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The Quadrilateral Coordinated With a Circle that Forms Pascal Points and its Properties

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Abstract. In the present paper, the concept of "a quadrilateral coordinated with a circle that forms Pascal points" ("coordinated quadrilateral" for short) is defined as a quadrilateral for which there exists a circle that forms Pascal points on the sides of the quadrilateral, and for which it holds that the following four points are collinear: the point of intersection of the extensions of the two opposite sides of the quadrilateral, the center of the circle, and the two Pascal points formed by it.

We investigate and prove the properties of this quadrilateral. These properties may be divided into two sets: (i) properties of the straight lines, line segments, and angles associated with the coordinated quadrilateral and (ii) properties of the circles associated with the coordinated quadrilateral. In addition, we show a method for constructing the coordinated quadrilateral.

Key Words: coordinated quadrilateral, circle that forms Pascal points, collinearity of points, geometric construction

MSC 2020: 51M04 (primary), 51M05, 51M15, 51N20

1 Introduction

In order to define the quadrilateral coordinated with a circle that forms Pascal points, we shall recall the definition of Pascal points and the circle that forms Pascal points. These concepts are the basis for the theory of the convex quadrilateral and a circle that forms Pascal points on its sides (see [2–7]).

A circle that forms Pascal points (see [2])

For a convex quadrilateral $ABCD$, in which E is the point of intersection of the diagonals and F is the point of intersection of the extensions of sides BC and AD, a circle that forms Pascal points is any circle that passes through points E and F and also through interior points of sides BC and AD (see Figure 1).

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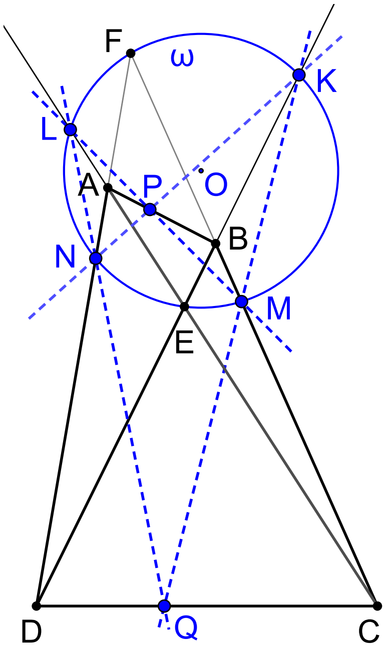

Figure 1: P and Q are Pascal points formed by circle ω.

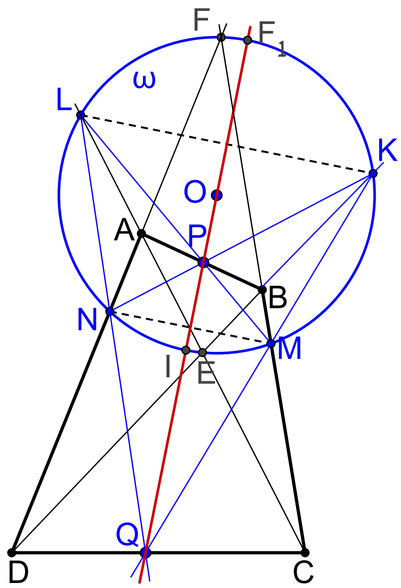

Figure 2: Line PQ passes through the center O of circle ω which forms P and Q.

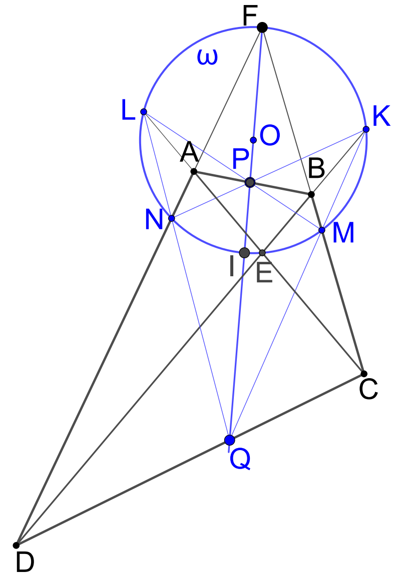

Figure 3: Points P, Q, O and F are collinear, therefore $ABCD$ is a coordinated quadrilateral.

Pascal points on the sides of a quadrilateral (see [2])

Let ω be a circle that forms Pascal points, and $M = \omega \cap BC$, $N = \omega \cap AD$. Also, let K and L be the points of intersection of ω with the extensions of diagonals BD and AC, respectively (see Figure 1). We further denote $P = KN \cap LM$ and $Q = KM \cap LN$.

There holds: $P \in [AB]$, $Q \in [CD]$.

Points P and Q are called *Pascal points* formed by circle ω on sides AB and CD.

If for a given convex quadrilateral, there exists one circle that forms Pascal points on sides AB and CD, then there is an infinite number of such circles (see [4, Proof of Theorem 1]). In this set of circles there will be one single circle ω where the Pascal points formed by it are collinear with the center of the circle.

For this circle, several additional properties hold (see [2, Theorems 6-9], [3], [5, Theorems 3, 5, 6]). In particular, let us note the following two properties:

Property (i) Chords KL and MN of circle ω are parallel to each other (see [2, Theorem 6]);

Property (ii) An inversion with respect to circle ω transforms points P and Q into one another (see [2, Theorems 7–8]). That is to say, points P, Q, F_1, and I constitute a harmonic quadruple where F_1 and I are the points of intersection of circle ω with straight line PQ (see Figure 2).

We shall now define the concept of "a quadrilateral coordinated with the circle that forms Pascal points" (for short, "coordinated quadrilateral").

Definition 1. Let $ABCD$ be a convex quadrilateral in which E is the point of intersection of its diagonals, and F is the point of intersection of the extensions of sides AD and BC.

If for this quadrilateral there is a circle ω (whose center is O) which forms Pascal points P and Q, and for which the four points F, O, P, and Q are collinear, then quadrilateral $ABCD$ is coordinated with circle ω (see Figure 3).

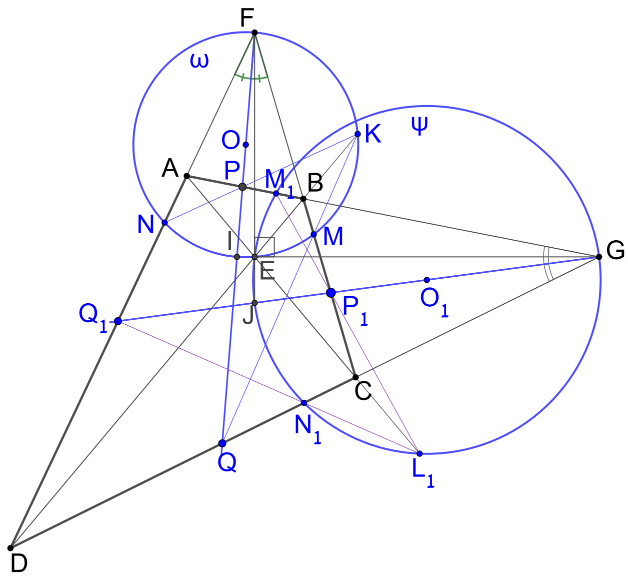

Figure 4: $\measuredangle FEG = 90°$. Ray FP bisects angle AFB, and ray GP_1 bisects angle BGC.

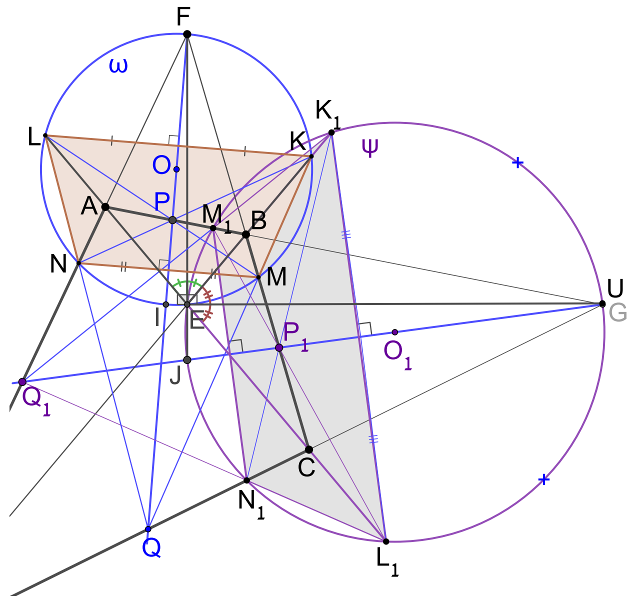

Figure 5: $KLNM$ is an isosceles trapezoid and therefore point F is the midpoint of arc $\widehat{KL}$. $K_1L_1N_1M_1$ is an isosceles trapezoid and therefore point U is the midpoint of the arc $\widehat{K_1L_1}$.

This article contains three sections:

In Section 2 we study the collinearity properties of the points, the properties of the angles, and the properties of the ratios of the lengths of segments of sides associated with a given coordinated quadrilateral.

In Section 3 we study the properties of the circles associated with a given coordinated quadrilateral.

In Section 4 we show a way to construct a coordinated quadrilateral and prove the correctness of the construction.

2 Properties of lines and angles associated with a given coordinated quadrilateral

Theorem 1. *Let $ABCD$ be a quadrilateral (in which $E = [AC] \cap [BD]$, $F = AD \cap BC$) that is coordinated with circle ω (E, $F \in \omega$) forming Pascal points P and Q on sides AB and CD, respectively (where $M = \omega \cap [BC]$, $N = \omega \cap [AD]$, $K = \omega \cap BD$, $L = \omega \cap AC$). Also, let G be the point of intersection of the continuation of sides AB and CD, and let ψ be the circle passing through points E and G. P_1 and Q_1 are the Pascal points formed by ψ on sides BC and AD; P_1 and Q_1 are collinear with the center O_1 of ψ (where $M_1 = \psi \cap [AB]$, $N_1 = \psi \cap [CD]$, $K_1 = \psi \cap BD$, $L_1 = \psi \cap AC$). Then*

(i) Straight line P_1Q_1 passes through point G.

(ii) $\measuredangle FEG = 90°$ (see Figure 4)*;*

(iii) Ray FP bisects angle AFB, and ray GP_1 bisects angle BGC.

Proof. (i)–(ii) According to Property (i) in the introduction, quadrilateral $KLNM$ is a trapezoid inscribed in circle ω, therefore it is an isosceles trapezoid in which the diagonals intersect at point P and the extensions of sides KM and LN intersect at point Q (see Figure 5).

