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A Sequence of Malfatti Circles

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Abstract. Given a Euclidean triangle, a smaller triangle is formed by joining the centers of the three Malfatti circles of the given triangle. A sequence of triangles is obtained by continuing this process. The limit of the sequence of triangles is a point. The limits of the inner angles of these triangles are $\pi/3$.

Key Words: Malfatti circles, limit

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1 Introduction

Given a Euclidean triangle T_0, it generates another triangle T_1 via some geometric construction. The triangle T_1 generates a triangle T_2 via the same construction. Continuing this process, one obtains a sequence of triangles $\{T_n\}_{n=0}^{\infty}$. Studying the limit of the sequence usually is an interesting question [2–6].

In this paper, a sequence of triangles is generated involving Malfatti circles.

Italian mathematician Gian Francesco Malfatti (1731–1807) posted the following problem in 1803 [1, Page 147].

To draw within a given triangle three circles each of which is tangent to the other two and to two sides of the triangle.

These three circles are called the *Malfatti circles* of the given triangle.

Let $A_0B_0C_0$ denote a Euclidean triangle. It has three Malfatti circles with the centers A_1, B_1, and C_1 respectively. The circle $\odot A_1$ is tangent to the sides A_0B_0 and A_0C_0; the circle $\odot B_1$ is tangent to the sides A_0B_0 and B_0C_0; and the circle $\odot C_1$ is tangent to the sides A_0C_0 and B_0C_0.

The triangle $A_1B_1C_1$ has three Malfatti circles with the centers A_2, B_2, and C_2 respectively. In general, for $n \geq 0$, the triangle $A_nB_nC_n$ has three Malfatti circles with the centers A_{n+1}, B_{n+1}, and C_{n+1} respectively. The process produces a sequence of triangles $\{A_nB_nC_n\}$.

Theorem 1. *The limit of the sequence of triangles $\{A_nB_nC_n\}$ is a point.*

Theorem 2. *The limits of the inner angles of the sequence of triangles $\{A_nB_nC_n\}$ are $\pi/3$.*

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To verify the statements about the sequence of triangles, we will prove the following equivalent ones about the sequence of radii of Malfatti circles. For $n \geq 1$, suppose the radii of the circles $\odot A_n$, $\odot B_n$ and $\odot C_n$ are p_n, q_n and r_n respectively. Then the sequence $\{(p_n, q_n, r_n)\}$ has the following properties.

Theorem 3.

$$\lim_{n\to\infty}(p_n, q_n, r_n) = (0, 0, 0).$$

Theorem 4.

$$\lim_{n\to\infty} p_n : q_n : r_n = 1 : 1 : 1.$$

2 Properties of Malfatti Circles

To prove the theorems, we need to establish some basic properties of Malfatti circles. Suppose a triangle ABC has three Malfatti circles $\odot A'$, $\odot B'$ and $\odot C'$. The circle $\odot A'$ is tangent to the sides AB and AC; the circle $\odot B'$ is tangent to the sides AB and BC; and the circle $\odot C'$ is tangent to the sides AC and BC.

The lengths of the sides of the triangle ABC are $|BC| = a, |AC| = b$ and $|AB| = c$. The circles $\odot A'$, $\odot B'$, and $\odot C'$ have radii p, q, and r. Suppose that the inscribed circle of the triangle ABC is $\odot O$ and its radius is ρ.

Lemma 5. $\max\{p, q, r\} < \rho$.

Proof. Any Malfatti circle is the incircle of a triangle which has sides parallel to the triangle ABC and is strictly contained in the triangle ABC. □

Lemma 6. *If $a > b$, then*

$$\frac{a+c-b}{b+c-a} > \left(\frac{q}{p}\right)^{1+\alpha} > 1, \quad \forall\alpha \in (0, \sqrt{2}).$$

Proof. The radii of Malfatti circles can be calculated by [7, Formula (3)])

$$p = \frac{(1+\tan\frac{B}{4})(1+\tan\frac{C}{4})}{(1+\tan\frac{A}{4})} \cdot \frac{\rho}{2},$$
$$q = \frac{(1+\tan\frac{C}{4})(1+\tan\frac{A}{4})}{(1+\tan\frac{B}{4})} \cdot \frac{\rho}{2},$$
$$r = \frac{(1+\tan\frac{A}{4})(1+\tan\frac{B}{4})}{(1+\tan\frac{C}{4})} \cdot \frac{\rho}{2}.$$

If $a > b$, then $A > B$. The above formulas imply $p < q$.

Since O is on the angle-bisector of the angle A,

$$\rho = \frac{1}{2}(b+c-a)\tan\frac{A}{2}.$$

Since O is on the angle-bisector of the angle B,

$$\rho = \frac{1}{2}(a+c-b)\tan\frac{B}{2}.$$

The inequality

$$\frac{a+c-b}{b+c-a} > \left(\frac{q}{p}\right)^{1+\alpha}$$

is equivalent to

$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}} > \left(\frac{q}{p}\right)^{1+\alpha} = \frac{(1+\tan\frac{A}{4})^{2+2\alpha}}{(1+\tan\frac{B}{4})^{2+2\alpha}},$$

which is equivalent to

$$\frac{(1+\tan\frac{B}{4})^{2+2\alpha}}{\tan\frac{B}{2}} > \frac{(1+\tan\frac{A}{4})^{2+2\alpha}}{\tan\frac{A}{2}}, \quad \text{for } 0 < B < A < \pi,$$

which can be derived from the fact that

$$\frac{(1+\tan x)^{2+2\alpha}}{\tan(2x)} = \frac{(1+\tan x)^{3+2\alpha}(1-\tan x)}{2\tan x}$$

is a decreasing function as $x \in (0, \pi/4)$.

