Journal for Geometry and Graphics
Volume 27 (2023), No. 2, 195–206

Analytic Solution of Castillon's Problem, Another Approach

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Abstract. We solve Castillon's problem using inversion in complex circle. This approach leads to an analytic solution to the problem similar but distinct from the standard solution due to Lagrange a Carnot. While the standard solution parametrizes the given circle by stereographic projection, the proposed approach makes use of circular inversions. Though the two approaches appear fairly different, each of them leads finally to a group of Möbius transformations, in one case over real and in the other over complex numbers.

Key Words: Castillon's problem, inversion in circle, complex circle

MSC 2020: 51M04 (primary), 51N10

1 Introduction

The well-known problem of classical geometry is formulated as follows [6]:

Castillon's Problem. Given a circle in the real plane and distinct points p_1, p_2, …, p_n not on the circle, inscribe a polygon $a_1 a_2 \dots a_n$ in the circle so that the side $a_i a_{i+1}$ ($a_{n+1} = a_1$) passes through the point p_i.

Castillon's problem, also known as Cramer-Castillon, is considered difficult, although it has already been solved. A nice description of the history of the problem and its various solutions can be found, for example, in [1].

Already in antiquity, for $n = 3$, the problem was dealt with by Pappus, but it was named after other mathematicians: Cramer, who presented it to Castillon. The latter solved the problem by means of synthetic geometry. Another solution, also within the realm of synthetic geometry, can be found, for example, in [6].

Of interest to us is the analytical solution first presented for $n = 3$ by Lagrange and then modified and generalized for arbitrary n by Carnot [3, par. 330, Problem XLV on p. 383]. However, we will approach the problem not through the standard projective plane, but rather through the lens of spherical geometry.

ISSN 1433-8157/

The main tool for finding a solution will be the inversion in circle. We will show that the inversion can be defined not only for the case of a real one but also for a complex circle. We will then use such a generalized inversion for our solution. Restriction of the inversion to the set relevant to our problem coincides with a Möbius transformation. This allows us to conveniently compose several such mappings. As a result, we can easily compute the solution as a fixed point of a composed transformation.

Finally, we briefly compare our solution with the standard analytical solution by Carnot.

2 Preliminaries

2.1 Inversion in Circle

The map is determined by a given fixed circle and is defined for all points except the center of the circle.

Definition 2.1. Let $\mathcal{C}$ be a circle in $\mathbb{R}^2$ with the center p and radius r and let $q \in \mathbb{R}^2$ be a point distant from p. The image of q in the inversion in the circle $\mathcal{C}$ is such a point q' that

- q' is on $\overleftrightarrow{pq}$,
- $(q-p) \cdot (q'-p) = r^2$.

The inversion is obviously an involution. It is a bijective transformation of $\mathbb{R}^2 - \{p\}$. From the definition we straightforwardly derive an analytic formula for the inversion:

$$q' = p + \left(\frac{r}{|q-p|}\right)^2 (q-p). \tag{1}$$

Let us recall some properties of circular inversion important for our work. We are considering the inversion in the circle $\mathcal{C}$ centered at p.

- Fixed points are points of the circle $\mathcal{C}$.
- A straight line passing through the center p of $\mathcal{C}$ is invariant.
- The image of a straight line not passing through the center of $\mathcal{C}$ is a circle passing through p, except the point p. If the straight line intersects $\mathcal{C}$, its image also contains the intersection points.
- A circle not passing through p is mapped to a circle.
- A circle orthogonal to the circle $\mathcal{C}$ is invariant.

A more detailed description of these and other properties can be found in the literature [4, 5, 8, 9].

2.2 Outline of the Solution

Let the points p_1, p_2, p_3 do not lie on the circle $\mathcal{K}$ and let $\triangle a_1a_2a_3$ be a solution, i.e.:

- $a_1, a_2, a_3 \in \mathcal{K}$,
- $p_i \in \overleftrightarrow{a_i a_{i+1}}$.