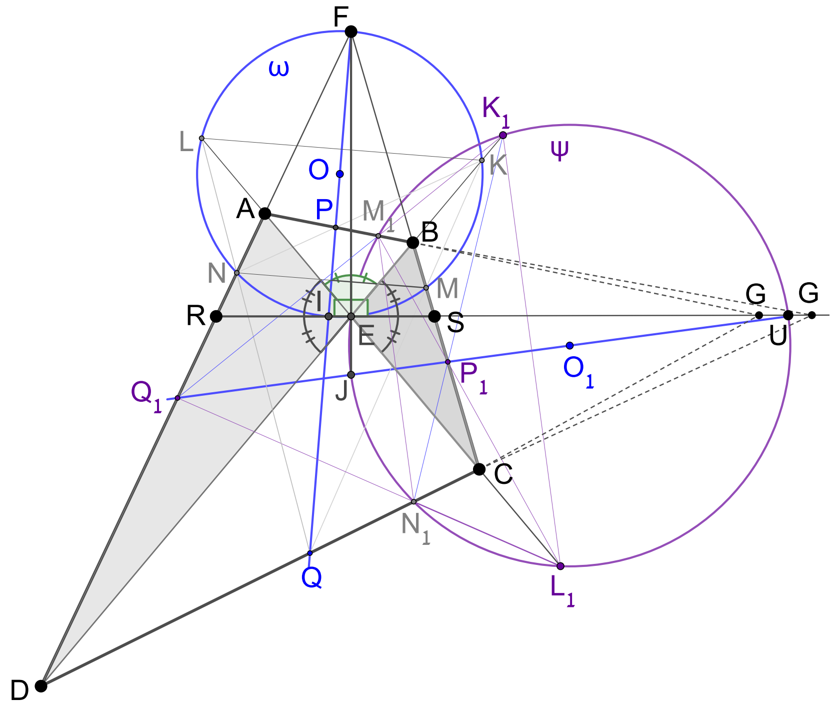

Figure 6: Straight lines AB and DC intersect at a point which belongs to line RS.

Therefore, straight line PQ bisects bases MN and KL of the trapezoid and is perpendicular to them. From this, it follows that diameter IF of circle ω is perpendicular to chord KL and bisects arc $\widehat{KL}$ (at point F) and $\widehat{MN}$ (at point I). From the fact that $\widehat{FL} = \widehat{KF}$, it follows that $\measuredangle KEF = \measuredangle FEL$. In other words, EF bisects angle KEL.

Similarly, since center O_1 of circle ψ is collinear with Pascal points P_1 and Q_1, it follows that $K_1L_1 \parallel N_1M_1$ and $K_1M_1 = L_1N_1$. In other words, quadrilateral $K_1L_1N_1M_1$ is an isosceles trapezoid.

We denote by J and U the points of intersection of straight line P_1Q_1 with circle ψ (see Figure 5), and we obtain $P_1Q_1 \perp K_1L_1$. Therefore, point U is the midpoint of arc $\widehat{K_1L_1}$. Hence, $\measuredangle K_1EU = \measuredangle UEL_1$, and therefore EU bisects $\measuredangle K_1EL_1$.

Since angles KEL and K_1EL_1 are supplementary adjacent angles, their angle bisectors are perpendicular to each other. In other words, $\measuredangle FEU = 90°$. In addition, $\measuredangle FEI = 90°$ (an inscribed angle that subtends diameter FI of circle ω). Therefore $\measuredangle IEU$ is a straight angle, and points I, E, and U lie on a single straight line.

We denote by R and S the points of intersection of straight-line EU with sides AD and BC, respectively. In triangle AED there holds (1) segment ER is an angle bisector of interior angle AED, and (2) segment EF is an angle bisector of exterior angle AEB (see Figure 6). Therefore, points F, A, R, and D form a harmonic quadruple. In other words, there holds: $(D, A; R, F) = -1$.

Similarly, in triangle BEC there holds (1) segment ES is an angle bisector of interior angle BEC, and (2) segment EF is an angle bisector of exterior angle BEA. Therefore, points F, B, S, and C form a harmonic quadruple. I.e., $(C, B; S, F) = -1$.

We have thus obtained that for each of the two straight lines FD and FC there are four points – F, A, R, D and F, B, S, C (point F is common the two quadruples) which satisfy the following equality of cross-ratios: $(D, A; R, F) = (C, B; S, F)$.

Therefore straight lines AB, RS, and DC intersect at a single point (see [10, Exercise 233]). Since it has been given that straight lines AB and DC intersect at point G, straight line RS also passes through point G. From the definition of points R and S, it follows that straight line RS passes through point U.

Let us assume that U and G are two different points of straight line RS. In this case, either G is an interior point of chord EU (i.e., G lies within circle ψ) or G lies on the continuation of chord EU (i.e., G lies outside circle ψ) (see Figure 6).

However, both cases lead to a contradiction to the data that point G belongs to circle ψ. Therefore, points U and G must coincide. It thus follows that the line P_1Q_1 passes through point G (we proved item (i) of Property 1), and it also follows that $\measuredangle FEG = 90°$ (we proved item (ii)).

(iii) In the proof of (a), we saw that point I is the midpoint of $\widehat{MN}$of circle ω. Therefore $\measuredangle MFI = \measuredangle NFI$. In other words, FI bisects angle MFN, which means that ray FP being the bisector of angle AFB (see Figure 5). Similarly, point J is the midpoint of $\widehat{M_1N_1}$ of circle ψ, and therefore $\measuredangle M_1GJ = \measuredangle N_1GJ$. In other words, ray GP_1 is the bisector of angle BGC (see Figure 6). □

Theorem 2. *In addition to the data from Theorem 1, we mark eight points of intersection as follows* (see Figure 7)*:* $T = PQ \cap P_1Q_1$*,* $V = NM \cap N_1M_1$*,* $Z = \omega \cap FG$*,* $Z_1 = \psi \cap FG$*,* $S = NM \cap PQ$*,* $W = NM \cap FE$*,* $W_1 = N_1M_1 \cap EG$*,* $S_1 = N_1M_1 \cap P_1Q_1$*. There holds:*

(i) Quadrilaterals $TSVS_1$*,* $SWEI$*, and* EW_1S_1J *are cyclic.*

(ii) Straight line MN *passes through point* G*, and straight line* N_1M_1 *passes through point* F*.*

(iii) Straight line IW *passes through point* Z*, and straight line* JW_1 *passes through point* Z_1*.*

Proof. *(i)* In proving Theorem 1 we have seen that $NM \perp PQ$, $N_1M_1 \perp P_1Q_1$, and $\measuredangle FEG = 90°$. Therefore in each of the quadrilaterals – $TSVS_1$, $SWEI$, and EW_1S_1J – there are two opposite right angles. Therefore, these quadrilaterals are cyclic.

(ii) From the proof of Theorem 1, it follows that points I, E, and G are collinear. Therefore straight-line IE passes through point G, in other words straight lines AB and IE intersect at point G. Let us prove that straight lines NM and IE also intersect at point G. To do this, let us denote the point of intersection of straight lines NM and IE by X and then prove that points X and G coincide.

Let us use the method of complex numbers in plane geometry. We choose a system of coordinates so that circle ω is the unit circle (center O of circle ω is located at the origin, and the radius is $OE = 1$). In this system, the equation of the unit circle is $z \cdot \overline{z} = 1$, where z and $\overline{z}$ are the complex coordinate and the complex conjugate of the coordinate of an arbitrary point Z located on circle ω. We denote the complex coordinates of points E, F, I, K, L, M, and N as e, f, i, k, l, m, and n, respectively. These points are located on the unit circle, and therefore there holds:

$$\overline{e} = \frac{1}{e}, \quad \overline{f} = \frac{1}{f}, \quad \overline{l} = \frac{1}{l}, \quad \overline{k} = \frac{1}{k}, \quad \overline{l} = \frac{1}{l}, \quad \overline{m} = \frac{1}{m} \quad \text{and} \quad \overline{n} = \frac{1}{n}. \tag{1}$$

FI is a diameter of the circle ω, therefore holds that

$$i = -f. \tag{2}$$

In addition, since center O and Pascal points P and Q are collinear, it follows that segments KL and MN are parallel to each other and therefore (see [2, Theorem 6])

$$mn = kl. \tag{3}$$

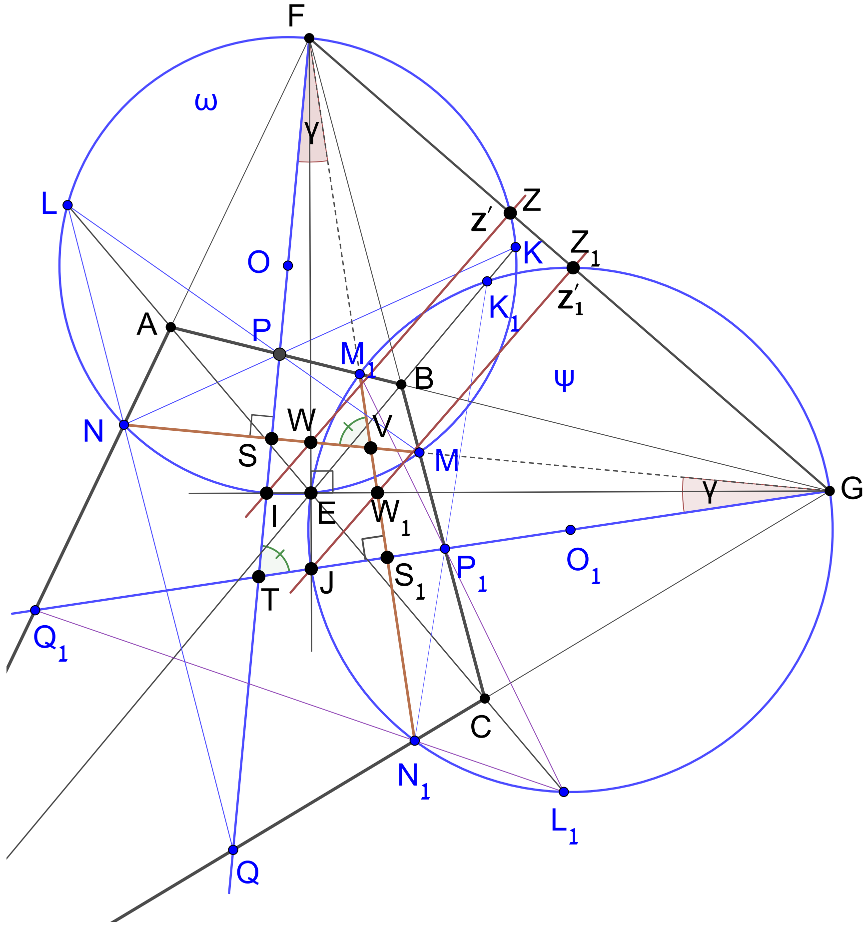

Figure 7: Straight lines NM and N_1M_1 pass through points G and F respectively.

The equation of straight-line PQ that passes through center O of unit circle ω is

$$p\overline{z} = \overline{p}z \tag{4}$$

(where z is the complex coordinate of an arbitrary point Z that belongs to straight line PQ).

