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Central and Twin Tetrahedra

Hidefumi Katsuura

San Jose State University, San Jose, USA
hidefumi.katsuura@sjsu.edu

Abstract. Given a tetrahedron T, the tetrahedron T' constructed by connecting the four centroids of its faces is called the *central tetrahedron* of T. A tetrahedron T can be inscribed in a parallelepiped W so that the edges of T are the diagonals of the faces of W. By drawing the remaining six diagonals on the faces of the parallelepiped W, we obtain a new tetrahedron $T^\star$, and call it the *twin tetrahedron* of T. Let $S^\star$ and $S^{\star\prime}$ be the circumcenters of $T^\star$ and $T^{\star\prime}$, respectively. We will prove that all tetrahedra T, T', $T^\star$, and $T^{\star\prime}$ have the centroid in common, say P, and the five points S, $S^{\star\prime}$, P, S', and $S^\star$ are collinear in this order such that $\overrightarrow{S'S^\star} = 2\overrightarrow{PS'}$, $\overrightarrow{SP} = 3\overrightarrow{PS'}$, $\overrightarrow{SS'} = 2\overrightarrow{S'S^\star}$, and $\overrightarrow{SS^\star} = 3\overrightarrow{S'S^\star}$. Moreover, we prove that (1) T' and $T^{\star\prime}$ are twins, and (2) if the tetrahedron T is orthocentric, then T, T', $T^\star$, $T^{\star\prime}$ are orthocentric with orthocenters $S^\star$, $S^{\star\prime}$, S, and S', respectively.

Key Words: central tetrahedron, twin tetrahedron, centroid, circumcenter, orthocentric tetrahedron, orthocenter

MSC 2020: 51M04

1 Introduction

Let $\mathcal{V}_s$ be the volume of a sphere, $\mathcal{V}_{cy}$ the volume of the cylinder that tangentially contains the sphere, and $\mathcal{V}_{co}$ the volume of the circular cone with the same base as the cylinder and inscribed in the cylinder as in Figure 1. About 2200 years ago, Archimedes of Syracuse discovered that the ratio of the volumes equals $\mathcal{V}_{co} : \mathcal{V}_s : \mathcal{V}_{cy} = 1 : 2 : 3$. (For additional $1 : 2 : 3$ relations, see [6].)

Euler in 1767 discovered a $1 : 2 : 3$ relation as in the following famous theorem.

Theorem 1 (Euler)**.** *In any triangle, the orthocenter* (H)*, the centroid* (G)*, and the circumcenter* (O) *are collinear, with* $\overline{HG} = 2(\overline{GO})$ *and* $\overline{HO} = 3(\overline{GO})$*.*

The line OH is called the *Euler line* of a triangle, and William Dunham dedicates almost the entirety of Chapter 7 to this theorem in his book [4]. Unlike for a triangle, the four altitudes of a tetrahedron do not usually concur, so an "orthocenter" for a tetrahedron does not exist in general. Hence, Theorem 1 does not extend to a general tetrahedron.

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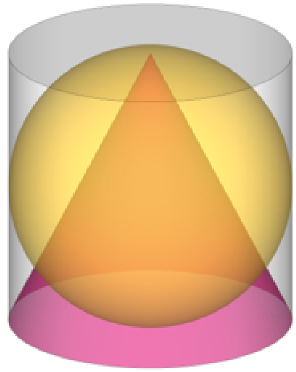

Figure 1: $\mathcal{V}_{co} : \mathcal{V}_s : \mathcal{V}_{cy} = 1 : 2 : 3$.

However, a theorem related to Theorem 1 concerning the *nine-point circle* of a triangle was discovered by Poncelet and C. B. Brianchon in 1821, and again by K. W. Feuerbach a year later (see page 147 of [4]). The statement of the theorem can be found on page 147 of [4] or in section 1.8 of [3]. A succinct proof of this theorem and of Theorem 1 can be found on pages 19–20 of [2]. We are interested in the simplified version of this theorem as stated in the next theorem.

