or equivalently

$$e_1\tilde{x}^2 + e_2\tilde{y}^2 = (\lambda_1^2 - \lambda_3^2)(1+\lambda_1^2)a^2b^2. \tag{49}$$

Since $e_1(-\delta) \geq 0$, we give the following cases:

- If $e_1 < 0$, $e_2 < 0$ i.e. $|\lambda_3| > |\lambda_1|$, then C_3 is an ellipse inscribed to C_1, C_2.
- If $e_1 > 0$, $e_2 > 0$ i.e. $|\lambda_3| < \frac{b|\lambda_1|}{a}$, then C_3 is an ellipse circumscribed to C_1, C_2.
- If $e_1 > 0$, $e_2 < 0$ i.e. $\frac{b|\lambda_1|}{a} < |\lambda_3| < |\lambda_1|$, then C_3 is a hyperbola circumscribed to C_1, C_2.
- If $e_1 = 0$ i.e. $|\lambda_3| = |\lambda_1|$, then C_3 degenerates to a double line, in particular to line RS.
- If $e_2 = 0$ i.e. $|\lambda_3| = \dfrac{b|\lambda_1|}{a}$, then C_3 degenerates to two common tangent lines of C_1, C_2 vertical to RS.

In case $|\lambda_3| \neq |\lambda_1|$, we get the canonical form of the equation of C_3 (s. (8)):

$$\frac{\tilde{x}^2}{r^2} + \frac{(b^2\lambda_1^2 - a^2\lambda_3^2)}{a^2b^2(\lambda_1^2 - \lambda_3^2)}\tilde{y}^2 = 1. \tag{50}$$

Equation (50) states a well known result: *If C_3 is an ellipse inscribed to the circle C_2, then its major axis is equal to the diameter $2r$ of C_2. If C_3 is an ellipse (resp. hyperbola) circumscribed to C_2, then its minor (resp. major) axis is equal to the diameter $2r$ of C_2. If C_3 degenerates to two tangent lines, then the distance between the lines is equal to the diameter $2r$ of C_2.* So, the following holds:

Proposition 5. *Let C_1 and C_2 be an ellipse and a circle respectively with common centre O intersecting at four points A, B, C, D. Let MN, RS be chords of C_1, C_2 respectively passing through O and forming a harmonic pencil with two of the chords of intersection, i.e. $O(A,B,M,R) = -1$. Then, the diameter RS of C_2 is one of the axes of the double contact conic C_3 of C_1, C_2, in case C_3 is non-degenerate.*

7 Double Contact Conics in Couples

Let C_1, C_2 be two ellipses with common centre O intersecting at four points A, B, C, D. Let MN, RS be chords of C_1, C_2 respectively passing through O and forming a harmonic pencil with two of the chords of intersection, i.e. $O(A,B,M,R) = -1$. We proved that there is a unique conic C_3 passing through M, N, R, S and having double contact with C_1 and C_2 at M, N and R, S, respectively.

Let now R_1, S_1 be the intersection points of MN and C_2 and M_1, N_1 be the intersection points of RS and C_1. Then, M_1N_1, R_1S_1 also form a harmonic pencil with the chords of intersection AC, BD. So, there is a unique conic, say C_3', passing through M_1, N_1, R_1, S_1

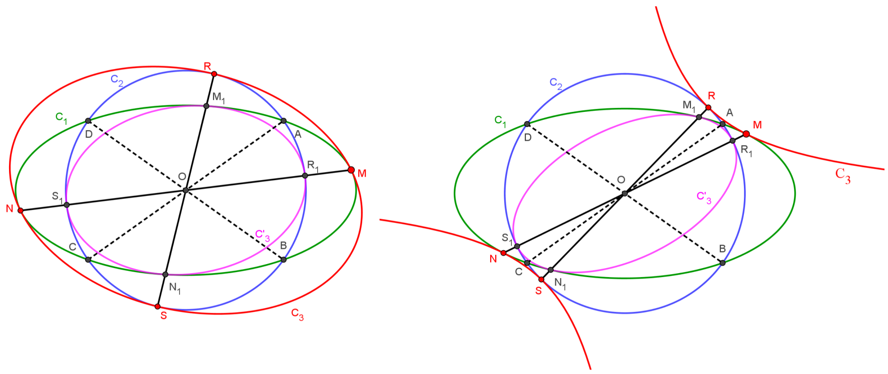

Figure 18: Couple of double contact conics C_3, C_3'

and having double contact with C_1 and C_2 at R_1, S_1 and M_1, N_1 respectively. Figure 18 shows the couple of conics C_3, C_3'.