If $\mathcal{C}_1$ is a circle orthogonal to $\mathcal{K}$, then inversion by the circle $\mathcal{C}_1$ leaves the circle $\mathcal{K}$ invariant. Moreover, if the center of $\mathcal{C}_1$ lies on the line $\overleftrightarrow{a_1a_2}$, then the image of the point a_1 in the inversion by $\mathcal{C}_1$ is necessarily the point a_2.

Similarly, if $\mathcal{C}_2$ resp. $\mathcal{C}_3$ are orthogonal to $\mathcal{K}$ and their centers are on $\overleftrightarrow{a_2a_3}$ resp. $\overleftrightarrow{a_3a_1}$, inversion by $\mathcal{C}_2$ maps a_2 to a_3 and inversion by $\mathcal{C}_3$ maps a_3 to a_1. The composition $f_3 \circ f_2 \circ f_1$ (f_i is the inversion by $\mathcal{C}_i$) then maps the point a_1 to itself.

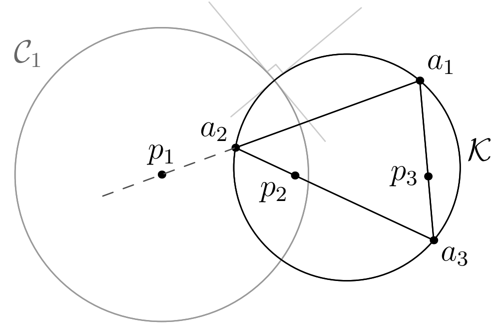

Figure 1: A solution $a_1a_2a_3$ of the problem given by the points p_1, p_2, p_3.

Let us modify the Castillon problem by adding the condition, that the points p_1, p_2 and p_3 lie on the *outside* of the circle $\mathcal{K}$. Then obviously the properties we require from the circle $\mathcal{C}_1$ can be satisfied, for example, by a circle centered at point p_1 with an appropriate radius. The same applies to the remaining two circles.

So the problem is solved in case the points p_1, p_2 a p_3 lie on the outside of the given circle $\mathcal{K}$:

1. Let $\mathcal{C}_i$ be the circle centered in p_i and orthogonal to the circle $\mathcal{K}$.
2. The point a_1 is found as a fixed point of the map $f_3 \circ f_2 \circ f_1$ (f_i being the inversion by $\mathcal{C}_i$).

3 Solving the Problem

3.1 Inner Product in Complex Plane

The proposed procedure for solving the problem works fine provided that the points p_1, p_2, and p_3 lie outside the given circle. Naturally one would like to solve it also in case when one or more of the given points are inside the circle. However, when we try to apply our method to a point inside $\mathcal{K}$, we need to construct a circle centered at this interior point that is orthogonal to $\mathcal{K}$. Unfortunately, such a circle does not exist in the real plane. Nevertheless, we can try to carry out the computations in this case anyway:

Example 1. Given the unit circle $\mathcal{K}\colon x^2 + y^2 - 1$ and a point $p = (2/3, 1/3)$ inside the circle, we want to find a circle $\mathcal{C}$ centered in p and orthogonal to $\mathcal{K}$.

First, we find a point of intersection $q = \mathcal{K} \cap \mathcal{C}$. As we want the circles to be orthogonal, it must hold

$$(q - p) \perp (q - O),$$

where $O = (0, 0)$. If we denote the coordinates of q by (q_x, q_y), this leads to the system of equations

$$\begin{aligned} q_x(q_x - 2/3) + q_y(q_y - 1/3) &= 0 \\ q_x^2 + q_y^2 &= 1 \end{aligned} \tag{2}$$

with no real and two complex conjugated solutions. Out of two such points we choose e.g.

$$q = \left(\frac{6-2i}{5}, \frac{3+4i}{5}\right).$$

The square of the radius of $\mathcal{C}$ (i.e. the square of the distance of q and p) is $-4/9$. So the circle centered in p and of the radius $\frac{2}{3}i$ is orthogonal to $\mathcal{K}$.