Theorem 2. *Let A', B', and C' be the midpoints of the edges BC, AC, and AB, respectively, of a triangle ABC. If O and O' are the circumcenters of the triangles ABC and $A'B'C'$, respectively, and if H is the orthocenter of the triangle ABC, then O' is the midpoint of the segment OH.*

Hence, with some computations using the two Theorems 1 and 2, we have the next corollary that resembles Theorem 1.

Corollary 1. *If O and O' are the circumcenters of the triangles ABC and $A'B'C'$, respectively, and if G is the centroid of the triangle ABC, then O, G, O' are collinear points with $\overline{OG} = 2(\overline{GO'})$, and $\overline{OO'} = 3(\overline{GO'})$.*

Since Corollary 1 does not mention the orthocenter, we were intrigued by the thought of finding a related theorem on a tetrahedron by interpreting the points A', B', and C' in Theorem 2 to be the centroids of the edges BC, AC, and AB, respectively. And we investigated a tetrahedron.

Definition 1. We denote a tetrahedron $ABCD$ by T throughout this paper. Let A', B', C', D' be the centroids of the faces BCD, ACD, ABD, and ABC, respectively, of a tetrahedron $ABCD$. The tetrahedron $A'B'C'D'$ is called the *central tetrahedron* of the tetrahedron T, and we denote the central tetrahedron $A'B'C'D'$ by T'. The circumcenters of T and T' are denoted by S and S', respectively. The segments AA', BB', CC' and DD' intersect. This intersection, denoted by P, is the *centroid* of the tetrahedron T.

Definition 2. Let us inscribe the tetrahedron $ABCD$ into a parallelepiped so that edges of the tetrahedron are the diagonals of the six faces of the parallelepiped. We label the diagonally opposite vertices of A, B, C, D of the parallelepiped by A^*, B^*, C^*, and D^*, respectively. (See Figure 2.) Hence, for example, the faces AD^*BC^* and A^*DB^*C of the parallelepiped are determined by the planes parallel to the lines AB and CD. We will call the tetrahedron $A^*B^*C^*D^*$ the *twin* of the tetrahedron $ABCD$, and we denote it by T^*. They are mirror images of each other. Since the twin of the tetrahedron $A^*B^*C^*D^*$ is the tetrahedron $ABCD$, we have $T^{**} = T$. Let S^* and $S^{*\prime}$ be the circumcenters of the twin tetrahedra T^* and its central tetrahedron $T^{*\prime}$, respectively.

We will show that P, the centroid of T, is also the centroid of T', T^*, and $T^{*\prime}$, and prove that S, P, and S' are collinear points such that $\overrightarrow{SP} = 3\overrightarrow{PS'}$ in Theorem 3. In Theorem 4, we will prove that P is the midpoint of the segment SS^* so that $\overrightarrow{SS'} = 2\overrightarrow{S'S^*}$ and $\overrightarrow{SS^*} = 3\overrightarrow{S'S^*}$. In addition, $S^{*\prime}$ is the midpoint of the segment SS', and $\overrightarrow{SS^{*\prime}} = \overrightarrow{S'S^*}$. (See Figure 3.) As a corollary, we will prove that T' and $T^{*\prime}$ are twin tetrahedra, i.e., $T^{*\prime} = T'^*$ (Corollary 2).

Definition 3. If four altitudes of a tetrahedron intersect, the tetrahedron is said to be *orthocentric*, and the intersection is called the *orthocenter*.

In Section 3, we will prove that if the tetrahedron T is orthocentric, then T', T^*, and $T^{*\prime}$ are all orthocentric tetrahedra. And the orthocenters of the tetrahedra T, T', T^*, and $T^{*\prime}$ are S^*, $S^{*\prime}$, S, and S', respectively.

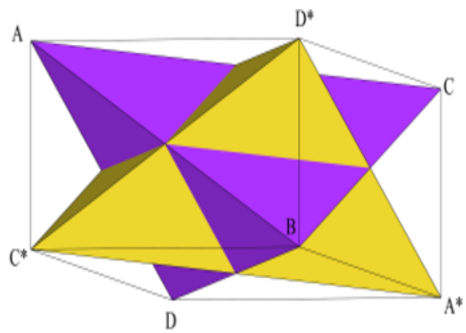

Figure 2: Two tetrahedra T and T^* are shown.