We use the following property (see [11, p. 181]): Let $T(t)$, $Q(q)$, $R(r)$, and $S(s)$ be four points on the unit circle, and let $U(u)$ be the point of intersection of straight lines TQ and RS. Then for coordinate u and its conjugate, $\overline{u}$, there holds:

$$\overline{u} = \frac{t+q-r-s}{tq-rs} \quad \text{and} \quad u = \frac{qrs+trs-tqs-tqr}{re-tq}. \tag{5}$$

In our case, since $P = KN \cap LM$, we can express the complex coordinate of point P and the conjugate of the coordinate by using the complex coordinates of points K, L, M, and N, as follows:

$$\overline{p} = \frac{n+k-m-l}{nk-ml} \quad \text{and} \quad p = \frac{mnl+mkl-mnk-nkl}{ml-nk}.$$

By Equation (3), the expression for p can be simplified to

$$p = \frac{mn(m+l-n-k)}{ml-nk}.$$

We substitute expressions for p and $\overline{p}$ in Equation (4) and obtain:

$$\frac{mn(m+l-n-k)}{ml-nk}\overline{z} = \frac{n+k-m-l}{nk-ml}z.$$

Hence:

$$\overline{z} = \frac{1}{mn}z. \tag{6}$$

Point F lies on straight line PQ. Therefore, $\overline{f} = \frac{1}{mn}f \implies mn = \frac{1}{\overline{f}}f$, and finally

$$mn = kl = f^2. \tag{7}$$

Now, let us find the complex coordinate of point X and its conjugate. First we find the equations of straight lines MN and IE, which are defined using pairs of points that belong to unit circle ω. For straight line MN, we obtain $z + mn\overline{z} = m + n \implies \overline{z} = -\frac{1}{mn}z + \frac{m+n}{mn}$, or, according to Equation (7), we thus obtain

$$\overline{z} = -\frac{1}{f^2}z + \frac{m+n}{f^2}. \tag{8}$$

For straight-line IE, we obtain $z + ie\overline{z} = i + e \implies \overline{z} = -\frac{1}{ie}z + \frac{i+e}{ie}$, or, according to Equation (2),

$$\overline{z} = \frac{1}{fe}z + \frac{f-e}{fe}. \tag{9}$$

We then solve the system of Equations (8) and (9): $\frac{1}{fe}z + \frac{f-e}{fe} = -\frac{1}{f^2}z + \frac{m+n}{f^2}$, and hence $\left(\frac{1}{fe} + \frac{1}{f^2}\right)z = \frac{m+n}{f^2} - \frac{f-e}{fe}$. Finally, for the coordinate of intersection point X, we obtain:

$$z_x = x = \frac{em + en + ef - f^2}{f+e}.$$

Therefore, the complex conjugate of x is:

$$\overline{x} = \frac{\frac{1}{em} + \frac{1}{en} + \frac{1}{ef} - \frac{1}{f^2}}{\frac{1}{f} + \frac{1}{e}} = \frac{\frac{f^2n + f^2m + fmn - emn}{ef^2mn}}{\frac{f+e}{fe}},$$

and using Equation (7) we obtain

$$\overline{x} = \frac{m+n+f-e}{f^2+fe}.$$

We now prove that points A, B, and X are collinear. For that, it is sufficient to prove that the following equality holds for these points (see [11, p. 156]):

$$a(\overline{b} - \overline{x}) + b(\overline{x} - \overline{a}) + x(\overline{a} - \overline{b}) = 0. \tag{10}$$

We use Equations (5) and express the complex coordinates (and their complex conjugates) of points A and B through the coordinates of the points located on unit circle ω.

$\overline{a} = \dfrac{f+n-e-l}{fn-el}$ and $a = \dfrac{eln + efl - fln - efn}{el - fn}$ (because $A = FN \cap EL$, see Figure 6);

$$\bar{b} = \frac{f+m-e-k}{fm-ek} \quad \text{and} \quad b = \frac{ekm+efk-fkm-efm}{ek-fm} \text{ (because } B = FM \cap EK\text{)}.$$

Consider the left-hand side of Equation (10). If we substitute the expressions for a, $\overline{a}$, b, $\overline{b}$, x, and $\overline{x}$, we obtain

$$\frac{eln+efl-fln-efn}{el-fn} \times \left(\frac{f+m-e-k}{fm-ek} - \frac{m+n+f-e}{f^2+fe}\right)+ \frac{ekm+efk-fkm-efm}{ek-fm} \times \left(\frac{m+n+f-e}{f^2+fe} - \frac{f+n-e-l}{fn-el}\right)+ \frac{em+en+ef-f^2}{f+e} \times \left(\frac{f+n-e-l}{fn-el} - \frac{f+m-e-k}{fm-ek}\right) \overset{\text{denote}}{=} \frac{A}{(f^2+fe)(el-fn)(fm-ek)} + \frac{B}{(f^2+fe)(fn-el)(ek-fm)} + \frac{C}{(f+e)(fn-el)(fm-ek)}.$$

The last three fractions on the left-hand side of Equation (10) are obtained after adding the fractions in the parentheses and denoting the numerators by A, B, and C.

The common denominator of these fractions is $f(f+e)(el-fn)(fm-ek)$. Hence, the last expression can be brought in the following form:

$$\frac{A+B-fC}{f(f+e)(el-fn)(fm-ek)}.$$

After opening the parenthesis, replacing products mn and kl with f^2, and collecting similar terms in each expression, we obtain:

$$A = e^2f^2n^2 - e^3fln - ef^4n - ef^3lm + 2e^2f^3l + e^2f^3m + 2e^2f^3n - e^3f^2l - 2ef^5 - ef^2lm^2 - e^3f^3 + 2e^2f^2lm - ef^3n^2 + e^2f^2ln + f^5n + f^4lm - 2ef^4l - e^2f^3k - e^2fkn^2 + ef^3kn + ef^4m + e^3fkn - e^2f^4,$$

$$B = -2e^2f^3k + e^3fkm + ef^3kn - e^2f^2m^2 + ef^4m - 2e^2f^2kn + e^3f^2k + ef^2kn^2 - 2e^2f^3m - e^2f^3n + e^3f^3 + 2ef^4k - e^2f^2km - f^4kn + ef^3m^2 + 2ef^5 - f^5m + e^2f^4 - ef^4n + e^2flm^2 + e^2f^3l - e^3flm - ef^3lm,$$

$$-fC = -ef^3m^2 + e^2f^2m^2 + ef^2lm^2 + e^2f^2km - e^3fkm - 2ef^4k - 2e^2f^2lm - e^2flm^2 + e^3flm + 2ef^4l + 2e^2f^2kn + e^2fkn^2 - e^3fkn + ef^3n^2 - e^2f^2n^2 - ef^2kn^2 - e^2f^2ln + e^3fln - 2ef^4m + e^2f^3m + 2ef^3lm + 3e^2f^3k - e^3f^2k + 2ef^4n - e^2f^3n - 2ef^3kn - 3e^2f^3l + e^3f^2l + f^5m - f^4lm - f^5n + f^4kn.$$

It can be ascertained that the sum of these three expressions is 0. Therefore, Equation (10) holds. Therefore straight line AB passes through point X (where $X = NM \cap IE$).

In summary, we obtained that straight line IE intersects straight line AB at points X and G. Therefore these points coincide. Likewise, it also follows that straight lines AB, CD, IE, P_1Q_1, and MN intersect at point G.

Similarly, if we choose ψ as the unit circle, it can be proven that the straight line N_1M_1 passes through point F.

(iii) In triangle IFG, segments FE and GS are altitudes to sides IG and IF, respectively. These segments intersect at point W (see Figure 8), therefore straight line IW contains the

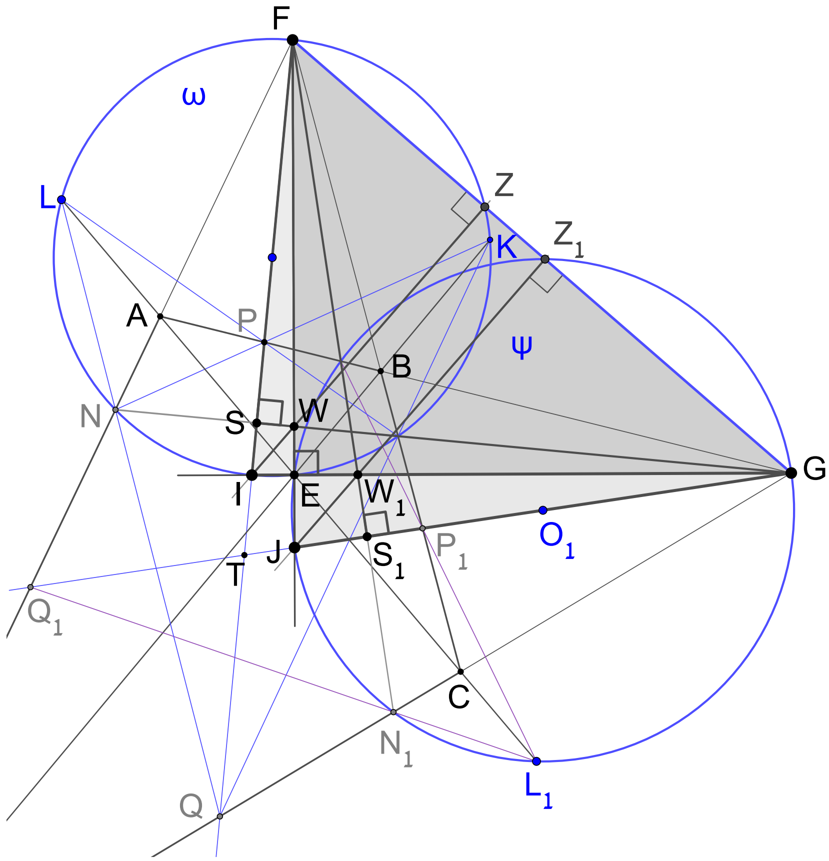

Figure 8: The intersection points of lines IW and JW_1 with line FG belong to circles ω and ψ respectively.

altitude to side FG (we denote this altitude by IZ'), i.e., $IZ' \perp FG$. On the other hand, in circle ω, angle FZI is a right angle (inscribed angle subtending the diameter), therefore there holds also $IZ \perp FG$. Hence points Z and Z' coincide and there holds $IW \perp FG$.

Similarly, in triangle JFG, segments GE and FS_1 are altitudes to sides JF and JG, respectively. These segments intersect at point W_1, and therefore straight line JW_1 contains the altitude to the third side (denoted by IZ_1'). On the other hand, in circle ψ, angle JZ_1G is a right angle. Therefore points Z_1 and Z_1' coincide, that is, $JW_1 \perp FG$. □

Other properties resulting from Theorems 1 and 2 (see Figure 7).

Property (1) The angle between angle bisectors FP and GP_1 is equal to the angle between straight lines MN and M_1N_1.

Property (2) The angle between straight lines FP and M_1N_1 is equal to the angle between straight lines GP_1 and MN.

Proof of Property (1): Since quadrilateral $TSVS_1$ is cyclic, it follows that $\measuredangle PTP_1 = \measuredangle SVF$.

Proof of Property (2): We denote $\measuredangle TFS_1 = \gamma$. In right-angled triangle TFS_1, there holds: $\measuredangle FTS_1 = 90° - \gamma$. Therefore, in right-angled triangle GST there holds $\measuredangle STG = 90° - \gamma$. It result $\measuredangle TGS = \gamma$, and $\measuredangle TFS_1 = \measuredangle TGS$.

The following property describes the ratios of segments and the lengths of segments associated with Pascal points.