We see that we could try to solve Castillon's problem even for points given inside the circle. To do so, we need to use a circle with a real center and a radius such that its square is a negative number. This leads to the definition of a symmetric complex inner product and a complex circle.

Definition 3.1. *Symmetric complex inner product* is such a map $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$, $(u, v) \mapsto u \cdot v$ that

1. $u \cdot v = v \cdot u$
2. $(u + w) \cdot v = (u \cdot v) + (w \cdot v)$,
   $(ku) \cdot v = k(u \cdot v)$, where $k \in \mathbb{C}$
3. $u \cdot u > 0$ for $u \in \mathbb{R}^2$ and $u \neq 0$.

Vectors u and v are *orthogonal*, if $u \cdot v = 0$. A nonzero vector u is *isotropic*, if $u \cdot u = 0$.

If $u = (u_x, u_y), v = (v_x, v_y) \in \mathbb{C}$, then clearly

$$u \cdot v = u_x v_x + u_y v_y \tag{3}$$

is a symmetric complex inner product. Observe, that a symmetric complex inner product as opposed to the Hermitian product is not positive definite: the vector $u = (i, 1)$ is isotropic and for $v = (i, 0)$ it holds $v \cdot v = -1$.

On the other hand, given any symmetric complex inner product there is a basis of the vector space $\mathbb{C}^2$ such that the inner product is computed as in (3): such basis can be obtained by the orthogonalization process and subsequent normalization, provided the given basis does not contain any isotropic vector.

Since now we assume our basis to be orthonormal.

Remark 1. Since we consider a symmetric scalar product instead of the standard Hermitian one in the complex plane (which refers to the plane of points whose coordinates are complex numbers), the computation of orthogonal complements leads to solving linear equations and, therefore, to standard linear algebra. So the orthogonal complement of a nonzero vector in $\mathbb{C}^2$ is a one-dimensional vector subspace.

We now briefly review and verify notions familiar from the real plane and extend to the plane over the complex numbers.

Let l be the line in $\mathbb{C}^2$ defined by

$$ax + by + c = 0, \quad a, b, c \in \mathbb{C}, \ (a, b) \neq (0, 0), \tag{4}$$

i.e. the set of points satisfying (4). Analogously let us consider the line l' given by $a'x + b'y + c' = 0$.

- The lines l and l' are *parallel*, if $ab' - ba' = 0$. Hence, two lines are parallel, if they coincide or they have no point in common.
- Lines l and l' are *orthogonal*, if $(a, b) \cdot (a', b') = 0$. As there are isotropic vectors in $\mathbb{C}^2$, a line can be orthogonal to itself, e.g. the line given by $x + iy = 0$.
- From Remark 1 it follows that for a line l and a point p there exists the unique line passing through p and orthogonal to l, and also the unique line passing through p and parallel to l. Sometimes these two lines coincide – it happens when l is orthogonal to itself.

3.2 Inversion in a Complex Circle

We wish to work with inversion in circles that do not pass through any real point, and contain only complex points. Circles of complex points are defined as follows:

Definition 3.2. *A complex circle* in $\mathbb{C}^2$ centered at a point $c = (c_x, c_y) \in \mathbb{R}^2$ (a real point) is the set of points in $\mathbb{C}^2$ satisfying the polynomial equation

$$(x - c_x)^2 + (y - c_y)^2 + v^2 = 0,$$

where $v \in \mathbb{R}^+$. The purely imaginary number $r = iv$ is *the radius* of the complex circle.

- Let $\mathcal{C}$ be a circle or a complex circle and let $p \in \mathcal{C}$. *The tangent line* to $\mathcal{C}$ at p is the unique line t such that p is the unique point both on $\mathcal{C}$ and t.
- Let $\mathcal{C}$ be a complex circle, let $\mathcal{K}$ be a real circle and let $p \in \mathcal{C} \cap \mathcal{K}$. Then $\mathcal{C}$ and $\mathcal{K}$ are *orthogonal*, if the tangent lines to $\mathcal{C}$ and $\mathcal{K}$ at p are orthogonal.