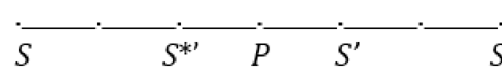

Figure 3: The positions of five points are shown.

2 The Main Theorem

We will prove our theorems, Theorems 3 and 4, using the Cartesian coordinates similar to the way Euler proved Theorem 1 as described in Chapter 7 of [4].

Theorem 3. *Let T be a tetrahedron. Recall T', S, S', and P in Definition 1. Then the following are true:*

(1) The point P is also the centroid of the tetrahedron T'.

(2) The points S, P, and S' are collinear, with $\overrightarrow{SP} = 3\overrightarrow{PS'}$. (See Figure 3, and compare this to Corollary 1.)

Proof. We will use the Cartesian coordinates to prove this theorem. Let T be a tetrahedron $ABCD$. Let $A = (1,0,0)$, $B = (a,b,0)$, $C = (c,d,e)$, $D = (0,0,0)$ for some numbers a, $b \neq 0$, c, d, $e \neq 0$. (See Figure 4).

A calculation $\frac{1}{4}(\overrightarrow{DA} + \overrightarrow{DB} + \overrightarrow{DC} + \overrightarrow{DD}) = \left\langle \frac{1+a+c}{4}, \frac{b+d}{4}, \frac{e}{4} \right\rangle$ proves that $P = \left(\frac{1+a+c}{4}, \frac{b+d}{4}, \frac{e}{4} \right)$ is the coordinate of the centroid of T.[1]

The vertices of the central tetrahedron T' are given by

$$A' = \left(\frac{a+c}{3}, \frac{b+d}{3}, \frac{e}{3}\right), \quad B' = \left(\frac{1+c}{3}, \frac{d}{3}, \frac{e}{3}\right), \quad C' = \left(\frac{1+a}{3}, \frac{b}{3}, 0\right), \quad D' = \left(\frac{1+a+c}{3}, \frac{b+d}{3}, \frac{e}{3}\right).$$

[1]We denote vector coordinates by angled brackets and point coordinates by round brackets.

A calculation $\frac{1}{4}\left(\overrightarrow{DA'}+\overrightarrow{DB'}+\overrightarrow{DC'}+\overrightarrow{DD'}\right)=\left\langle \frac{1+a+c}{4},\frac{b+d}{4},\frac{e}{4}\right\rangle$ proves that $P=\left(\frac{1+a+c}{4},\frac{b+d}{4},\frac{e}{3}\right)$ is also the centroid of T'.

The midpoints E, I, J of the edges DA, DB, and DC, respectively, are given by $E=\left(\frac{1}{2},0,0\right)$, $I=\left(\frac{a}{2},\frac{b}{2},0\right)$, $J=\left(\frac{c}{2},\frac{d}{2},\frac{e}{2}\right)$.

Then, equations of the planes through E, I, J normal to the edges DA, DB, and DC, respectively, are given by

$$x=\tfrac{1}{2},\quad a(x-\tfrac{a}{2})+b(y-\tfrac{b}{2})=0,\quad c(x-\tfrac{c}{2})+d(y-\tfrac{d}{2})+e(y-\tfrac{e}{2})=0.$$

The intersection of these three planes is the circumcenter S of the tetrahedron $ABCD$, and we have

$$S=\left(\tfrac{1}{2},\tfrac{a^2+b^2-a}{2b},\tfrac{b(c^2+d^2+e^2-c)-d(a^2+b^2-a)}{2be}\right).$$

Let E', I', J' be the midpoints of the edges $D'A'$, $D'B'$, and $D'C'$, respectively. Then

$$E'=\left(\tfrac{1+2a+2c}{6},\tfrac{b+d}{3},\tfrac{e}{3}\right),\qquad I'=\left(\tfrac{2+a+2c}{6},\tfrac{b+2d}{6},\tfrac{e}{3}\right),\qquad J'=\left(\tfrac{2+2a+c}{6},\tfrac{3b+d}{6},\tfrac{e}{6}\right).$$