It is enough to show that

$$f(y) = \frac{(1+y)^{3+2\alpha}(1-y)}{2y}$$

is a decreasing function as $y \in (0,1)$.

Note that $f(y) > 0$ as $y \in (0,1)$. And

$$\frac{f'(y)}{f(y)} = (\ln f(y))' = \frac{3+2\alpha}{1+y} - \frac{1}{1-y} - \frac{1}{y} = \frac{-(3+2\alpha)y^2 + (2+2\alpha)y - 1}{(1+y)(1-y)y}.$$

Since the discriminant of the numerator $g(y) = -(3+2\alpha)y^2 + (2+2\alpha)y - 1$ is $\Delta = (2+2\alpha)^2 - 4(3+2\alpha) = 4\alpha^2 - 8 < 0$, $\forall \alpha \in (0, \sqrt{2})$, the numerator $g(y) < 0$, $\forall y \in \mathbb{R}$. Hence $f'(y) < 0$ as $y \in (0,1)$. Therefore $f(y)$ is decreasing as $y \in (0,1)$. □

3 Proof of Theorems

Proof of Theorem 3. Applying Lemma 5

$$\max\{p,q,r\} < \rho = \frac{1}{2}\sqrt{\frac{(b+c-a)(a+c-b)(a+b-c)}{a+b+c}}$$

to the sequence of triangles $\{A_nB_nC_n\}$, we have

$$\begin{aligned}
\max\{p_n, q_n, r_n\} &< \frac{1}{2}\sqrt{\frac{2p_{n-1}2q_{n-1}2r_{n-1}}{2(p_{n-1}+q_{n-1}+r_{n-1})}} \\
&= \frac{1}{2}\sqrt{\frac{4p_{n-1}q_{n-1}r_{n-1}}{p_{n-1}+q_{n-1}+r_{n-1}}} \\
&= \sqrt{\frac{p_{n-1}q_{n-1}r_{n-1}}{p_{n-1}+q_{n-1}+r_{n-1}}} \\
&\le \sqrt{\frac{p_{n-1}q_{n-1}r_{n-1}}{3(p_{n-1}q_{n-1}r_{n-1})^{\frac{1}{3}}}} \\
&= \frac{(p_{n-1}q_{n-1}r_{n-1})^{\frac{1}{3}}}{\sqrt{3}}.
\end{aligned}$$

The above inequality is independent of the index. Replacing the index n by $n-1$, we get

$$\begin{aligned} & \max\{p_{n-1}, q_{n-1}, r_{n-1}\} < \frac{(p_{n-2}q_{n-2}r_{n-2})^{\frac{1}{3}}}{\sqrt{3}} \\ \Longrightarrow \quad & p_{n-1}q_{n-1}r_{n-1} < \frac{p_{n-2}q_{n-2}r_{n-2}}{3\sqrt{3}} \\ \Longrightarrow \quad & \frac{(p_{n-1}q_{n-1}r_{n-1})^{\frac{1}{3}}}{\sqrt{3}} < \frac{(p_{n-2}q_{n-2}r_{n-2})^{\frac{1}{3}}}{(\sqrt{3})^2}. \end{aligned}$$

Therefore

$$\begin{aligned} \max\{p_n, q_n, r_n\} &< \frac{(p_{n-1}q_{n-1}r_{n-1})^{\frac{1}{3}}}{\sqrt{3}} \\ &< \frac{(p_{n-2}q_{n-2}r_{n-2})^{\frac{1}{3}}}{(\sqrt{3})^2} \\ &\cdots \\ &< \frac{(p_1q_1r_1)^{\frac{1}{3}}}{(\sqrt{3})^{n-1}}. \end{aligned}$$

So we have $\lim_{n\to\infty}(p_n, q_n, r_n) = 0$. □

Proof of Theorem 4. If $|B_0C_0| = |A_0C_0|$, then $p_1 = q_1$. Therefore $|B_1C_1| = |A_1C_1|$ which implies $p_2 = q_2$. In general, $p_n = q_n$ for any $n \geq 1$. Thus $\lim_{n\to\infty} p_n : q_n = 1 : 1$.

If $|B_0C_0| > |A_0C_0|$, Lemma 6 implies that $q_1 > p_1$. Then

$$|B_1C_1| = q_1 + r_1 > p_1 + r_1 = |A_1C_1|.$$

Applying Lemma 6 to the triangle $A_1B_1C_1$, we have $q_2 > p_2$ and

$$\frac{q_1}{p_1} = \frac{|A_1B_1| + |B_1C_1| - |A_1C_1|}{|A_1B_1| + |A_1C_1| - |B_1C_1|} > \left(\frac{q_2}{p_2}\right)^{1+\alpha}, \quad \forall \alpha \in (0, \sqrt{2}).$$

Continuing this argument, we can see that

$$\left(\frac{q_1}{p_1}\right)^{(\frac{1}{1+\alpha})^{n-1}} > \frac{q_n}{p_n} > 1.$$

Therefore

$$1 = \lim_{n\to\infty} \left(\frac{q_1}{p_1}\right)^{(\frac{1}{1+\alpha})^{n-1}} \geq \lim_{n\to\infty} \frac{q_n}{p_n} \geq 1.$$

Hence

$$\lim_{n\to\infty} \frac{q_n}{p_n} = 1.$$

If $|B_0C_0| < |A_0C_0|$, consider the sequence $\{\frac{p_n}{q_n}\}$. Same arguments show that its limit for $n \to \infty$ equals 1. □

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