Theorem 3. *In addition to the data of Theorem 2, we denote the following angles* (see Figure 9)*:* $\measuredangle AFP = \measuredangle PFB = \phi$, $\measuredangle S_1FP_1 = \alpha$, $\measuredangle TFS_1 = \gamma$, $\measuredangle DFQ = \beta$, $\measuredangle BGP_1 = \measuredangle P_1GC = \varphi$, $\measuredangle PGS = \delta$, $\measuredangle TGS = \gamma$, $\measuredangle MGC = \theta$. *Therefore:*

(a) (i) $\dfrac{AP}{PB} = \dfrac{\cos(\phi+\delta)}{\cos(\phi-\delta)}$, *(ii)* $\dfrac{P_1T}{TQ_1} = \dfrac{\cos(\phi+\gamma)}{\cos(\phi-\gamma)}$, *(iii)* $\dfrac{CQ}{QD} = \dfrac{\cos(\phi+\theta)}{\cos(\phi-\theta)}$,

(iv) $\dfrac{CP_1}{P_1B} = \dfrac{\cos(\varphi+\alpha)}{\cos(\varphi-\alpha)}$, *(v)* $\dfrac{PT}{TQ} = \dfrac{\cos(\varphi+\gamma)}{\cos(\varphi-\gamma)}$, *(vi)* $\dfrac{AQ_1}{Q_1D} = \dfrac{\cos(\varphi+\beta)}{\cos(\varphi-\beta)}$.

(b) (i) $AB = FP\sin\phi\left(\dfrac{1}{\cos(\phi-\delta)} + \dfrac{1}{\cos(\phi+\delta)}\right)$,

(ii) $PQ = FT\sin\phi\left(\dfrac{1}{\cos(\phi-\gamma)} + \dfrac{1}{\cos(\phi+\gamma)}\right)$,

(iii) $CD = FQ\sin\phi\left(\dfrac{1}{\cos(\phi-\theta)} + \dfrac{1}{\cos(\phi+\theta)}\right)$,

(iv) $BC = GP_1\sin\varphi\left(\dfrac{1}{\cos(\varphi-\alpha)} + \dfrac{1}{\cos(\varphi+\alpha)}\right)$,

(v) $P_1Q_1 = GT\sin\varphi\left(\dfrac{1}{\cos(\varphi-\gamma)} + \dfrac{1}{\cos(\varphi+\gamma)}\right)$,

(vi) $AC = GQ_1\sin\varphi\left(\dfrac{1}{\cos(\varphi-\beta)} + \dfrac{1}{\cos(\varphi+\beta)}\right)$.

(c) Pascal-point pairs P, Q *and* P', Q' *divide the pairs of opposite sides in a quadrilateral by ratios satisfying the following equality:*

$$\frac{AP}{PB}\cdot\frac{CQ}{QD} = \frac{CP_1}{P_1B}\cdot\frac{AQ_1}{Q_1D}.$$

Proof. *(a) Proof of formula a(i).* From the data of Theorem 3, there holds that for the angles of triangles PGS and FAP:

$\measuredangle FPA = \measuredangle GPS = 90^\circ - \delta$ and $\measuredangle FAP = 180^\circ - \phi - (90^\circ - \delta) = 90^\circ + \delta - \phi$.

Therefore, in $\triangle AFP$, there holds

$$\frac{AP}{\sin\phi} = \frac{FP}{\sin(90^\circ+\delta-\phi)} \quad\Longrightarrow\quad AP = \frac{FP\sin\phi}{\cos(\phi-\delta)}. \tag{11}$$

For the angles of triangle FAB, we obtain $\measuredangle FBA = 180^\circ - 2\phi - (90^\circ + \delta - \phi) = 90^\circ - (\phi+\delta)$. Hence, in $\triangle FPB$, there holds

$$\frac{PB}{\sin\phi} = \frac{FP}{\sin(90^\circ-(\delta+\phi))} \quad\Longrightarrow\quad PB = \frac{FP\sin\phi}{\cos(\phi+\delta)} \tag{12}$$

From Equations (11) and (12), it follows that

$$\frac{AP}{PB} = \frac{\cos(\phi+\delta)}{\cos(\phi-\delta)}.$$

Proof of formula a(ii). For the angles of triangles FTS_1 and FTQ_1, we obtain:

$\measuredangle FTS_1 = 90^\circ - \gamma$, and therefore $\measuredangle FQ_1T = 90^\circ - \gamma - \phi = 90^\circ - (\gamma+\phi)$.

Hence, in $\triangle FTQ_1$ there holds

$$\frac{TQ_1}{\sin\phi} = \frac{FT}{\sin(90^\circ+\gamma-\phi))} \quad\Longrightarrow\quad TQ_1 = \frac{FT\sin\phi}{\cos(\phi-\gamma))} \tag{13}$$

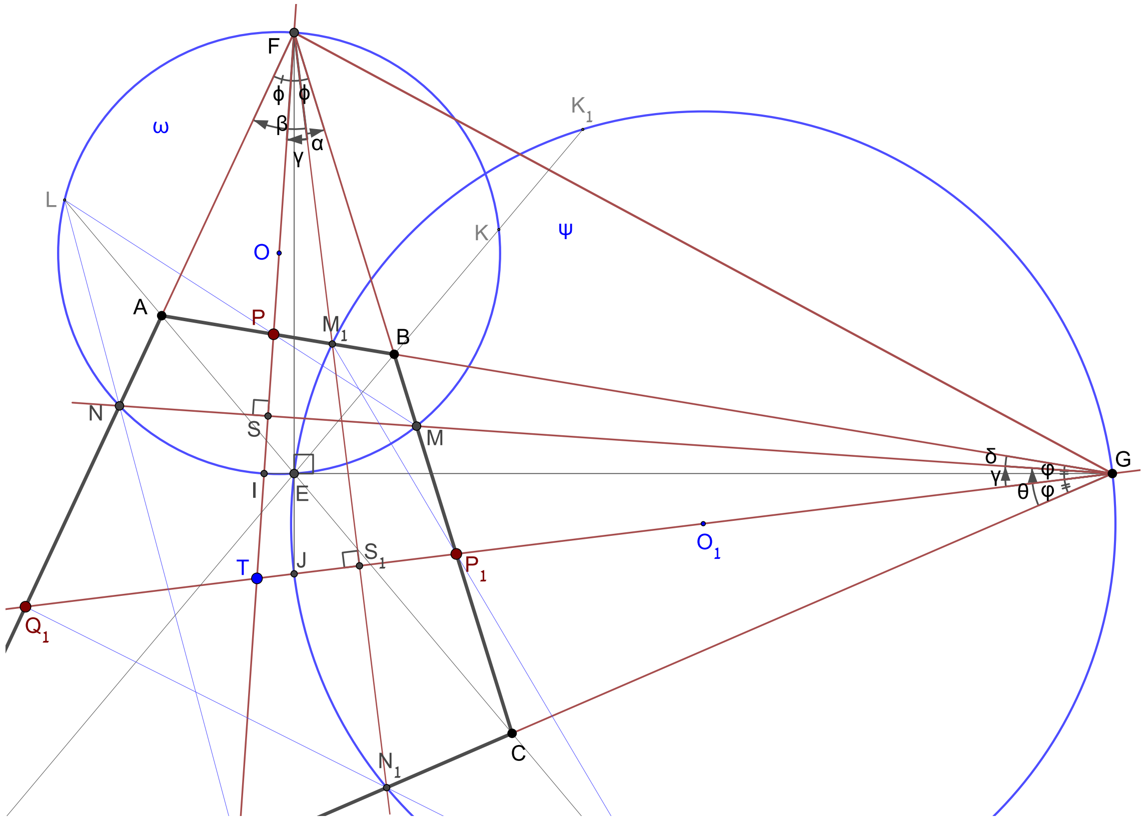

Figure 9: $\measuredangle TFS_1 = \measuredangle TGS = \gamma \implies \beta - \alpha = \theta - \delta = \gamma$

For the angles of triangle FQ_1P_1, there holds:

$\measuredangle FP_1Q_1 = 180^\circ - 2\phi - (90^\circ + \gamma - \phi) = 90^\circ - (\phi + \gamma)$. Therefore, in $\triangle FTP$, there holds

$$\frac{P_1T}{\sin\phi} = \frac{FT}{\sin\left(90^\circ - (\phi+\gamma)\right)} \implies P_1T = \frac{FT\sin\phi}{\cos\left(\phi+\gamma\right)} \tag{14}$$

From Equations (13) and (14) it follows that

$$\frac{P_1T}{TQ_1} = \frac{\cos(\phi-\delta)}{\cos(\phi+\delta)}.$$

Proof of formula a(iii). For the angles of triangles GSQ, FDQ, and FDC, we obtain: $\measuredangle SQG = 90^\circ - \theta$, $\measuredangle FDQ = 90^\circ - \theta - \phi$, and $\measuredangle FCD = 180^\circ - 2\phi - (90^\circ - \theta - \phi) = 90^\circ + \theta - \phi$. Hence, in $\triangle FDQ$:

$$\frac{QD}{\sin\phi} = \frac{FQ}{\sin\left(90^\circ - (\theta+\phi)\right)} \implies QD = \frac{FQ\sin\phi}{\cos\left(\phi+\theta\right)} \tag{15}$$

and in $\triangle FCQ$:

$$\frac{CQ}{\sin\phi} = \frac{FQ}{\sin\left(90^\circ + (\theta-\phi)\right)} \implies CQ = \frac{FQ\sin\phi}{\cos\left(\phi-\theta\right)} \tag{16}$$

From Equations (15) and (16) it follows that

$$\frac{CQ}{QD} = \frac{\cos(\phi+\theta)}{\cos(\phi-\theta)}.$$

Since the proofs of formulas *a(iv)–a(vi)* are similar to each other, we shall only give the proof for formula *a(vi)*.

Proof of formula a(vi). For the angles of triangles GS_1M_1, FBM_1, and BCG, we obtain $\measuredangle GM_1S_1 = 90° - \varphi$, $\measuredangle GBC = \measuredangle FBM_1 = (90° - \varphi) - \alpha$, and $\measuredangle GCB = 180° - 2\varphi - (90° - \varphi - \alpha) = 90° + \alpha - \varphi$. Therefore, in $\triangle GBP_1$ there holds that

$$\frac{P_1B}{\sin\varphi} = \frac{GP_1}{\sin(90° - (\varphi + \alpha))} \quad \Longrightarrow \quad P_1B = \frac{GP_1 \sin\varphi}{\cos(\varphi + \alpha)} \tag{17}$$

and in $\triangle GCP_1$,

$$\frac{CP_1}{\sin\varphi} = \frac{GP_1}{\sin(90° + \alpha - \varphi)} \quad \Longrightarrow \quad CP_1 = \frac{GP_1 \sin\varphi}{\cos(\varphi - \alpha)} \tag{18}$$

From Equations (17) and (18) it follows that

$$\frac{CP_1}{P_1B} = \frac{\cos(\varphi + \theta)}{\cos(\varphi - \theta)}.$$

(b) The formulas of this section are obtained by summing the appropriate pairs of formulas from Section *(a)*: formula *b(i)* is obtained by adding formulas (11) and (12), formula *b(ii)* is obtained by adding formulas (13) and (14), and so forth.