Since $\overrightarrow{A'D'}=\frac{1}{3}\langle 1,0,0\rangle$, $\overrightarrow{B'D'}=\frac{1}{3}\langle a,b,0\rangle$, $\overrightarrow{C'D'}=\frac{1}{3}\langle c,d,e\rangle$, the equations of the planes through E', I', J' normal to the edges $D'A'$, $D'B'$, and $D'C'$ are given, respectively, as

$$x=\tfrac{1+2a+2c}{6},\quad a\left(x-\tfrac{2+a+2c}{6}\right)+b\left(y-\tfrac{b+2d}{6}\right)=0,\quad c\left(x-\tfrac{2+2a+c}{6}\right)+d\left(y-\tfrac{2b+d}{6}\right)+e\left(z-\tfrac{e}{6}\right)=0.$$

The intersection of these three planes is the circumcenter S' of the tetrahedron $A'B'C'D'$, and we have

$$S'=\left(\tfrac{1+2a+2c}{6},\tfrac{-a^2+b^2+2bd+a}{6b},\tfrac{b(-c^2-d^2+e^2+c)+d(a^2+b^2-a)}{6be}\right).$$

Since

$$\overrightarrow{SS'}=\left\langle\tfrac{a+c-1}{3},\tfrac{-2a^2-b^2+bd+2a}{3b},\tfrac{b(-2c^2-2d^2-e^2+2c)+d(2a^2+2b^2-2a)}{3be}\right\rangle,$$

a vector equation of the line SS' is given by

$$\begin{aligned}\langle x,y,z\rangle&=\left\langle\tfrac{1}{2},\tfrac{a^2+b^2-a}{2b},\tfrac{b(c^2+d^2+e^2-c)-d(a^2+b^2-a)}{2be}\right\rangle\\&\quad+t\left\langle\tfrac{a+c-1}{3},\tfrac{-2a^2-b^2+bd+2a}{3b},\tfrac{b(-2c^2-2d^2-e^2+2c)+d(2a^2+2b^2-2a)}{3be}\right\rangle,\quad t\in\mathbb{R}.\end{aligned}$$

Here, $\langle x,y,z\rangle_{t=0}=\overrightarrow{DS}$ and $\langle x,y,z\rangle_{t=1}=\overrightarrow{DS'}$.

If we let $t=\frac{3}{4}$ in the vector equation of the line SS', we have

$$\begin{aligned}\langle x,y,z\rangle&=\left\langle\tfrac{1}{2},\tfrac{a^2+b^2-a}{2b},\tfrac{b(c^2+d^2+e^2-c)-d(a^2+b^2-a)}{2be}\right\rangle\\&\quad+\tfrac{3}{4}\left\langle\tfrac{a+c-1}{3},\tfrac{-2a^2-b^2+bd+2a}{3b},\tfrac{b(-2c^2-2d^2-e^2+2c)+d(2a^2+2b^2-2a)}{3be}\right\rangle\\&=\left\langle\tfrac{1+a+c}{4},\tfrac{b+d}{4},\tfrac{e}{4}\right\rangle\\&=\overrightarrow{DP}.\end{aligned}$$

This proves that the points S, P, and S' are collinear, and $\overrightarrow{SP}=3\overrightarrow{PS'}$ since $t=\frac{3}{4}$ in the vector equation of the line SS' gives $\overrightarrow{SP}$. $\square$

Theorem 4. *Recall S^* and $S^{*\prime}$ are the circumcenters of the tetrahedra T^* and $T^{*\prime}$, respectively. Then, the centroid P of the tetrahedron T is the centroid of T^* and $T^{*\prime}$. The points S, $S^{*\prime}$, P, S', and S^* are collinear in this order, with the point P being the midpoint of the segment SS^*, and $\overrightarrow{S'S^*}=\overrightarrow{S^{*\prime}}=\overrightarrow{SS^{*\prime}}$, $\overrightarrow{SS'}=2\overrightarrow{S'S^*}$, and $\overrightarrow{SS^*}=3\overrightarrow{S'S^*}$ (see Figure 3).*

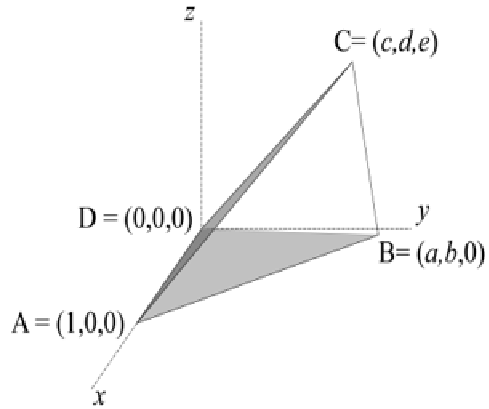

Figure 4: This is to locate points A, B, C, D.