Remark 2. If p is a point on a complex circle $\mathcal{C}$ with the center c, then the tangent line to $\mathcal{C}$ at p is orthogonal to the line $\overleftrightarrow{cp}$.

Hence a circle $\mathcal{K}$ centered at k and a complex circle $\mathcal{C}$ centered at c are orthogonal if and only if $\overleftrightarrow{cp}$ and $\overleftrightarrow{kp}$ are orthogonal, where $p \in \mathcal{C} \cap \mathcal{K}$.

In the following we will use the inversion also in complex circle. Fortunately, Definition 2.1 of the inversion stated in the beginning works well also for complex circles. For the sake of completeness, we rephrase it:

Definition 3.3. The image of the point $q \in \mathbb{C}^2$ in inversion in a (real or complex) circle $\mathcal{C}$ centered in $p \in \mathbb{R}^2$ $(q \neq p)$ with a radius r is such a point $q' \in \mathbb{C}^2$ that

- q' lies on the line $\overleftrightarrow{pq}$,
- $(q - p) \cdot (q' - p) = r^2$.

The condition $(q - p) \cdot (q' - p) = r^2$ at the end of the definition is a linear equation for the coordinates of point q', thus determining a line. This line and the line $\overleftrightarrow{pq}$ are not parallel. It follows that the point q' is uniquely determined by this definition.

Example 2. Let $\mathcal{C}$ be the complex circle given by

$$x^2 + y^2 + 1.$$

Let $q = (q_x, q_y) \neq (0, 0)$ be any point in the plane. Its image in inversion in $\mathcal{C}$ is the point q' on the line through q and O such that $(q - O) \cdot (q' - O) = -1$, the square of the radius of $\mathcal{C}$. So we get

$$q' = \left(-\frac{q_x}{q_x^2 + q_y^2}, -\frac{q_y}{q_x^2 + q_y^2} \right). \tag{5}$$

Any circle centered in a real point with the square of its radius being a negative real number is an image of $\mathcal{C}$ from Example 2 in an affine transformation over the real numbers (shifting by a real vector and scaling by a positive real number). Therefore, this example sufficiently illustrates the behavior of the plane in the inverse in all the circles of interest to us.

In particular, we get that for a real point p also its image p' is a real point. To visualize the effect of the map, let us restrict ourselves to the real points and their images. We see that the inversion by a complex circle with purely imaginary radius r is the composition of the inversion by the concentric real circle of radius $|r|$ and reflection through the center of the circle. Hence, also the inversion in a complex circle maps lines and circles to lines and circles.

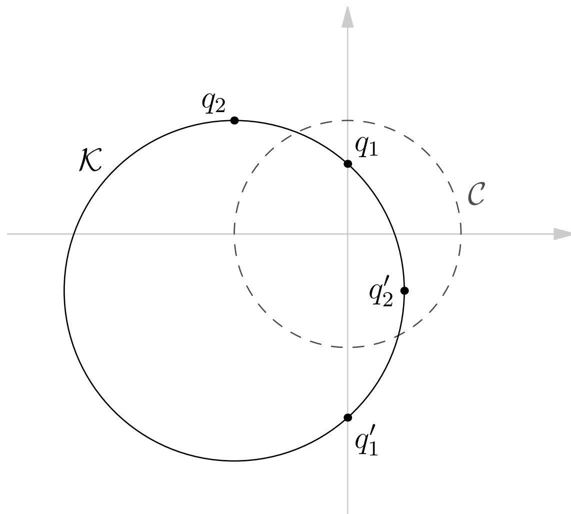

Figure 2: The real circle $\mathcal{K}$ is orthogonal to the complex circle $\mathcal{C}$, hence the inversion in $\mathcal{C}$ leaves $\mathcal{K}$ invariant.

