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Equalities and Inequalities in the Excircle Quadrilateral of a Convex Quadrilateral

Ovidiu T. Pop[1], Mario Dalcín[2]

[1]*National College "Mihai Eminescu" Satu Mare, Romania*
ovidiutiberiu@yahoo.com

[2]*'ARTIGAS' Secondary School Teachers Institute-CFE Montevideo, Uruguay*
mdalcin00@gmail.com

Abstract. In this paper, we will demonstrate some identities and inequalities that occur in the quadrilateral determined by the centers of the excircles in a convex quadrilateral.

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1 Introduction

In this section, we recall some known results that occur in quadrilaterals.

In a given convex quadrilateral $ABCD$, we note the lengths of the sides by $a = AB$, $b = BC$, $c = CD$, $d = DA$, A, B, C, D the angle measures, F the area and s to it's semi-perimeter. If the quadrilateral $ABCD$ is cyclic, we note by $\mathcal{C}_{(O,R)}$ the circumscribed circle, where O is the center and R is the radius of this circle. If the quadrilateral $ABCD$ is tangential, we note by $\mathcal{C}_{(I,r)}$ the inscribed circle, where I is the center and r is the radius of this circle. A quadrilateral $ABCD$ is bicentric if and only if it is cyclic and tangential. Its study was started by Nicolas Fuss in 1794, see [2], and continues to the present days as can be seen in [1], Chapter 6 of [5] and [4].

Let $ABCD$ be a convex quadrilateral. The circle tangent to side AB and tangent to the extensions of its two adjacent sides, is called the excircle of the quadrilateral corresponding to the side AB. Let $\mathcal{C}_{(I_a,r_a)}$ this circle, where I_a is the center and r_a is the radius. Similarly are defined $\mathcal{C}_{(I_b,r_b)}$, $\mathcal{C}_{(I_c,r_c)}$, $\mathcal{C}_{(I_d,r_d)}$ the excircles of $ABCD$ tangents to the sides b, c, d respectively (see Figure 1).

The following equalities and inequalities referring to cyclic and bicentric quadrilaterals are proved in [5].

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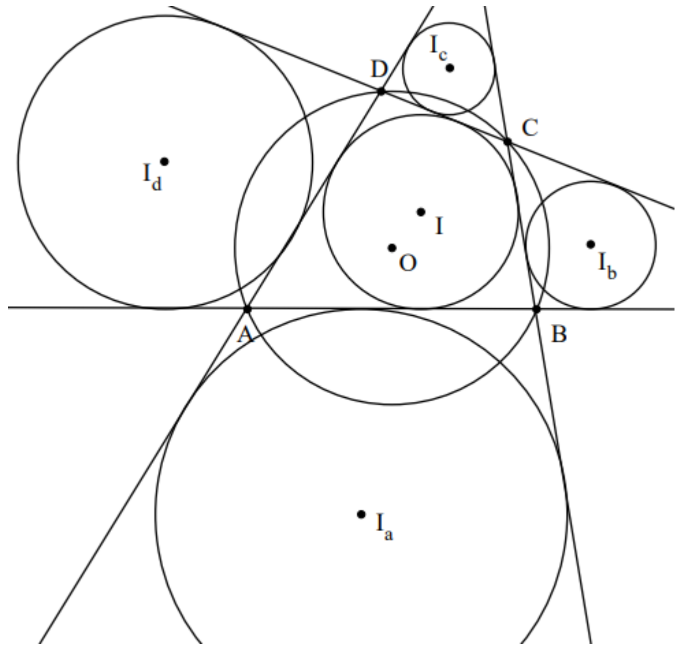

Figure 1: Bicentric quadrilateral with excircles

Theorem 1.1. *In a bicentric quadrilateral $ABCD$, the following equalities hold*

$$OI^2 = R^2 + \left(r - \sqrt{4R^2 + r^2}\right)r, \tag{1}$$

$$OI_a^2 = R^2 + \left(\sqrt{4R^2 + r^2} - r\right)r_a, \tag{2}$$

$$F = rs = \sqrt{abcd}. \tag{3}$$

Theorem 1.2. *Let $ABCD$ be a bicentric quadrilateral. The following inequality holds*

$$2\sqrt{2r\left(\sqrt{4R^2 + r^2}\right) - r} \leq 2. \tag{4}$$

If $R = r\sqrt{2}$, then $ABCD$ is a square, both circles are concentric and (4) holds.
If $R \neq r\sqrt{2}$, then the equality holds in (4) if and only if $ABCD$ is an isosceles trapezoid.
Moreover, we have

$$s \leq \sqrt{4R^2 + r^2} + r \tag{5}$$

which becomes equality if $ABCD$ is orthodiagonal.