Proof. Let

$$A^* = \left(\tfrac{a+c-1}{2}, \tfrac{b+d}{2}, \tfrac{e}{2}\right),\ \ B^* = \left(\tfrac{1-a+c}{2}, \tfrac{d-b}{2}, \tfrac{e}{2}\right),\ \ C^* = \left(\tfrac{a-c+1}{2}, \tfrac{b-d}{2}, -\tfrac{e}{2}\right),\ \ D^* = \left(\tfrac{a+c+1}{2}, \tfrac{b+d}{2}, \tfrac{e}{2}\right).$$

Then check that

$$\overrightarrow{DA^*} + \overrightarrow{DB^*} = \langle c, d, e\rangle = \overrightarrow{DC}, \quad \overrightarrow{DA^*} + \overrightarrow{DC^*} = \langle a, b, 0\rangle = \overrightarrow{DB},$$
$$\overrightarrow{DB^*} + \overrightarrow{DC^*} = \langle 1, 0, 0\rangle = \overrightarrow{DA}, \quad \text{and} \quad \overrightarrow{DA^*} + \overrightarrow{DB^*} + \overrightarrow{DC^*} = \left\langle \tfrac{a+c+1}{2}, \tfrac{b+d}{2}, \tfrac{e}{2}\right\rangle = \overrightarrow{DD^*}.$$

From these, we can see that the vertices D, A^*, B, C^*, D^*, A, B^*, C form a parallelepiped. Hence, $A^*B^*C^*D^*$ is the twin tetrahedron T^* of T.

The midpoint of D^*A^* is $\left(\frac{a+c}{2}, \frac{b+d}{2}, \frac{e}{2}\right)$, and $\overrightarrow{D^*A^*} = \langle -1, 0, 0\rangle$. So, the normal plane to the edge D^*A^* through its midpoint is

(i) $x = \frac{a+c}{2}$.

The midpoint of D^*B^* is $\left(\frac{c+1}{2}, \frac{d}{2}, \frac{e}{2}\right)$, and $\overrightarrow{D^*B^*} = \langle -a, -b, 0\rangle$. So, the normal plane to the edge D^*B^* through its midpoint is

(ii) $ax + by = \frac{ac+bd+a}{2}$.

The midpoint of D^*C^* is $\left(\frac{a+1}{2}, \frac{b}{2}, 0\right)$, and $\overrightarrow{D^*C^*} = \langle -c, -d, -e\rangle$. So, the normal plane to the edge D^*C^* through its midpoint is

(iii) $cx + dy + ez = \frac{ac+bd+c}{2}$.

By solving the system of equations (i)–(iii), we obtain the circumcenter S^* of the tetrahedron T^* as

$$S^* = \left(\tfrac{a+c}{2}, \tfrac{a+bd-a^2}{2b}, \tfrac{b(c-c^2-d^2+d(a^2+b^2-a)}{2be}\right).$$

From the proof of Theorem 3, the vector equation of the line SS' is

$$\langle x, y, z\rangle = \left\langle \tfrac{1}{2}, \tfrac{a^2+b^2-a}{2b}, \tfrac{b(c^2+d^2+e^2-c)-d(a^2+b^2-a)}{2be}\right\rangle$$
$$+ t\left\langle \tfrac{a+c-1}{3}, \tfrac{-2a^2-b^2+bd+2a}{3b}, \tfrac{b(-2c^2-2d^2-e^2+2c)+d(2a^2+2b^2-2a)}{3be}\right\rangle, \quad t \in \mathbb{R}.$$