Example 3. Consider the inversion in a complex circle $\mathcal{C}$ from Example 2. Let us study the circle $\mathcal{K}$ with center in $p = (-1, -1/2)$ and radius $3/2$. For an intersection

$$q = \left(\frac{4-3i}{5}, \frac{2+6i}{5}\right)$$

of the two circles we have $(q-p)\cdot(q-O) = 0$, therefore $\mathcal{C}$ and $\mathcal{K}$ are orthogonal.

We can find images of some other points of the circle $\mathcal{K}$, e.g.

$$q_1 = \left(0, \frac{\sqrt{5}-1}{2}\right), \quad q_2 = (-1, 1)$$

and we find out that they are mapped back to the circle $\mathcal{K}$ (see Figure 2).

Let $\mathcal{C}$ be a complex circle centered in p. The properties of inversion in $\mathcal{C}$ largely mirror the properties of standard inversion:

(i) Each straight line passing through the center p is invariant, since the image of q by definition lies on $\overleftrightarrow{pq}$.

(ii) Fixed points of the inversion are the points of $\mathcal{C}$ – we easily check this by examining the images of the intersections of the circle with the line passing through its center, p.

Theorem 3.4. *Let $\mathcal{C}$ be a complex circle. A circle orthogonal to $\mathcal{C}$ is invariant in the inversion in $\mathcal{C}$.*

Proof. Let $\mathcal{K}$ be a circle orthogonal to $\mathcal{C}$ and let $p \in \mathcal{K} \cap \mathcal{C}$. From Remark 2 we get that the tangent line to $\mathcal{K}$ in p passes through the center of $\mathcal{C}$, therefore is by (i) invariant. Furthermore, both p and the other intersection point of $\mathcal{K}$ and $\mathcal{C}$ are fixed points of the inversion according to (ii). A circle is uniquely determined by two points and the tangents at those points, therefore the image of the circle $\mathcal{K}$ coincides with the circle $\mathcal{K}$. $\square$

Thanks to this statement about inversion in a complex circle, we can use exactly the same approach to solve Castillon's problem as we proposed in Subsection 2.2.

4 On the Side of Computations

In the formula (5) for inversion in the unit complex circle, just like for inversion in a real circle, no complex numbers appear. We obtain the inversion in any complex circle by combining the inversion (5) with translations by real vectors and scalings by real scaling factors. Thus, the inversion in any complex circle (i.e., a circle with a real center and a purely imaginary radius) maps a real point to a real point. Therefore, we can restrict our calculations to the real plane.

If n points are given in Castillon's problem, then we need to compose n such inversions. If we were to use the formula (5), already for $n = 3$, it would lead to polynomials of degree 6. We want to avoid this complication.

For this purpose, we interpret the real plane as the Gauss plane, where a point $p \in \mathbb{R}^2$ with coordinates (p_x, p_y) is represented by the complex number $p_x + ip_y$. For a complex number $z = z_x + iz_y$, $\overline{z}$ denotes the complex conjugate $z_x - iz_y$.

Now we demonstrate that the restriction of the inversion in a circle (both real and complex) to a suitable and relevant subset of the plane can be expressed as a Möbius transformation.

Proposition 4.1. *Let $\mathcal{K}$ be the real unit circle $x^2 + y^2 = 1$ and let the point $p \in \mathbb{R}^2$ be not on $\mathcal{K}$. Let further $\mathcal{C}$ be a circle centered in the real point p and with a radius r. Then $\mathcal{C}$ and $\mathcal{K}$ are orthogonal if and only if*

$$\overline{p}p - r^2 = 1.$$

(The center p is interpreted as a complex number in the Gauss plane.)