The following inequalities also hold

$$2\sqrt{2r\left(\sqrt{4R^2 + r^2} - r\right)} \leq s \leq \sqrt{4R^2 + r^2} + r. \tag{6}$$

If $R = r\sqrt{2}$, then both inequalities become equalities and $ABCD$ is a square.
If $R \neq r\sqrt{2}$, then at least one of the inequalities (6) is strict.
The inequality of Fejes-Tóth holds too

$$R \geq r\sqrt{2}. \tag{7}$$

Theorem 1.3. *For any bicentric quadrilateral $ABCD$, the following equality hold*

$$r_a = \frac{ar}{c} \tag{8}$$

and its analogues.

Moreover r_a, r_b, r_c *and* r_d *are the roots of the equation*

$$x^4 - 2\left(\sqrt{4R^2+r^2} - r\right)x^3 + \left(s^2 + 2r^2 - 4r\sqrt{4R^2+r^2}\right)x^2 - 2r^2\left(\sqrt{4R^2+r^2} - r\right)x + r^4 = 0, \tag{9}$$

$$r_a + r_b + r_c + r_d = 2\left(\sqrt{4R^2+r^2} - r\right), \tag{10}$$

$$r_ar_b + r_ar_c + r_ar_d + r_br_c + r_br_d + r_cr_d = s^2 + 2r^2 - 4r\sqrt{4R^2+r^2}, \tag{11}$$

$$r_ar_br_c + r_ar_br_d + r_ar_cr_d + r_br_cr_d = 2r^2\left(\sqrt{4R^2+r^2} - r\right), \tag{12}$$

$$r_ar_br_cr_d = r^4, \tag{13}$$

$$r_a^2 + r_b^2 + r_c^2 + r_d^2 = 16R^2 - 2s^2 + 4r^2, \tag{14}$$

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} + \frac{1}{r_d} = \frac{2\left(\sqrt{4R^2+r^2} - r\right)}{r^2}, \tag{15}$$

and

$$\frac{1}{r_ar_b} + \frac{1}{r_ar_c} + \frac{1}{r_ar_d} + \frac{1}{r_br_c} + \frac{1}{r_br_d} + \frac{1}{r_cr_d} = \frac{s^2 + 2r^2 - 4r\sqrt{4R^2+r^2}}{r^4}. \tag{16}$$

Theorem 1.4. *In the cyclic quadrilateral* $ABCD$*, we have the following relations and their analogues*

$$F = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{17}$$

$$\cos\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{ad+bc}}, \tag{18}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}, \tag{19}$$

$$r_a = \frac{a}{\tan\frac{A}{2}\tan\frac{B}{2}}. \tag{20}$$

2 Characterizations of the quadrilateral $I_aI_bI_cI_d$

In this section, we will demonstrate some properties of the quadrilateral $I_aI_bI_cI_d$.

Theorem 2.1. *If* $ABCD$ *is a bicentric quadrilateral, then*

$$OI_a^2 + OI_b^2 + OI_c^2 + OI_d^2 = 8R^2 + 4OI^2, \tag{21}$$

where I_a, I_b, I_c, I_d *are the centers of the excircles corresponding to sides* a, b, c, d *respectively.*