If we let $t = \frac{3}{2}$ in this equation, we have

$$\begin{aligned}\langle x, y, z\rangle &= \left\langle \tfrac{1}{2}, \tfrac{a^2+b^2-a}{2b}, \tfrac{b(c^2+d^2+e^2-c)-d(a^2+b^2-a)}{2be}\right\rangle \\ &\quad + \tfrac{3}{2}\left\langle \tfrac{a+c-1}{3}, \tfrac{-2a^2-b^2+bd+2a}{3b}, \tfrac{b(-2c^2-2d^2-e^2+2c)+d(2a^2+2b^2-2a)}{3be}\right\rangle \\ &= \left\langle \tfrac{a+c}{2}, \tfrac{a+bd-a^2}{2b}, \tfrac{b(c-c^2-d^2)+d(a^2+b^2-a)}{2be}\right\rangle.\end{aligned}$$

This shows that S^* is on the line SS', and $\overrightarrow{SS^*} = \frac{3}{2}\overrightarrow{SS'}$.

Since $\frac{1}{4}(\overrightarrow{DA^*} + \overrightarrow{DB^*} + \overrightarrow{DC^*} + \overrightarrow{DD^*}) = \left\langle \frac{1+a+c}{4}, \frac{b+d}{4}, \frac{e}{4} \right\rangle$, P is the centroid of T^* from the proof of Theorem 3.

All things considered, S^* correspond to $t = \frac{3}{2}$ in the vector equation of SS', S' corresponds to $t = 1$, and P corresponds to $t = \frac{3}{4}$. Hence, this proves that P is the midpoint of SS^* since $\frac{3}{4} = \frac{1}{2} \cdot \frac{3}{2}$.

By Theorem 3, P is also the centroid of $T^{*\prime}$. And we can apply Theorem 3 to the tetrahedron T^* in place of the tetrahedron T, and the tetrahedron $T^{*\prime}$ in place of the tetrahedron T'. This allows us to replace (S by S^*) and (S' by $S^{*\prime}$) in Theorem 3(2) to obtain $\overrightarrow{S^*P} = 3\overrightarrow{PS^{*\prime}}$. Since $\overrightarrow{SP} = 3\overrightarrow{PS'}$, and $\overrightarrow{SP} = \overrightarrow{S^*P} = 3\overrightarrow{PS^{*\prime}}$, this shows that $PS^{*\prime} = PS'$. Hence, $S^{*\prime}$ corresponds to $t = \frac{1}{2}$ in the vector equations of SS'. Therefore, we have $\overrightarrow{SS^{*\prime}} = \overrightarrow{S^{*\prime}S'} = \overrightarrow{S'S^*}$ and $\overrightarrow{SS'} = 2\overrightarrow{S'S^*}$, and $\overrightarrow{SS^*} = 3\overrightarrow{S'S^*}$. This completes the proof of statement (1) of this theorem. □

Remark 1. A tetrahedron $ABCD$ is *isosceles* if $AB = CD$, $AC = BD$, and $AD = BC$. The parallelepiped that circumscribed to the isosceles tetrahedra T and T^* is a rectangular box and all the points S, $S^{*\prime}$, P, S', and S^* are identical since it is known that the circumcenter and the centroid of an isosceles tetrahedron are the same (see page 97 of [1]).

Corollary 2. *The tetrahedron $T^{*\prime}$ is the twin tetrahedron of T'. In short, we have $T^{*\prime} = T'^*$.*

Proof. From the proof of Theorem 4, we have

$$A^* = \left(\tfrac{a+c-1}{2}, \tfrac{b+d}{2}, \tfrac{e}{2}\right),\ \ B^* = \left(\tfrac{1-a+c}{2}, \tfrac{d-b}{2}, \tfrac{e}{2}\right),\ \ C^* = \left(\tfrac{a-c+1}{2}, \tfrac{b-d}{2}, -\tfrac{e}{2}\right),\ \ D^* = \left(\tfrac{a+c+1}{2}, \tfrac{b+d}{2}, \tfrac{e}{2}\right).$$

From these, we have

$$A^{*\prime} = \left(\tfrac{a+c+3}{6}, \tfrac{b+d}{6}, \tfrac{e}{6}\right),\ B^{*\prime} = \left(\tfrac{3a+c+1}{6}, \tfrac{3b+d}{6}, \tfrac{e}{6}\right),\ C^{*\prime} = \left(\tfrac{a+3c+1}{6}, \tfrac{b+3d}{6}, \tfrac{3e}{6}\right),\ D^{*\prime} = \left(\tfrac{a+c+1}{6}, \tfrac{b+d}{6}, \tfrac{e}{6}\right).$$