Proof. Let q be an intersection of $\mathcal{C}$ and $\mathcal{K}$. These two circles are orthogonal to each other if and only if $(q - O) \perp (q - p)$, i.e.

$$(q - O) \cdot (q - p) = 0. \tag{6}$$

Passing to the coordinates $p = (p_x, p_y)$, $q = (q_x, q_y)$, from the fact that $q \in \mathcal{K}$ we get

$$p_x q_x + p_y q_y = 1. \tag{7}$$

On the other hand, we can rewrite (6) as follows:

$$((q - p) + (p - O)) \cdot (q - p) = 0.$$

Since $q \in \mathcal{C}$, together with (7) we obtain

$$r^2 - (p_x^2 + p_y^2) + 1 = 0,$$

which is the condition in the proposition.

Notice, that if the point p lies outside of the unit circle, this is actually the standard Pythagorean theorem. If the point p lies inside the unit circle, the radius of the circle $\mathcal{C}$ is a purely imaginary number. □

Theorem 4.2. *Let $p \in \mathbb{R}^2$ be not on the real unit circle. Let $\mathcal{C}$ be a circle (real or complex) centered at p and orthogonal to the unit circle. If the point z lies on the real unit circle, then it's image z' in inversion in $\mathcal{C}$ is*

$$z' = \frac{z - p}{\overline{p}z - 1}.$$

Proof. Let r be the radius of $\mathcal{C}$, so we have $r^2 \in \mathbb{R}^+$, if p is outside of the unit circle, and $r^2 \in \mathbb{R}^-$, if p is inside the unit circle. The inversion in $\mathcal{C}$ is the composition of three more elementary maps:

1. The translation by the vector $-p$. The point z is mapped to the point $z - p$.
2. The inversion in the circle centered in $O = (0, 0)$ with radius r. By the inversion, the point z is mapped to the point $r^2/\overline{z}$. Notice, that the formula is valid both for real and purely imaginary radius of the circle.
3. The translation by the vector p.

As the result, $z \in \mathbb{R}^2$ is mapped to the point

$$z' = \frac{r^2}{\overline{z - p}} + p = \frac{r^2 + p\overline{z} - p\overline{p}}{\overline{z} - \overline{p}}.$$

We are interested only in the points on the unit circle, which are such points z, that $z\overline{z} = 1$. Moreover, if we apply the Proposition 4.1, we get the formula stated in the Theorem. □

When solving Castillon's problem, we are interested only in the points on the given unit circle. From the Theorem 4.2 we see, that the restriction of the inversion to points on the unit circle is described as a Möbius transformation. This is very convenient, since these functions (transformations of the Gaussian plane) form a group. Moreover, due to their matrix representation they are convenient to work with: the transformation

$$z \mapsto z' = \frac{z - p_j}{\overline{p_j} z - 1}$$

describing the restriction to the unit circle of the inversion in a circle centered in $p_j = (p_{jx}, p_{jx}) = p_{j1} + ip_{j2}$ is represented by the matrix

$$\begin{pmatrix} 1 & -p_j \\ \overline{p_j} & -1 \end{pmatrix}. \tag{8}$$

The matrix of the product of inversions given by p_1, p_2, ..., p_n is then found as the product of the matrices of this type. The fixed point a_1 that we look for, is a solution to the equation

$$x = \frac{a_1 x + a_2}{a_3 x + a_2 v},$$

where the matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

is of the type

$$\begin{pmatrix} s & -t \\ \overline{t} & -\overline{s} \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} s & t \\ \overline{t} & \overline{s} \end{pmatrix}$$

for odd resp. even n.

5 Examples

We illustrate our solution on two examples: finding a triangle for three given points and finding a pentagon for five given points, that is inscribed in the unit circle $x^2 + y^2 = 1$.

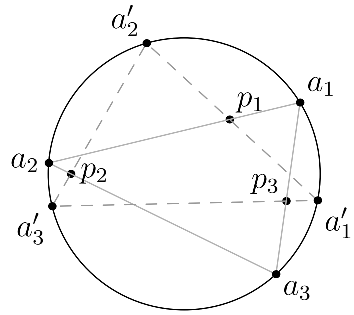

Figure 3: Two solutions are found for the given points p_1, p_2, p_3.