Proof. Taking (2) into account, we have

$$OI_a^2 + OI_b^2 + OI_c^2 + OI_d^2 = R^2 + (\sqrt{4R^2+r^2} - r)r_a + R^2 + (\sqrt{4R^2+r^2} - r)r_b + R^2 + (\sqrt{4R^2+r^2} - r)r_c + R^2 + (\sqrt{4R^2+r^2} - r)r_d,$$

from where

$$OI_a^2 + OI_b^2 + OI_c^2 + OI_d^2 = 4R^2 + \left(\sqrt{4R^2+r^2} - r\right)(r_a + r_b + r_c + r_d)$$

and by (10) it results in that

$$\begin{aligned} OI_a^2+OI_b^2+OI_c^2+OI_d^2 &= 4R^2+\left(\sqrt{4R^2+r^2}-r\right)\left[2\left(\sqrt{4R^2+r^2}-r\right)\right] \\ &= 4R^2+8R^2+2r^2-4r\sqrt{4R^2+r^2}+2r^2 = 8R^2+4\left(R^2+r^2-r\sqrt{4R^2+r^2}\right). \end{aligned}$$

From the identity above and (1), we get the identity from (21). □

Lemma 2.1. *In the cyclic quadrilateral $ABCD$, we have the following relations*

$$\tan\frac{A}{2}=\frac{(s-a)(s-d)}{F}, \tag{22}$$

$$r_a=\frac{aF}{(s-a)(a+c)} \tag{23}$$

and analogues.

Proof. From (19) we have

$$\tan\frac{A}{2}=\sqrt{\frac{(s-a)^2(s-d)^2}{(s-a)(s-b)(s-c)(s-d)}}$$

and taking (17) into account, the relation (22) follows. Replacing (22) and analogue in (20) we obtain

$$r_a=\frac{a}{\frac{(s-a)(s-d)}{F}+\frac{(s-b)(s-a)}{F}}$$

from where (23) results. □

Lemma 2.2. *In the cyclic quadrilateral $ABCD$, we have the following relation and its analogues*

$$I_cI_d=\frac{s(ad+bc)\sqrt{ab+cd}}{(a+c)(b+d)}\cdot\frac{1}{\sqrt{(s-c)(s-d)}}. \tag{24}$$

Proof. Let $I_cE\perp DC$, $E\in DC$ (see Figure 2). In triangle I_cDE we have $\sin(\widehat{I_cDE})=\frac{I_cE}{I_cD}$, equivalent with $\sin(\frac{\pi}{2}-\frac{D}{2})=\frac{r_c}{I_cD}$, from where $I_cD=\frac{rc}{\cos\frac{D}{2}}$ and analogous $I_dD=\frac{r_d}{\cos\frac{D}{2}}$. Because the points I_c, D and I_d are collinear, we have $I_cI_d=I_cD+I_dD=\frac{r_c+r_d}{\cos\frac{D}{2}}$.

Taking (23) into account, we have

$$\begin{aligned} I_cI_d &= \frac{\frac{cF}{(s-c)(c+a)}+\frac{dF}{(s-d)(d+b)}}{\cos\frac{D}{2}} \\ &= \frac{F\left[c(d+b)(a+b+c-d)+d(c+a)(a+b-c+d)\right]}{2(c+a)(d+b)(s-c)(s-d)\cos\frac{D}{2}} \\ &= \frac{F(acd+bcd+abc+abd+b^2c+bc^2+a^2d+ad^2)}{2(c+a)(d+b)(s-c)(s-d)\sqrt{\frac{(s-a)(s-b)}{ab+cd}}} \\ &= \frac{\sqrt{(s-a)(s-b)(s-c)(s-d)}\cdot 2s(ad+bc)\sqrt{ab+cd}}{2(c+a)(d+b)(s-c)(s-d)\sqrt{(s-a)(s-b)}} \\ &= \frac{s(ad+bc)\sqrt{ab+cd}}{(a+c)(b+d)\sqrt{(s-c)(s-d)}}, \end{aligned}$$