(c) Let us consider the products of the ratios by which points P and Q divide sides AB and CD. From formulas *a(i)* and *a(iii)* we obtain:

$$\frac{AP}{PB} \cdot \frac{CQ}{QD} = \frac{\cos(\phi + \delta)}{\cos(\phi - \delta)} \cdot \frac{\cos(\phi + \theta)}{\cos(\phi - \theta)} = \frac{\frac{1}{2}[\cos(\delta - \theta) + \cos(2\phi + \delta + \theta)]}{\frac{1}{2}[\cos(\theta - \delta) + \cos(2\phi - \delta - \theta)]} \underbrace{=}_{\delta+\theta=2\varphi}$$
$$= \frac{[\cos(\theta - \delta) + \cos(2\phi + 2\varphi)]}{[\cos(\theta - \delta) + \cos(2\phi - 2\varphi)]}.$$

Similarly, from formulas *a(iv)* and *a(vi)* for the product of the ratios by which points P_1 and Q_1 divide sites BC and AD, there holds:

$$\frac{CP_1}{P_1B} \cdot \frac{AQ_1}{Q_1D} = \frac{\cos(\varphi + \alpha)}{\cos(\varphi - \alpha)} \cdot \frac{\cos(\varphi + \beta)}{\cos(\varphi - \beta)} = \frac{\frac{1}{2}[\cos(\alpha - \beta) + \cos(2\varphi + \alpha + \beta)]}{\frac{1}{2}[\cos(\beta - \alpha) + \cos(2\varphi - \alpha - \beta)]} \underbrace{=}_{\alpha+\beta=2\phi}$$
$$= \frac{[\cos(\beta - \alpha) + \cos(2\varphi + 2\phi)]}{[\cos(\beta - \alpha) + \cos(2\varphi - 2\phi)]}.$$

Let us prove that the following equality holds: $\beta - \alpha = \theta - \delta$. For angle β there holds: $\beta = \phi + \gamma = \alpha + \gamma + \gamma = \alpha + 2\gamma$ (see Figure 9), therefore $\beta - \alpha = 2\gamma$.

For angle θ there holds: $\theta = \varphi + \gamma = \delta + \gamma + \gamma = \delta + 2\gamma$, therefore $\theta - \delta = 2\gamma$. Therefore the above equality holds, and therefore

$$\frac{[\cos(\theta - \delta) + \cos(2\phi + 2\varphi)]}{[\cos(\theta - \delta) + \cos(2\phi - 2\varphi)]} = \frac{[\cos(\beta - \alpha) + \cos(2\varphi + 2\phi)]}{[\cos(\beta - \alpha) + \cos(2\varphi - 2\phi)]},$$

it thus follows that

$$\frac{AP}{PB} \cdot \frac{CQ}{QD} = \frac{CP_1}{P_1B} \cdot \frac{AQ_1}{Q_1D}. \qquad \square$$

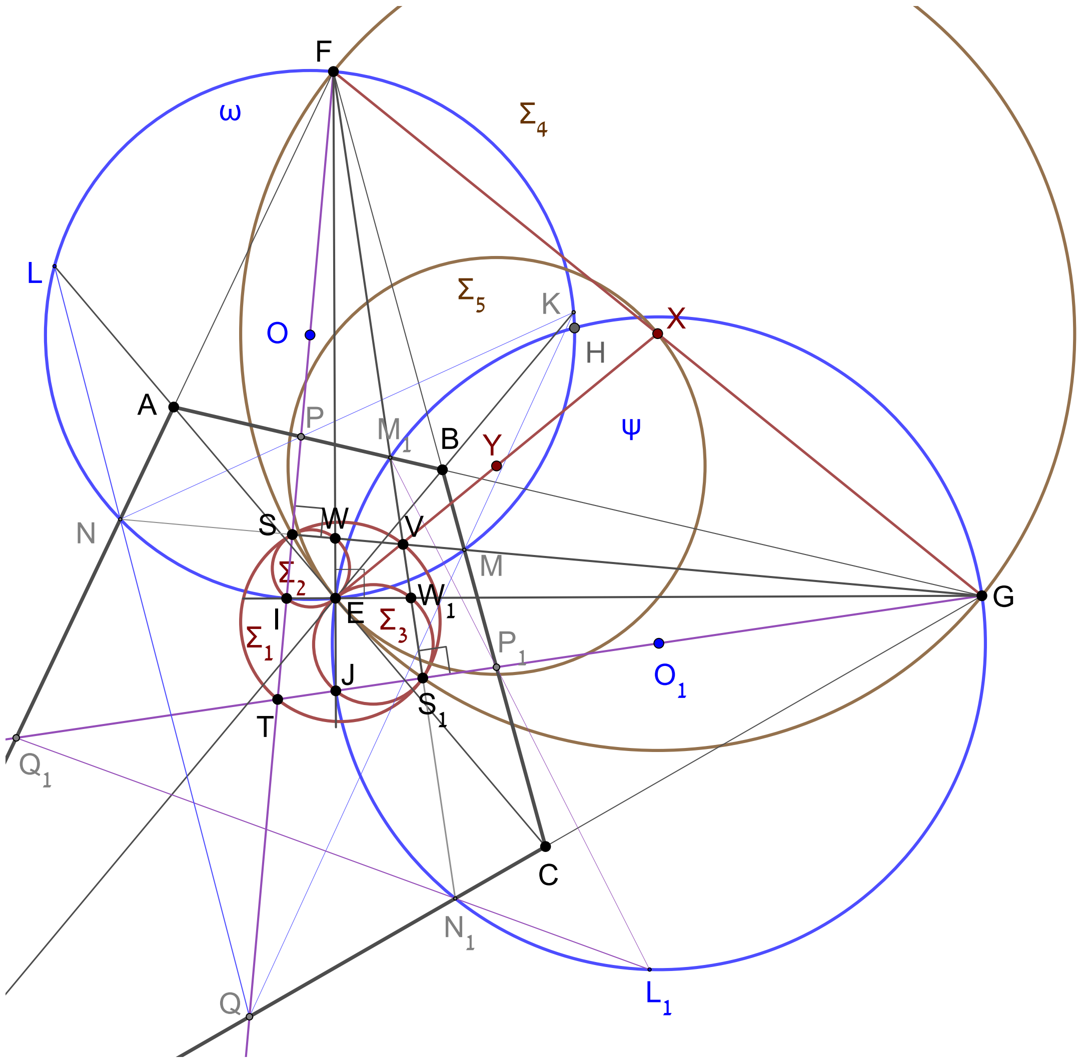

Figure 10: The circles Σ_1, Σ_2 and Σ_3 are tangent to each other in pairs. The circle Σ_4 is perpendicular to the circles Σ_1, Σ_2, and Σ_3. The circle Σ_5 is perpendicular to the circles Σ_2, Σ_3.

3 The properties of circles associated with a coordinated quadrilateral

Theorem 4. *In addition to the data of Theorem 2, let* (see Figure 10)*:*

Σ_1 *be a circle inscribing quadrilateral* $TSVS_1$*;*

Σ_2 *be a circle inscribing quadrilateral* $SWEI$*;*

Σ_3 *be a circle inscribing quadrilateral* EW_1S_1J*;*

Σ_4 *be a circle whose center* X *is the midpoint of segment* FG *and whose radius is segment* XE*;*

Σ_5 *be a circle whose center* Y *is the midpoint of segment* XE *and whose radius is segment* YE*. Then:*

(a) Circles Σ_1*,* Σ_2*, and* Σ_3 *are tangent to each other in pairs.*

(b) Circle Σ_4 *is perpendicular to circles* Σ_1*,* Σ_2*, and* Σ_3*.*

(c) Circle Σ_5 *is perpendicular to circles* Σ_2 *and* Σ_3*.*

(d) Circles Σ_4 *and* Σ_5 *are tangent to each other.*

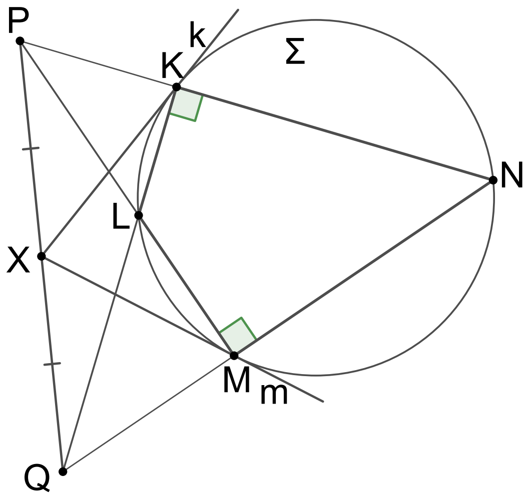

Figure 11: The tangents to circle Σ at points K and M intersect at the midpoint of segment PQ.

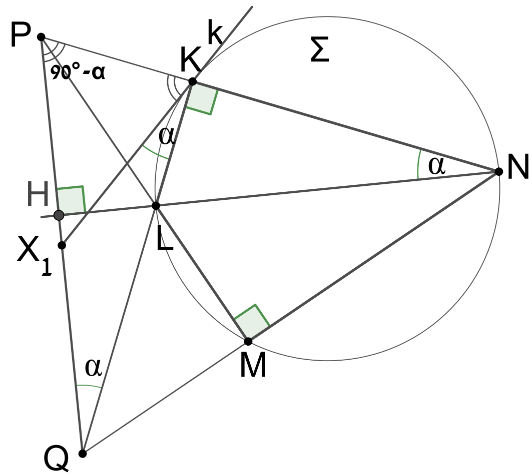

Figure 12: Triangles X_1KQ and X_1KP are isosceles, therefore point X_1 is midpoint of segment PQ.

Proof. First we prove the following lemma (see Figure 11):

Lemma. *Let $KLMN$ be a quadrilateral in which $\measuredangle K = \measuredangle M = 90°$; $KLMN$ inscribed in circle Σ; P is the point of intersection of the continuations of sides KN and LM, Q is the point of intersection of the continuations of sides KL and MN; X is the midpoint of segment PQ;*

Then: straight lines XK and XM are tangents to circle Σ at points K and M, respectively.

Proof of the lemma. Let us prove that the tangent k to circle Σ at point K passes through point X.

We denote by X_1 the point of intersection of k with segment PQ (see Figure 12). In triangle NPQ, segments QK and PM are altitudes to sides NP and NQ, respectively, which intersect at point L. Therefore, straight line NL contains the third altitude, NH, to side PQ of the triangle. We denote $\measuredangle HNP = \alpha$. Hence we obtain:

(i) $\measuredangle LKX_1 = \measuredangle QKX_1 = \measuredangle LNK = \alpha$ (The angle formed by a tangent and a chord ending at the point of contact is equal to one half the arc it intercepts on the circle);

(ii) $\measuredangle NPH = 90° - \alpha$;

(iii) $\measuredangle KQP = 90° - \measuredangle KPQ = 90° - (90° - \alpha) = \alpha$.

Therefore $\measuredangle QKX_1 = \measuredangle KQX_1$. In other words $\triangle QKX_1$ is an isosceles triangle in which

$$KX_1 = QX_1. \tag{19}$$

Angles PKX_1, X_1KQ, and QKN are complementary to 180°.

Therefore, $\measuredangle PKX_1 = 180° - \alpha - 90° = 90° - \alpha$. It follows that $\triangle PKX_1$ is an isosceles triangle in which

$$KX_1 = PX_1. \tag{20}$$

From Equations (19) and (20), we obtain that $PX_1 = QX_1$. In other words, X_1 is the midpoint of segment PQ and therefore points X and X_1 coincide.

In a similar manner we prove that the tangent at point M to the circle inscribing quadrilateral $KLMN$ also passes through the midpoint of segment PQ (i.e., through point X). □

We return to the proof of the theorem.