Obviously, the equation of C_3' is obtained by the equation (23) of C_3 substituting λ_3 by λ_4 (s. (6)). So, C_3' belongs to the one-parameter family of conics (23) too. In the general case one of the two conics C_3, C_3' is an ellipse inscribed to C_1, C_2 and the other one is an ellipse (resp. hyperbola) circumscribed to C_1, C_2. The major axis of the inscribed ellipse is harmonic conjugate to the minor (resp. major) axis of the circumscribed one with respect to the two chords of intersection. So, the following holds:

Proposition 6. *Let C_1, C_2 be two ellipses with common centre O intersecting at four points A, B, C, D. Every chord MN of C_1 passing through O corresponds to two double contact conics C_3, C_3' of C_1, C_2. In the general case one of the two conics is an ellipse inscribed to C_1, C_2 and the other one is an ellipse (resp. hyperbola) circumscribed to C_1, C_2. The major axis of the inscribed ellipse and the minor (resp. major) axis of the circumscribed one with the two chords of intersection are in hyperbolic involution with double lines the lines of the chords of intersection of C_1, C_2.*

8 Common Tangent Lines of C_1, C_2

In case $|\lambda_3| = \frac{b|\lambda_1|}{a}$ conic C_3 degenerates to the common tangent lines ε_1, ε_3 or ε_2, ε_4 of C_1, C_2. In the following we will give another construction of the common tangent lines of C_1, C_2 using a parallel homology. The general solution of the construction of the common tangents of two conics using methods of Projective Geometry is given in [5, p. 226–229].

We consider an ellipse C_1 and a concentric circle C_2 having four intersection points A, B, C, D. We will define a parallel homology that maps circle C_2 to ellipse C_1 in the following way:

We take one of the common diameters of C_1, C_2, say AC. Let EF be the diameter of C_2, which is perpendicular to AC and GH the diameter of C_1, which is conjugate to AC (s. Figure 19). We define a parallel homology with axis of homology AC, which maps point E to point G (resp. to point H). So, line EG (resp. EH) defines the direction of the parallel homology.

Since perpendicular diameters AC, EF of C_2 correspond to conjugate diameters AC, GH of C_1, then C_1, C_2 are in parallel homology. Let now KL be the diameter of C_2 vertical to EG (resp. EH) and ε_2, ε_4 (resp. ε_1, ε_3) the lines from K, L parallel to EG (resp. EH). Then lines ε_2, ε_4 (resp. ε_1, ε_3) are tangent lines of C_2, parallel to the direction of the homology.

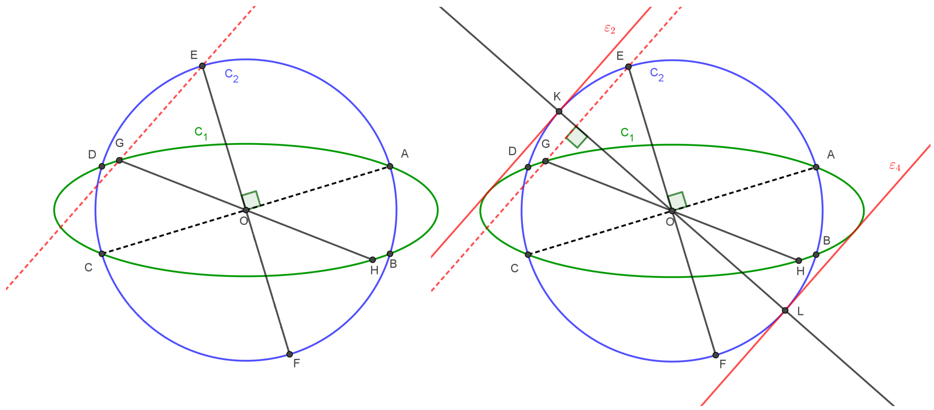

Figure 19: Construction of common tangents of C_1, C_2 using an homology

Therefore they are tangent lines to C_1, too. So, ε_2, ε_4 (resp. ε_1, ε_3) are common tangent lines of C_1, C_2. Hence the following holds:

Proposition 7. *Let C_1, C_2 be two ellipses with common centre O intersecting at four points A, B, C, D. Let GH, EF be the diameters of C_1, C_2 respectively conjugate to a common diameter of C_1, C_2, say AC. Every line which joins one end point of GH with one end point of EF defines the gradient of one couple of common tangent lines of C_1, C_2 and the line that joins the remaining end points of GH and EF defines the gradient of the other couple of common tangent lines of C_1, C_2.*

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