from where the identity (24) follows. □

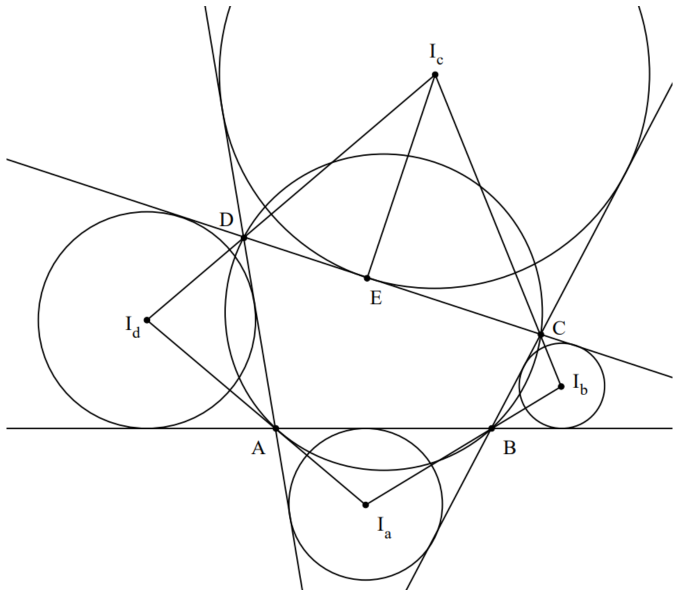

Figure 2: Convex quadrilateral with excircles

Theorem 2.2. *For any convex quadrilateral $ABCD$, the quadrilateral $I_aI_bI_cI_d$ is cyclic.*

Proof. In triangle DI_cC we have $\widehat{DI_cC} = \pi - \widehat{CDI_c} - \widehat{DCI_c} = \pi - \frac{\pi-D}{2} - \frac{\pi-C}{2} = \frac{C+D}{2}$ and similarly $\widehat{AI_aB} = \frac{A+B}{2}$. Because $A+B+C+D = 2\pi$, then $\widehat{DI_cC} + \widehat{AI_aB} = \frac{A+B+C+D}{2} = \pi$, so the quadrilateral $I_aI_bI_cI_d$ is cyclic. □

The following theorem is proved in [3, Theorem 6.2]. Here it is demonstrated in an original way.

Theorem 2.3. *If the quadrilateral $ABCD$ is cyclic, then the quadrilateral $I_aI_bI_cI_d$ is orthodiagonal.*

Proof. Taking (24) into account, we have

$$\begin{aligned} {I_aI_b}^2 + {I_cI_d}^2 &= \frac{s^2(ad+bc)^2(ab+cd)}{(a+c)^2(b+d)^2} \cdot \frac{1}{(s-a)(s-b)} + \frac{s^2(ad+bc)^2(ab+cd)}{(a+c)^2(b+d)^2} \cdot \frac{1}{(s-c)(s-d)} \\ &= \frac{s^2(ad+bc)^2(ab+cd)[(s-c)(s-d)+(s-a)(s-b)]}{(a+c)^2(b+d)^2F^2} \\ &= \frac{s^2(ad+bc)^2(ab+cd)[(a+b-c+d)(a+b+c-d)+(-a+b+c+d)(a-b+c+d)]}{4(a+c)^2(b+d)^2F^2} \\ &= \frac{s^2(ad+bc)^2(ab+cd)[(a+b)^2-(c-d)^2+(c+d)^2-(a-b)^2]}{4(a+c)^2(b+d)^2F^2} \\ &= \frac{s^2(ad+bc)^2(ab+cd)^2}{(a+c)^2(b+d)^2F^2} \end{aligned}$$

and similarly ${I_aI_d}^2 + {I_cI_b}^2$ has the same value.

Then ${I_aI_d}^2 + {I_cI_b}^2 = {I_aI_b}^2 + {I_cI_d}^2$, so the quadrilateral $I_aI_bI_cI_d$ is orthodiagonal. □

Corollary 2.1. *If the quadrilateral $ABCD$ is bicentric, then the quadrilateral $I_aI_bI_cI_d$ is cyclic and O is the center of circumscribed circle if and only if the quadrilateral $ABCD$ is a square.*

Proof. O is the center of the circumscribed circle, then $OI_a = OI_b = OI_c = OI_d$, and taking (2) into account we have

$$\sqrt{R^2 + (\sqrt{4R^2 + r^2} - r)r_a} = \sqrt{R^2 + (\sqrt{4R^2 + r^2} - r)r_b} = \sqrt{R^2 + (\sqrt{4R^2 + r^2} - r)r_c} = \sqrt{R^2 + (\sqrt{4R^2 + r^2} - r)r_d},$$

equivalent to $r_a = r_b = r_c = r_d$.