(a) For quadrilateral $TSVS_1$ (see Figure 10), the data of the lemma holds, namely: angles TSV and TS_1V are right angles, the continuations of sides TS and S_1V intersect at point F, the continuations of sides SV and TS_1 intersect at point G, and X is the midpoint of segment FG. Therefore, straight lines XS and XS_1 are tangent to circle Σ_1 at points S and S_1, respectively.

Similarly, in quadrilateral $SWEI$ there holds the following: angles ISW and IEW are right angles, the continuations of sides IS and EW intersect at point F, the continuations of sides SW and IE intersect at point G, and X is the midpoint of segment FG. Therefore straight lines XS and XE are tangents to circle Σ_2 at points S and E, respectively.

For quadrilateral EW_1S_1J there holds: angles JEW_1 and JS_1W_1 are right angles, the continuations of sides JE and S_1W_1 intersect at point F, the continuations of sides EW_1 and JS_1 intersect at point G, and X is the midpoint of segment FG. Therefore straight lines XS_1 and XE are tangent to circle Σ_3 at points S_1 and E, respectively.

We have thus obtained that (i) straight line XS is a common tangent to circles Σ_1 and Σ_2 at point S, and therefore these circles are internally tangent to each other (circle Σ_2 lies inside circle Σ_1); (ii) straight line XS_1 is a common tangent to circles Σ_1 and Σ_3 at point S_1, and therefore these circles are internally tangent to each other (circle Σ_3 lies inside circle Σ_1); and (iii) straight line XE is a common tangent to circles Σ_2 and Σ_3 at point E, and therefore these circles are externally tangent to each other.

(b) From the proofs of the lemma and of Section *(a)*, it follows that

$$XE = XS = XS_1 = XF = XG.$$

Therefore points E, S, S_1, F, and G lie on circle Σ_4. Since the tangents to circles Σ_1, Σ_2, and Σ_3 pass through the center X of the circle Σ_4, it follows that Σ_4 is perpendicular to each of the circles Σ_1, Σ_2, and Σ_3 (see [1, Theorem 5.51]).

(c) The center Y of the circle Σ_5 lies on the common tangent XE of the circles Σ_2 and Σ_3. Therefore circle Σ_5 is perpendicular to circles Σ_2 and Σ_3.

(d) Segment EX is both a diameter of circle Σ_4 and a radius of circle Σ_5. Therefore, the perpendicular to EX that passes through the point E (which is common to Σ_4 and Σ_5) is a common tangent of these circles. Therefore circles Σ_4 and Σ_5 are tangent to each other. □

Theorem 5. *In addition to the data of Theorem 2, let (see Figure 13): Σ_6 be a circle that passes through points T, O, and O_1; Σ_7 be a circle that passes through points F, P_1, and Q_1; Σ_8 be a circle that passes through points G, P, and Q.*

Then:

- *(a) Circles ω, ψ, Σ_6, Σ_7, and Σ_8 intersect at a single point (point H in Figure 13).*
- *(b) The circles ψ and Σ_7, ω and Σ_8 are perpendicular to each other in pairs.*
- *(c) The angle between circles ω and Σ_7 is equal to the angle between circles ψ and Σ_8.*
- *(d) The angle between circles ω and ψ is equal to angle FTG (the angle between straight lines PQ and P_1Q_1).*

Proof. *(a)* Let H denote the point of intersection of circles ω and ψ. Let us prove that quadrilateral $TOHO_1$ is cyclic. In circle ω there holds $OI = OH$. Therefore triangle OIH is an isosceles triangle. We denote $\measuredangle OIH = \measuredangle OHI = \alpha$, and hence $\measuredangle HOF = 2\alpha$ (see Figure 14).

Quadrilateral $EHGJ$ is inscribed in circle ψ, therefore $\measuredangle HGJ = \measuredangle HEF =: \beta$.

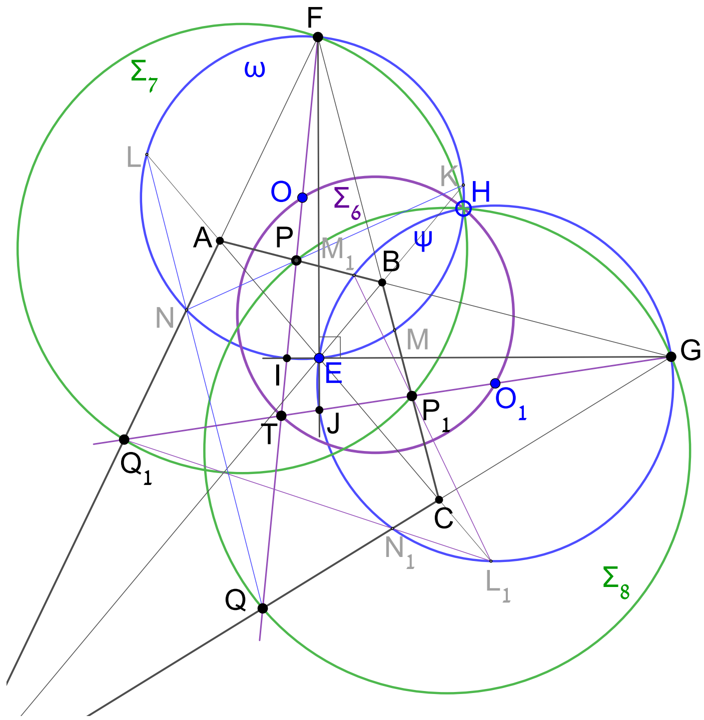

Figure 13: Circles ψ, ω, Σ_6, Σ_7 and Σ_8 intersect at a single point (at point H).

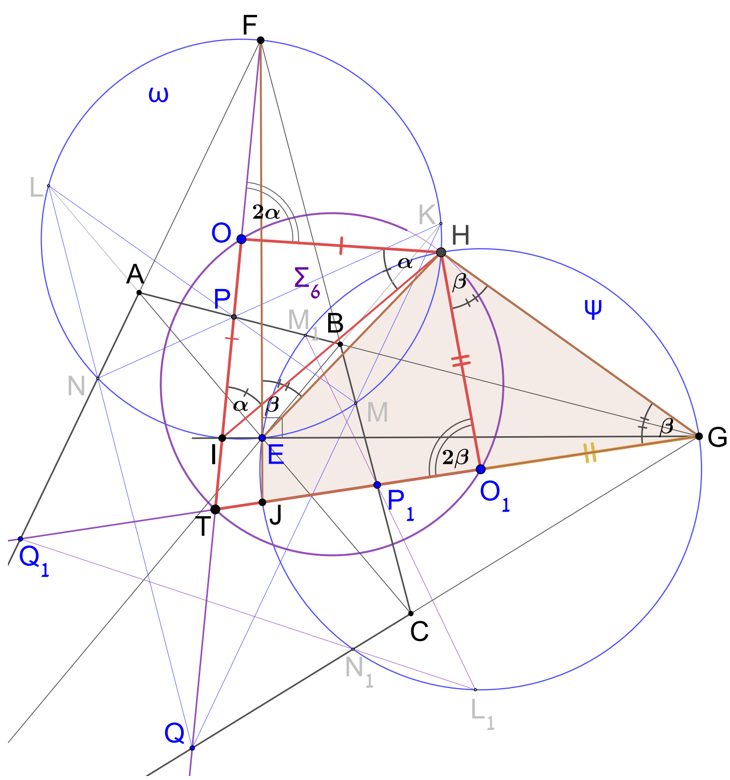

Figure 14: $\omega \cap \psi = H$; $TOHO_1$ is cyclic quadrilateral, therefore also circle Σ_6 passes through point H.

In addition, there holds that $O_1G = O_1H$, meaning that triangle O_1GH is an isosceles triangle and therefore $\measuredangle HO_1J = 2\beta$.

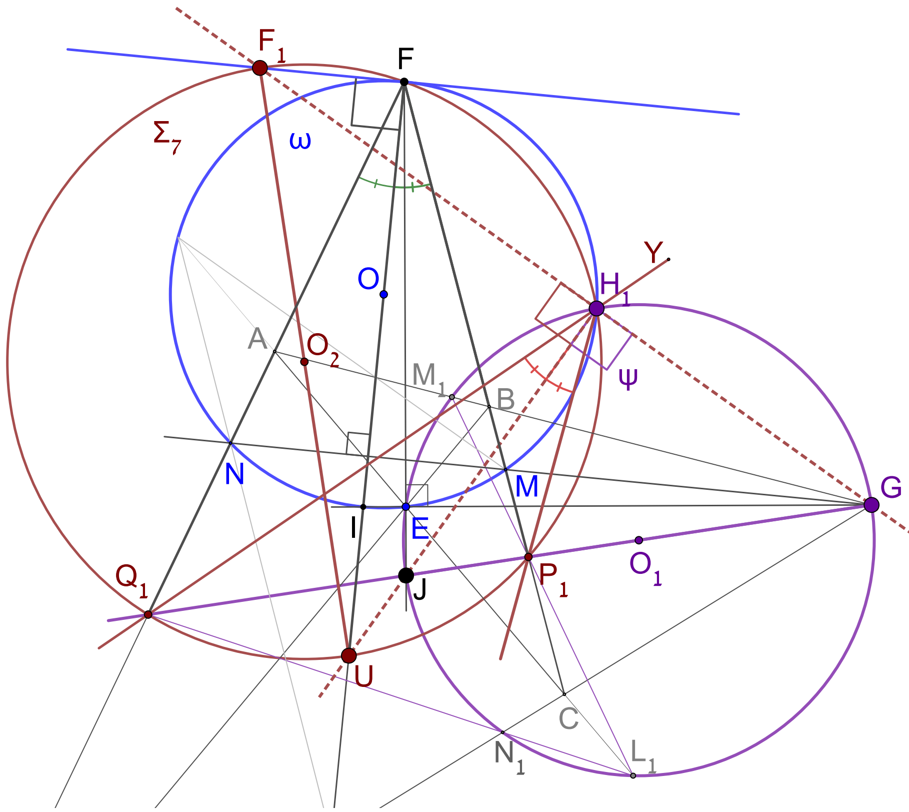

Figure 15: Points G, H_1, and F_1 are collinear. Points H_1, J, and U are collinear.

In circle ω, angles FIH and FEH are inscribed angles subtending the same arc FH. Therefore they are equal, i.e., $\alpha = \beta \Rightarrow 2\alpha = 2\beta$. Result from here $\measuredangle HOF = \measuredangle HO_1T$. It thus follows that quadrilateral $TOHO_1$ is cyclic. Therefore, circle Σ_6, which passes through the three vertices T, O, and O_1 of the quadrilateral, must also pass through the fourth vertex, H.

We shall now prove that circles ω, ψ, and Σ_7 intersect at point H.

Let H_1 denote the point of intersection (in addition to F) of circles ω and Σ_7, and let U denote the point of intersection (in addition to F) of straight line FI and circle Σ_7. We perform the following auxiliary construction: we draw a tangent to circle ω at point F, and denote by F_1 the point of its intersection with circle Σ_7 (see Figure 15).

In the geometric state obtained in Figure 15, the following properties hold (see [8, Property 4]):

(i) Points G, H_1, and F_1 are collinear.

(ii) Points H_1, J, and U are collinear.

Segment F_1U is a diameter in circle Σ_7 (because $\measuredangle F_1FU = 90°$). Therefore, angle F_1H_1U subtending the diameter is equal to 90°. Therefore we obtain $\measuredangle F_1H_1J = 90°$.