5.1 Three Points

Let

$$p_1 = (1/3, 2/5), \quad p_2 = (-5/6, 0), \quad p_3 = (3/4, -1/5).$$

Each of the points p_i determines the inversion f_i in the complex circle centered at p_i and orthogonal to the given unit circle. The restriction of f_i to the unit circle coincides with Möbius transformation of the Gauss plane (the points p_i being interpreted as complex numbers in the Gauss plane) and is encoded in the matrix M_i:

$$M_1 = \begin{pmatrix} 1 & -1/3 - 2/5i \\ 1/3 - 2/5i & -1 \end{pmatrix},$$
$$M_2 = \begin{pmatrix} 1 & 5/6 \\ -5/6 & -1 \end{pmatrix},$$
$$M_3 = \begin{pmatrix} 1 & -3/4 + 1/5i \\ 3/4 + 1/5i & -1 \end{pmatrix}.$$

The composed transformation $f_3 \circ f_2 \circ f_1$ is then represented by the matrix (the results of the computations are displayed in approximate decimal expansion)

$$M = M_3 M_2 M_1 = \begin{pmatrix} 2.073 - 0.867i & -2.192 - 0.394i \\ 2.192 - 0.394i & -2.073 - 0.867i \end{pmatrix}.$$

Finally, the point a_1 is found as a fixed point of the transformation:

$$\frac{(2.073 - 0.867i)a_1 + (-2.192 - 0.394i)}{(2.192 - 0.394i)a_1 + (-2.073 - 0.867i)} = a_1.$$

As this is a quadratic equation, we get two solutions:

$$a_1 = 0.851 + 0.525i, \qquad a_1' = 0.981 - 0.195i.$$

The point a_2 is the image $f_1(a_1)$, the point a_3 is the image $f_2(a_2)$:

$$\begin{array}{ll} a_2 = -0.997 + 0.080i & a_2' = -0.277 + 0.961i \\ a_3 = 0.675 - 0.738i & a_3 = -0.971 - 0.238i \end{array}$$

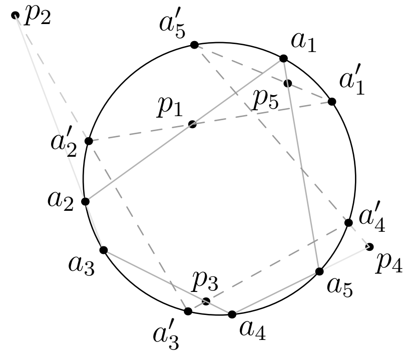

Figure 4: Two solutions are found for the given points $p_1, \ldots, p_5$.

5.2 Five Points

The method can by successfully applied for more points too. We find a pentagon $p_1p_2 \ldots p_5$ inscribed in the unit circle such that $p_i \in \overleftrightarrow{a_ia_{i+1}}$. Let

$$\begin{aligned} p_1 &= (-0.2, 0.4), \\ p_2 &= (-1.5, -1.2), \\ p_3 &= (-0.1, -0.9), \\ p_4 &= (1.1, -0.5), \\ p_5 &= (0.5, 0.7). \end{aligned}$$

The points p_2 and p_4 lie outside the circle, so each gives the inversion in a real circle, all other points determine the inversions in complex circles. The matrices M_i are constructed as in the previous example, The matrix M is then the product of all five of them. Then the first point of the pentagon is again found as a root of a quadratic equation, so in generic case, we again obtain two solutions:

$$a_1 = 0.469 + 0.883i, \qquad a'_1 = 0.824 - 0.566i.$$

For the rest of the pentagon we again compute $a_{i+1} = f_i(a_i)$:

$$\begin{aligned} a_2 &= -0.986 - 0.167i & a'_2 &= -0.961 + 0.277i \\ a_3 &= -0.852 - 0.524i & a'_3 &= -0.232 - 0.973i \\ a_4 &= 0.091 - 0.996i & a'_4 &= 0.946 - 0.323i \\ a_5 &= 0.733 - 0.681i & a'_5 &= -0.185 + 0.983i \end{aligned}$$

All computations were done using the computer algebra system Maxima [7].