From the proof of Theorem 3, we have

$$A' = \left(\tfrac{a+c}{3}, \tfrac{b+d}{3}, \tfrac{e}{3}\right), \quad B' = \left(\tfrac{1+c}{3}, \tfrac{d}{3}, \tfrac{e}{3}\right), \quad C' = \left(\tfrac{1+a}{3}, \tfrac{b}{3}, 0\right), \quad D' = \left(\tfrac{1+a+c}{3}, \tfrac{b+d}{3}, \tfrac{e}{3}\right).$$

From these, we have

$$\begin{aligned}
\overrightarrow{B'A^{*\prime}} &= \tfrac{1}{6}\langle a+c+3, b+d, e\rangle - \tfrac{1}{6}\langle 2c+2, 2d, 2e\rangle = \tfrac{1}{6}\langle a-c+1, b-d, -e\rangle, \\
\overrightarrow{B'B^{*\prime}} &= \tfrac{1}{6}\langle 3a+c+1, 3b+d, e\rangle - \tfrac{1}{6}\langle 2c+2, 2d, 2e\rangle = \tfrac{1}{6}\langle 3a-c-1, 3b-d, -e\rangle, \\
\overrightarrow{B'C^{*\prime}} &= \tfrac{1}{6}\langle a+3c+1, b+3d, 3e\rangle - \tfrac{1}{6}\langle 2c+2, 2d, 2e\rangle = \tfrac{1}{6}\langle a+c-1, b+d, e\rangle, \\
\overrightarrow{B'D^{*\prime}} &= \tfrac{1}{6}\langle a+c+1, b+d, e\rangle - \tfrac{1}{6}\langle 2c+2, 2d, 2e\rangle = \tfrac{1}{6}\langle a-c-1, b-d, -e\rangle, \\
\overrightarrow{B'A'} &= \tfrac{1}{6}\langle 2a+2c+2, 2b+2d, 2e\rangle - \tfrac{1}{6}\langle 2c+2, 2d, 2e\rangle = \tfrac{1}{6}\langle 2a-2, 2b, 0\rangle, \\
\overrightarrow{B'D'} &= \tfrac{1}{6}\langle 2a+2c+2, 2b+2d, 2e\rangle - \tfrac{1}{6}\langle 2c+2, 2d, 2e\rangle = \tfrac{1}{6}\langle 2a, 2b, 0\rangle, \\
\overrightarrow{B'C'} &= \tfrac{1}{6}\langle 2a+2, 2b, 0\rangle - \tfrac{1}{6}\langle 2c+2, 2d, 2e\rangle = \tfrac{1}{6}\langle 2a-2c, 2b-2d, -2e\rangle.
\end{aligned}$$

Hence, check that

$$\overrightarrow{B'C^{*\prime}} + \overrightarrow{B'D^{*\prime}} = \overrightarrow{B'A'}, \quad \overrightarrow{B'C^{*\prime}} + \overrightarrow{B'A^{*\prime}} = \overrightarrow{B'A'}, \quad \overrightarrow{B'A^{*\prime}} + \overrightarrow{B'D^{*\prime}} = \overrightarrow{B'C'},$$
$$\text{and} \quad \overrightarrow{B'D^{*\prime}} + \overrightarrow{B'A^{*\prime}} + \overrightarrow{B'C^{*\prime}} = \overrightarrow{B'B^{*\prime}}.$$

These four vector equations prove that $B'C^{*\prime}A'D^{*\prime}$, $B'C^{*\prime}D'A^{*\prime}$, and $B'A^{*\prime}C'D^{*\prime}$ are parallelograms, and $B'C^{*\prime}A'D^{*\prime}A^{*\prime}D'B^{*\prime}C'$ is a parallelepiped. Therefore, the twin parallelogram of T' is the tetrahedron $T^{*\prime}$. In other words, we have shown that $T^{*\prime} = T'^*$, $A^{*\prime} = A'^*$, $B^{*\prime} = B'^*$, $C^{*\prime} = C'^*$, $D^{*\prime} = D'^*$. □

3 An Orthocentric Tetrahedron

The next is a known theorem on an orthocentric tetrahedron.