For $\measuredangle JH_1G$, which is supplementary to angle $\measuredangle F_1H_1J$, there holds that $\measuredangle JH_1G = 90°$.

This means that point H_1 lies on the circle whose diameter is segment JG, i.e., $H_1 \in \psi$. From the fact that circles ω, ψ, and Σ_7 intersect at point H_1, and circles ω and ψ intersect at point H (in addition to E), it follows that circles ω, ψ, and Σ_7 intersect at point H.

We can similarly prove that circles ω, ψ, and Σ_8 intersect at point H.

(b) Segments JG and F_1U are diameters of circles ψ and Σ_7, respectively (see Figure 16). We shall prove that radii O_1H and O_2H of circles ψ and Σ_7, which have the common endpoint

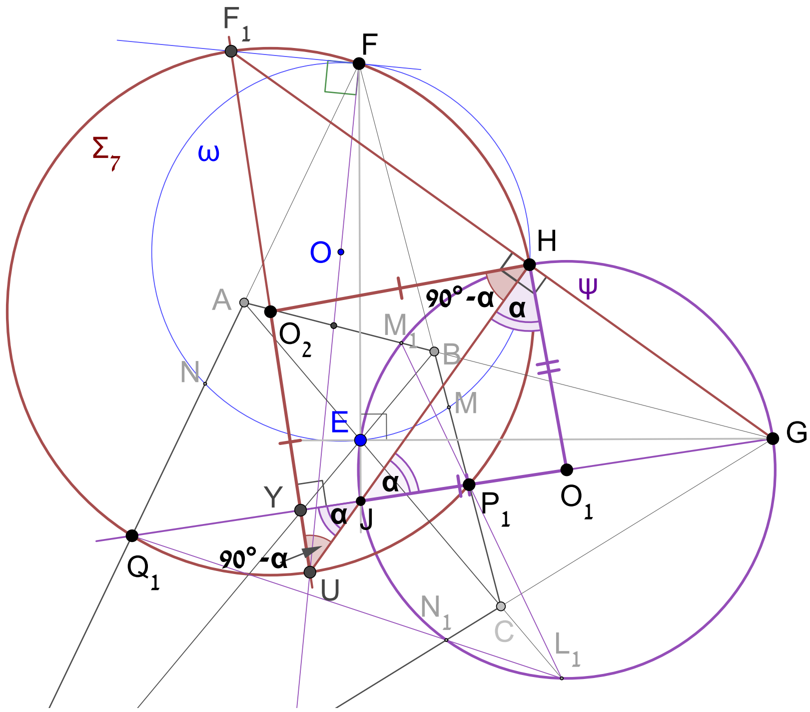

Figure 16: $\measuredangle O_2HO_1$ is an angle between radii O_2H and O_1H of circles Σ_7 and ψ. $\measuredangle O_2HO_1 = (90° - \alpha) + \alpha = 90°$. Circles Σ_7 and ψ are perpendicular.

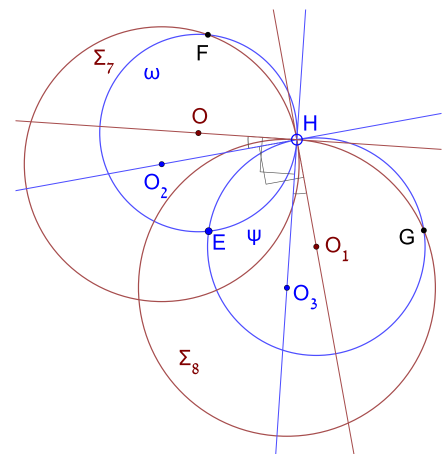

Figure 17: Straight lines O_2H, O_3H, O_1H, and OH are tangent to circles ψ, ω, Σ_7 and Σ_8. $\measuredangle OHO_2 = \measuredangle O_1HO_3$.

H, are perpendicular to each other.

In isosceles triangle O_1JH, there holds: $\measuredangle O_1JH = \measuredangle O_1HJ \coloneqq \alpha$.

Then $\measuredangle Q_1JU = \alpha$ (because points J, H, and U are collinear).

In circle Σ_7, point U is the midpoint of arc P_1Q_1. Therefore diameter F_1U is perpendicular to chord P_1Q_1. It thus follows that in right-angled triangle YJU, $\measuredangle JUY = 90° - \alpha$. Therefore, in isosceles triangle O_2UH, there holds that $\measuredangle O_2HU = \measuredangle O_2UH = 90° - \alpha$.

Finally, we obtain $\measuredangle O_2HO_1 = \measuredangle O_2HU + \measuredangle UHO_1 = 90° - \alpha + \alpha = 90°$. In other words, radii O_1H and O_2H of circles ψ and Σ_7 are perpendicular to each other and therefore circles ψ and Σ_7 are perpendicular to each other (see [9]).

We can similarly prove that circles ω and Σ_8 are perpendicular to each other.

(c) From Section *(b)* above, straight lines O_2H, O_3H, O_1H, and OH are tangent to circles ψ, ω, Σ_7 and Σ_8, respectively (see Figure 17). Therefore, angles $\measuredangle O_1HO_3$ and $\measuredangle O_1HO_2$ are the angles between circles ω and Σ_7 and between circles ψ and Σ_8, respectively.

For these angles there holds: $\measuredangle O_1HO_3 = \measuredangle O_1HO_2 - \measuredangle O_2HO_3 = 90° - \measuredangle O_2HO_3$, $\measuredangle OHO_2 = \measuredangle OHO_3 - \measuredangle O_2HO_3 = 90° - \measuredangle O_2HO_3$.

Therefore, $\measuredangle OHO_2 = \measuredangle O_1HO_3$. In other words, the angle between circles ω and Σ_7 is equal to the angle between circles ψ and Σ_8.

(d) In the geometric state in (see Figure 18) the following are shown:

(i) the three circles, ψ, ω, and Σ_6;

(ii) OE and OH are radii of circle ω and O_1E and O_1H are radii of circle ψ;

(iii) straight lines EX and EY are tangents to circles ω and ψ at E (the common point of these circles).

In cyclic quadrilateral $TOHO_1$, there holds that $\measuredangle OTO_1 + \measuredangle OHO_1 = 180°$. Based on the

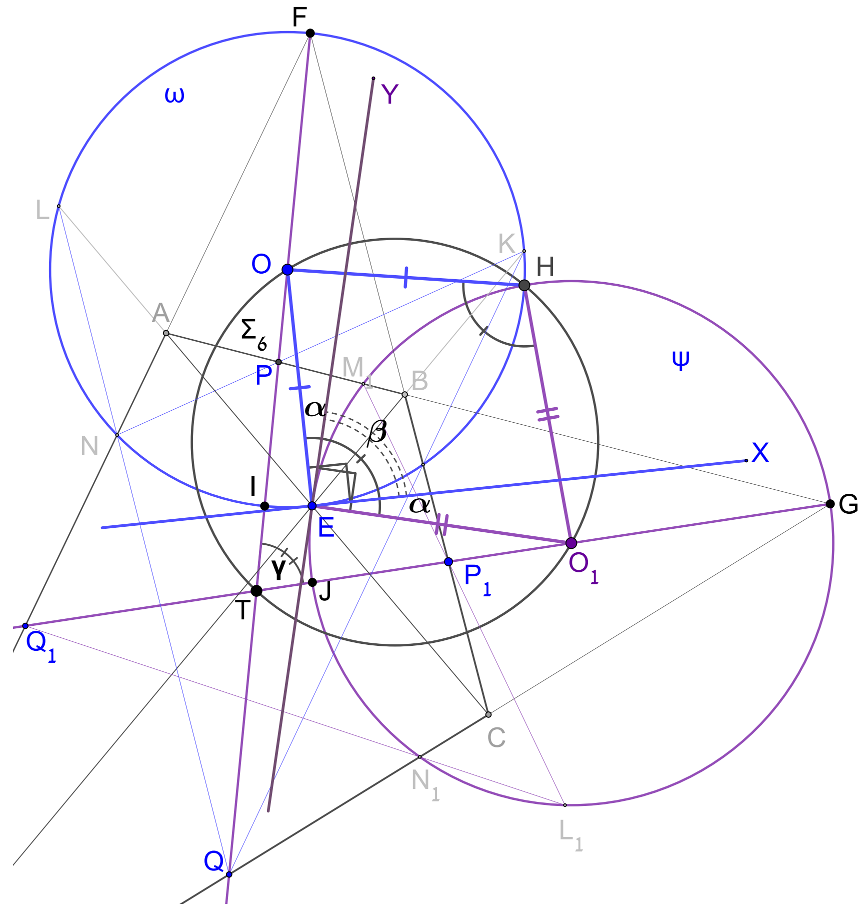

Figure 18: $\measuredangle OHO_1 = \measuredangle OEO_1$ and $\measuredangle OTO_1 + \measuredangle OEO_1 = 180°$. $\gamma + \alpha + \beta + \alpha = 180°$ and $\alpha + \beta = 90°$. Therefore $\beta = \gamma$.

equalities $OE = OH$ and $O_1E = O_1H$, it follows that quadrilateral $EOHO_1$ is a kite in which $\measuredangle OHO_1 = \measuredangle OEO_1$. Therefore there holds that

$$\measuredangle OTO_1 + \measuredangle OEO_1 = 180°. \tag{21}$$

$\measuredangle OEX$ and $\measuredangle O_1EY$ are right angles (an angle between a tangent and a radius to the point of tangency). It thus follows that $\measuredangle OEY = \measuredangle O_1EX =: \alpha$.

We denote $\measuredangle YEX = \beta$ and $\measuredangle OTO_1 = \gamma$. According to this notation, from Equation (21) it follows that

$$\gamma + 2\alpha + \beta = 180° \tag{22}$$

On the other hand, from the equality $\measuredangle OEX = \measuredangle OEY + \measuredangle YEX = \alpha + \beta = 90°$, it follows that

$$2\alpha + 2\beta = 180° \tag{23}$$

Comparing equalities 22 and 23, it follows that $\beta = \gamma$, where β is the angle between the tangents to circles ω and ψ at their point of intersection, E, and that γ is the angle between lines PQ and P_1Q_1. □

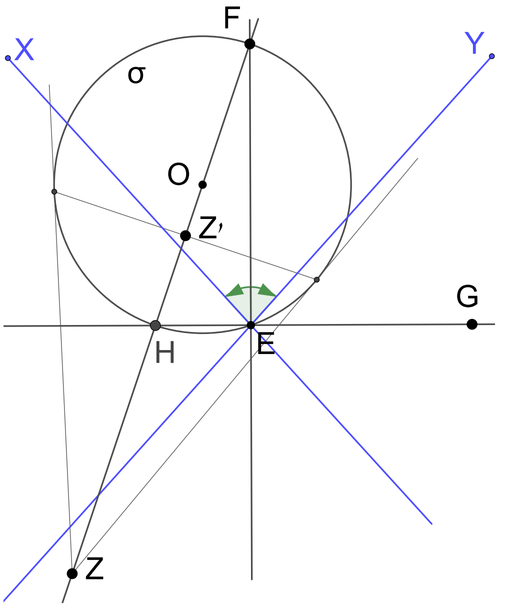

Figure 19: Steps 1–5 of construction.