Using the formulas from (8), we have $\frac{ar}{c} = \frac{br}{d} = \frac{cr}{a} = \frac{dr}{b}$. From $\frac{ar}{c} = \frac{cr}{a}$ we get $a = c$ and from $\frac{br}{d} = \frac{dr}{b}$ we obtain $b = d$.

Because $ABCD$ is cyclic and $a = c$ and $b = d$, we have equivalent to $ABCD$ is rectangle and because $ABCD$ is tangential we have $a = b$, so $ABCD$ is a square. □

Theorem 2.4. *Let $ABCD$ be a cyclic quadrilateral. Then $I_aI_bI_cI_d$ is tangential quadrilateral if and only if $a = c$ or $b = d$.*

Proof. According to Theorem 2.3 we have ${I_aI_b}^2 + {I_cI_d}^2 = {I_aI_d}^2 + {I_cI_b}^2$ and because $I_aI_bI_cI_d$ is tangential quadrilateral it results $I_aI_b + I_cI_d = I_aI_d + I_cI_b$.

From here it follows that $I_aI_b \cdot I_cI_d = I_aI_d \cdot I_cI_b$. Taking (24) into account we have

$$\frac{s^2(ad + bc)^2(ab + cd)}{(a + c)^2(b + d)^2F} = \frac{s^2(ab + cd)^2(ad + bc)}{(a + c)^2(b + d)^2F},$$

equivalent with $ab + cd = ad + bc$, equivalent with $(a - c)(d - b) = 0$ and the theorem is proved. □

Remark 2.1. The condition $a = c$ means that $B = C$ and $A = D$, so the inscribed quadrilateral $ABCD$ is an isosceles trapezium and similarly if $b = d$.

If $a = c$ the quadrilateral $I_aI_bI_cI_d$ is a right kite with $\widehat{I_dI_aI_b} = \widehat{I_bI_cI_d} = \frac{\pi}{2}$.

As $B = C$ and $A = D$, then $A + B = C + D = \pi$. In triangle AI_aB we have $\widehat{AI_aB} = \pi - \frac{\pi - A}{2} - \frac{\pi - B}{2} = \frac{\pi}{2}$. The triangles AI_aB and DI_cC are congruent and the triangles BI_bC and DI_dA are isosceles.

Remark 2.2. From Theorem 2.4 it does not follow that for $I_aI_bI_cI_d$ to be tangential, the quadrilateral $ABCD$ must be tangential.

3 Inequalities in the quadrilateral $I_aI_bI_cI_d$

In this section, we will give some old and new inequalities which take place in the quadrilateral $I_aI_bI_cI_d$. We get some known inequalities from [5, p. 172]. These inequalities are immediately obtained from (10) to (16) using the inequalities (6) and (7).

Theorem 3.1. *In any bicentric quadrilateral ABCD, the following inequalities hold*

$$4r \le r_a + r_b + r_c + r_d \le 3R\sqrt{2} - 2r, \tag{25}$$

$$6r^2 \le 2r(2\sqrt{4R^2+r^2} - 3r) \le \sum r_a r_b \le 2(2R^2 + 2r^2 - r\sqrt{4R^2+r^2}) \le 2(3R^2 - r\sqrt{4R^2+r^2}), \tag{26}$$

$$4r^3 \le \sum r_a r_b r_c \le R^2\left(\frac{3R\sqrt{2}}{2} - r\right), \tag{27}$$

$$r_a r_b r_c r_d \le \frac{R^4}{4}, \tag{28}$$

$$4(2R^2 - r\sqrt{4R^2+r^2}) \le \sum r_a^2 \le 4(4R^2 + 5r^2 - 4r\sqrt{4R^2+r^2}) \tag{29}$$

and

$$\frac{6}{r^2} \le \frac{2(2\sqrt{4R^2+r^2} - 3r)}{r^3} \le \sum \frac{1}{r_a r_b} \le \frac{2(2R^2 + 2r^2 - r\sqrt{4R^2+r^2})}{r^4}. \tag{30}$$