Theorem 5 (See Problem 312 on page 64 of [5].)**.** *A tetrahedron is orthocentric if, and only if, opposite edges of the tetrahedron are mutually perpendicular.*

The next theorem shows that an orthocentric tetrahedron has a collinear property similar to Theorem 1 of Euler.

Theorem 6. *The orthocenter of an orthocentric tetrahedron T is the circumcenter of its twin tetrahedron T^*.*

Proof. We use the same notations as in the proof of Theorem 3. By Theorem 5, since the tetrahedron $ABCD$ denoted by T is orthocentric, we must have

$$\begin{aligned}
&\overrightarrow{DC}\cdot\overrightarrow{AB} = \langle c,d,e\rangle\cdot\langle a-1,b,0\rangle = c(a-1)+bd=0,\\
&\overrightarrow{AC}\cdot\overrightarrow{DB} = \langle c-1,d,e\rangle\cdot\langle a,b,0\rangle = a(c-1)+bd=0, \quad \text{and}\\
&\overrightarrow{DA}\cdot\overrightarrow{BC} = \langle 1,0,0\rangle\cdot\langle c-a,d-b,e\rangle = c-a=0.
\end{aligned}$$

These imply that

(1) $c=a$, and $d=\frac{a-a^2}{b}$, where $b\neq 0, e\neq 0$.

Hence, $A=(1,0,0)$, $B=(a,b,0)$, $C=\left(a,\frac{a-a^2}{b},e\right)$, $D=(0,0,0)$.

Let S^* be the circumcenter of the twin tetrahedron T^* of T. By the proof of Theorem 4, we have

$$S^*=\left(\frac{a+c}{2},\frac{a+bd-a^2}{2b},\frac{b(c-c^2-d^2)+d(a^2+b^2-a)}{2be}\right)=\left(a,\frac{a-a^2}{b},\frac{(a-a^2)(a^2+b^2-a)}{b^2e}\right).$$

(2) Since $\overrightarrow{CS^*}=\left\langle 0,0,\frac{(a-a^2)(a^2+b^2-a)}{b^2e}-e\right\rangle$, it is normal to the plane ABD.

(3) We have $\overrightarrow{AS^*}=\left\langle a-1,\frac{a-a^2}{b},\frac{(a-a^2)(a^2+b^2-a)}{b^2e}\right\rangle=(a-1)\left\langle 1,\frac{-a}{b},\frac{-a(a^2+b^2-a)}{b^2e}\right\rangle=\frac{a-1}{be}\left\langle be,-ae,\frac{a}{b}(a-a^2-b^2)\right\rangle$. A normal vector of the plane BCD is given by $\overrightarrow{DB}\times\overrightarrow{DC}=\langle be,-ae,ad-bc\rangle=\left\langle be,-ae,\frac{a}{b}(a-a^2-b^2)\right\rangle$. Hence, the vector $\overrightarrow{AS^*}$ is normal to the plane BCD.

Similarly, we can see that the vectors $\overrightarrow{BS^*}$ and $\overrightarrow{CS^*}$ are normal to the planes ACD and ABD, respectively. Therefore, S^* is the orthocenter of T. □

Corollary 3. *Suppose T is an orthocentric tetrahedron. Then T', T^*, and $T^{*\prime}$ are all orthocentric tetrahedra. Moreover, the orthocenters of T and T' are S^* and $S^{*\prime}$, respectively. And the orthocenters of T^* and $T^{*\prime}$ are S and S', respectively.*

Proof. Since T' is similar to T, the tetrahedron T' is orthocentric. By Corollary 2, $T^{*\prime}$ is the twin of T'. Hence, the orthocenter of T' is $S^{*\prime}$ by Theorem 6. Since T^* is congruent to T, tetrahedra T^* and $T^{*\prime}$ are orthocentric. Hence, we can see that the orthocenters of T^* and $T^{*\prime}$ are $S^{**}=S$ and $S^{**\prime}=S'$, respectively. □

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