6 Conclusion

The solution presented in this paper is similar to the standard analytical solution [3, 10]. Let us briefly recall the latter one.

There, each vertex of the solution lying on the unit circle is represented by one real number using the stereographic projection: the center of the projection is at $(-1, 0)$ and the parameter u representing the vertex $a = (a_x, a_y)$ is the y-coordinate of the projection of a

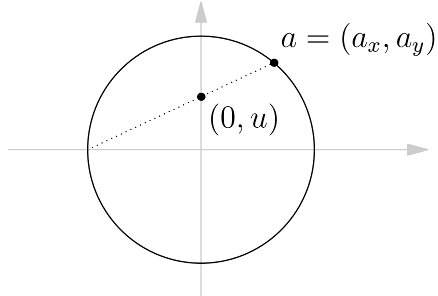

Figure 5: Stereographic projection.

to the y-axis. The given point $p_i = (p_{ix}, p_{iy})$ gives the transformation g_i that transforms the space of parameters of the points on the circle in such a way that the parameter u_i of the point a_i of the solution is mapped to the parameter u_{i+1} of the point a_{i+1}. So g_i is actually a transformation of the y-axis (i.e. parameter space). More precisely, it is the Möbius transformation given by the matrix

$$\begin{pmatrix} -p_{iy} & 1 - p_{ix} \\ -1 - p_{ix} & p_{iy} \end{pmatrix}. \tag{9}$$

These particular transformations are composed exactly as in our solution. The fixed point u of the composed transformation is then the parameter representing the first vertex a_1 of the sought polygon. So eventually a_1 is computed by stereographic projection with the center $(-1, 0)$:

$$u \mapsto \left(\frac{1 - u^2}{1 + u^2}, \frac{2u}{1 + u^2}\right).$$

Our solution resembles the analytical solution according to Lagrange and Carnot, and even the matrices representing the Möbius transformations corresponding to the given points exhibit a significant degree of similarity. However, upon closer inspection, we can see that the matrix (9) does not represent the circular inversion as in our solution.

The advantage of the well-known analytical solution seems to be that it operates with Möbius transformations over the real numbers, unlike our solution, where we have to multiply complex matrices. However, upon closer examination of the elementary computations we see that the number of costly operations, namely the multiplications of real numbers, is the same in both cases. Even the real numbers entering the calculations are the same: they are the coordinates of the given points.

Furthermore, in our solution, we do not need to perform any stereographic projection at the end because the fixed point is directly the desired point a_1 on the unit circle. To find it, we solve a quadratic equation over $\mathbb{C}$. In the standard analytical solution, on the other hand, a quadratic equation over the reals is solved for finding the parameter u.

Mentioning finally the last slight difference, the classical approach by Lagrange and Carnot is not perfectly symmetric as problems can arise in the case when the vertex a_1 of the solution would be the center of the stereographic projection $(-1, 0)$.

To conclude, on the computational side, both approaches are equal. The approach introduced in the paper gives a nice geometrical interpretation of computations that are carried out while solving the problem. It also does not introduce any special case that would need an extra treating in an implementation.

In [2], solving Castillon's problem using circular inversion is presented as unsatisfactory and complicated. However, introducing inversions in complex circles and interpreting each mapping as a Möbius transformation leads to a nice and efficient solution.

Acknowledgement

The author is grateful to the reviewer for valuable comments that helped to significantly raise the quality of the presentation.

This research was supported by KEGA 038UK-4/2024.

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Received October 30, 2023; final form January 25, 2024.