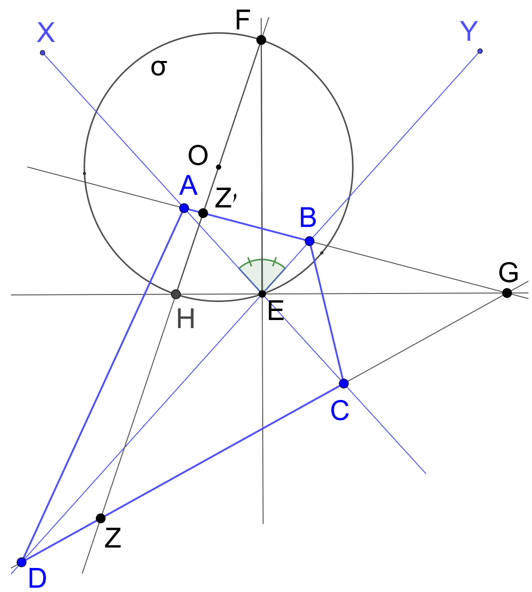

Figure 20: Steps 6–8 of construction.

4 Method for Constructing the Coordinated Quadrilateral

Stages of construction

1. Let σ be an arbitrary circle in which FH is a diameter and FE is a chord.
2. Draw straight lines FH, FE, and HE (see Figure 19).
3. Through point E draw straight lines EX and EY, so that the following condition holds $\measuredangle FEX = \measuredangle FEY$.
4. Let G be a point on straight line HE that lies outside circle σ, and let Z be a point on ray FH that lies outside circle σ.
5. Draw a point, Z', that corresponds to point Z by inversion relative to circle σ.
6. Draw straight lines GZ and GZ' through point G (see Figure 20).
7. Label the points of intersection of straight line GZ with straight lines EX and EY with C and D, respectively. Similarly, label the points of intersection of straight line GZ' with straight lines EX and EY with A and B, respectively.
8. Connect the following pairs of points using segments: A and B, B and C, C and D, D and A.

Quadrilateral $ABCD$ is a coordinated quadrilateral.

Proof that the quadrilateral constructed in accordance with Steps 1–8 above is a coordinated quadrilateral First, we must prove that the continuations of the sides AD and BC intersect at point F.

Step 3 of the construction assures that segment EF bisects angle AEB in $\triangle AEB$. In addition, given that $\measuredangle FEH = 90°$ (an inscribed angle subtending the diameter), it follows that $EV \perp EG$ (see Figure 21). Therefore, segment EG bisects angle BEC, which is exterior to $\triangle AEB$. Therefore, points A, V, B, and G constitute a harmonic quadruple.

In a similar manner, we prove that points D, W, C, and G also constitute a harmonic quadruple.

We obtained, that on each of the straight lines GA and GD (which intersect at point G) there are three additional points: A, V, B and D, W, C respectively, and the following

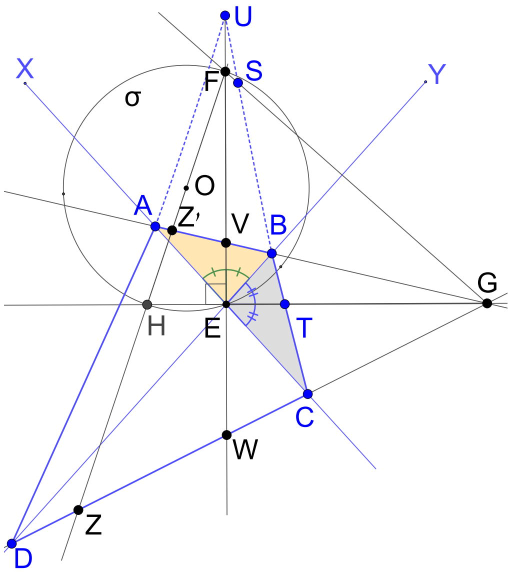

Figure 21: By the assumption, straight lines AD and BC intersect at the point belonging to the straight line FE (at point U, where $U \neq F$).

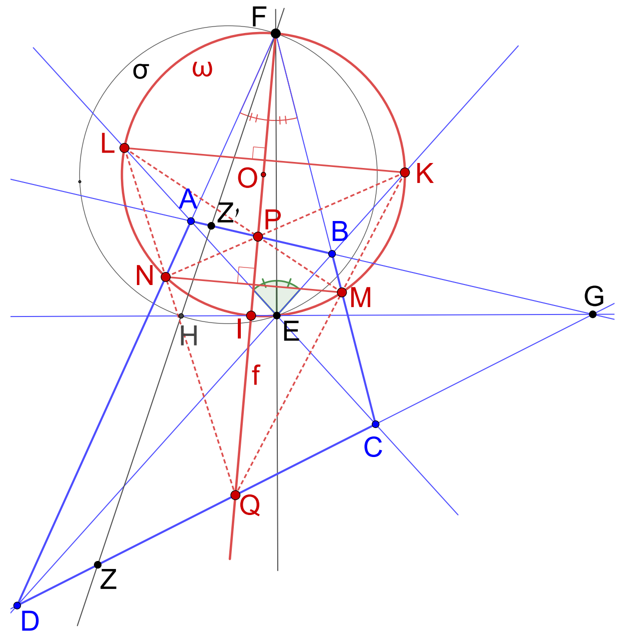

Figure 22: Te bisector f of angle AFB intersects straight lines HE, AB, and CD at I, P, and Q respectively. P and Q are Pascal points formed by the circle ω whose diameter is segment FI.

equality holds:

$$(A, B; V, G) = (D, C; W, G)(= -1).$$

Therefore straight lines DA, WV, and CB intersect at a single point (see [10, Exercise 233]), which is equivalent to the fact that straight lines AD and BC intersect at the point belonging to the straight line FE.

We denote the point of intersection of AD and BC by U. Let us assume that U is not the same as F. Without restricting the generality, let us assume that U lies in the continuation of chord FE (see Figure 21).

From Step 5 above, points Z and Z' belong to the straight line passing through diameter FH of circle σ, and they transform one into the other by inversion relative to this circle. Therefore, points Z, H, Z', and F constitute a harmonic quadruple. Therefore the four straight lines GZ, GH, GZ', and GF constitute a harmonic quadruple of straight lines passing through point G (see [10, Exercise 200]).

We denote by T and S the points of intersection of straight line BC with straight lines GH and GF. From the assumption, it is clear that $S \neq U$ (see Figure 21). Since points C, T, B, and S are the four points of intersection of straight line BC with the above harmonic quadruple of straight lines passing through point G, these points constitute a harmonic quadruple.

On the other hand, in triangle CEB, segment ET bisects angle CEB and segment EU bisects angle BEA, which is exterior to this triangle. Therefore points C, T, B, and U also constitute a harmonic quadruple.

To summarize: on straight line BC there are two harmonic quadruples S, B, T, C and U, B, T, C, which coincide at three points, and therefore must also coincide at the fourth point, in other words $S = U$. This contradicts with $S \neq U$, thereby showing that assumption $F \neq U$ is false. From this, it follows that straight lines AD and BC intersect at point F.

Let f denote the bisector of angle AFB (see Figure 22).

Now we prove that f passes through Pascal points P and Q (formed using the circle ω on sides AB and CD) and through O, the center of circle ω.

Let I, P, and Q denote the points of intersection of f with straight lines HE, AB, and CD, respectively (see Figure 22). Consider circle ω passing through points F, I, and E (in this circle FI is a diameter and FE is a chord). We denote by M and N the points of intersection of circle ω with sides BC and AD, respectively, and by K and L the points of intersection of circle ω with the continuations of diagonals BD and AC, respectively. Since $\widehat{NI} = \widehat{IM}$ (because FI bisects angle NFM), and since $\widehat{LF} = \widehat{FK}$ (because EF bisects angle LEK), it follows that FI is a mid-perpendicular of segments MN and KL. Therefore quadrilateral $KLMN$ is an isosceles trapezoid, and therefore the point of intersection of diagonals KN and LM and the continuations of legs KM and LN belongs to straight line FI (that is, to f).

On the other hand, based on the fundamental theorem of the Pascal point theory on the sides of the quadrilateral (see [2]), straight lines KN and LM intersect on side AB, and straight lines KM and LN intersect on side CD of quadrilateral $ABCD$. These points of intersection are the Pascal points formed by circle ω.

We have thus obtained that P and Q, the points of intersection of straight line FI with sides AB and CD, respectively, are also the points of intersection of straight lines KN and LM, and KM and LN, respectively. Therefore, points P and Q are the Pascal points formed using circle ω.

To summarize, for quadrilateral $ABCD$, in which the continuations of sides BC and AD intersect at point F, the diagonals intersect at point E, and circle ω (passing through points F, E, and I) forms Pascal points P and Q on sides AB and CD, there holds: The center O of ω and points F, P, and Q belong to straight line FI.

Therefore, by definition, $ABCD$ is a coordinated quadrilateral Q.E.D.

References

[1] H. S. M. Coxeter and S. L. Greitzer: *Geometry revisited*, vol. 19. Mathematical Association of America, 1967.

[2] D. Fraivert: *The theory of a convex quadrilateral and a circle that forms "Pascal points"-the properties of "Pascal points" on the sides of a convex quadrilateral.* Journal of Mathematical Sciences: Advances and Applications **40**, 1–34, 2016. http://dx.doi.org/10.18642/jmsaa_7100121666.

[3] D. Fraivert: *The theory of an inscribable quadrilateral and a circle that forms Pascal points.* Journal of Mathematical Sciences: Advances and Applications **42**, 81–107, 2016. http://dx.doi.org/10.18642/jmsaa_7100121742.

[4] D. Fraivert: *Properties of a Pascal points circle in a quadrilateral with perpendicular diagonals.* Forum Geometricorum **17**, 509–526, 2017. https://forumgeom.fau.edu/FG2017volume17/FG201748.pdf.

[5] D. Fraivert: *Properties of the Tangents to a Circle that Forms Pascal Points on the Sides of a Convex quadrilateral.* Forum Geometricorum **17**, 223–243, 2017. https://forumgeom.fau.edu/FG2017volume17/FG201726.pdf.

[6] D. Fraivert: *Pascal-points quadrilaterals inscribed in a cyclic quadrilateral.* The Mathematical Gazette **103**(557), 233–239, 2019. doi: 10.1017/mag.2019.54.

[7] D. Fraivert: *A Set of Rectangles Inscribed in an Orthodiagonal Quadrilateral and Defined by Pascal-Points Circles.* Journal for Geometry and Graphics **23**, 5–27, 2019. https://www.heldermann.de/JGG/JGG23/JGG231/jgg23002.htm.

[8] D. Fraivert: *Properties of Harmonic Quadruples That Transform One Into the Other by Perspective Projection Whose Center Lies at a Point on a Circle.* Global Journal of Advanced Research on Classical and Modern Geometries **12**(2), 304–315, 2023.

[9] D. Fraivert and M. Stupel: *Necessary and sufficient conditions for orthogonal circles.* International Journal of Mathematical Education in Science and Technology **53**(10), 2837–2848, 2022. doi: 10.1080/0020739X.2021.1945153.

[10] J. Hadamard: *Lessons in geometry: plane geometry*, vol. 1. American Mathematical Society Providence, 2008.

[11] Z. Skopets: *Geometric miniatures.* Moscow, Russia: Prosveschenie, 1990.

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