Theorem 3.2. *In any bicentric quadrilateral ABCD, for $\beta \le \frac{8\sqrt{2}}{3}$ and $\delta \ge 4$ the following inequality hold*

$$4r + \beta\left(R - r\sqrt{2}\right) \le 4r + \frac{8\sqrt{2}}{3}\left(R - r\sqrt{2}\right) \le r_a + r_b + r_c + r_d \le 4r + 4\left(R - r\sqrt{2}\right) \le 4r + \delta\left(R - r\sqrt{2}\right). \tag{31}$$

Proof. According to (10) it is true that $\sum r_a = 2(\sqrt{4R^2+r^2} - r)$. We find α, β, γ and δ such that

$$\alpha r + \beta R \le 2\left(\sqrt{4R^2+r^2} - r\right) \le \gamma r + \delta R. \tag{32}$$

Inequalities (32) hold if and only if $0 \le 2\sqrt{4R^2+r^2} - 2r - \alpha r - \beta R$ and $0 \ge 2\sqrt{4R^2+r^2} - 2r - \gamma r - \delta R$, equivalent to

$$0 \le 2\sqrt{4\left(\frac{R}{r}\right)^2 + 1} - 2 - \alpha - \beta\frac{R}{r}$$

and $0 \ge 2\sqrt{4\left(\frac{R}{r}\right)^2 + 1} - 2 - \gamma - \delta\frac{R}{r}$. According to (7) we have $r\sqrt{2} \le R$, equivalent to $\sqrt{2} \le \frac{R}{r}$. Equality holds if and only if $ABCD$ is square. We note $x = \frac{R}{r}$ and then $x \ge \sqrt{2}$. The inequalities above become

$$0 \le 2\sqrt{4x^2+1} - 2 - \alpha - \beta x \tag{33}$$

and

$$0 \ge 2\sqrt{4x^2+1} - 2 - \gamma - \delta x. \tag{34}$$

Let $f, g\colon [\sqrt{2}, +\infty) \to \mathbb{R}$ be functions defined by $f(x) = 2\sqrt{4x^2+1} - 2 - \alpha - \beta x$ and $g(x) = 2\sqrt{4x^2+1} - 2 - \gamma - \delta x$. We put the condition that in (33) and (34) equality occurs for $x = \sqrt{2}$, equivalent to $f(\sqrt{2}) = 0$ and $g(\sqrt{2}) = 0$, equivalent to

$$\alpha = 4 - \beta\sqrt{2} \tag{35}$$

and

$$\gamma = 4 - \delta\sqrt{2}. \tag{36}$$

We have $f'(x) = \frac{8x}{\sqrt{4x^2+1}} - \beta$, $g'(x) = \frac{8x}{\sqrt{4x^2+1}} - \delta$ and $f''(x) = g''(x) = \frac{8}{(4x^2+1)\sqrt{4x^2+1}} > 0$, for any $x \in [\sqrt{2}, +\infty)$, so f' and g' are an increasing functions. We put the conditions that $f'(\sqrt{2}) \geq 0$, equivalent to

$$\beta \leq \frac{8\sqrt{2}}{3}. \tag{37}$$

On the other hand, for the function g' we put the condition $\lim_{x\to+\infty} g'(x) \leq 0$, equivalent to $\lim_{x\to+\infty}\left(\frac{8x}{\sqrt{4x^2+1}} - \delta\right) \leq 0$, equivalent to $4 - \delta \leq 0$, so

$$\delta \geq 4. \tag{38}$$

Because f' is an increasing function, it result that $f'(x) \geq f'(\sqrt{2}) \geq 0$ for any $x \in [\sqrt{2}, +\infty)$, so $f'(x) \geq 0$ for any $x \in [\sqrt{2}, +\infty)$.

x	$\sqrt{2}$ $\qquad$ $+\infty$
$f'(x)$	$+$
$f(x)$	$\nearrow$

From the function variation table, it follows that $f(x) \geq 0$, for any $x \in [\sqrt{2}, +\infty)$. So, from (32), (33), (35) and (37) results that $(4 - \beta\sqrt{2})r + \beta R \leq 2(\sqrt{4R^2 + r^2} - r)$ for any $\beta \leq \frac{8\sqrt{2}}{3}$, equivalent to $4R + \beta(R - r\sqrt{2}) \leq 2(\sqrt{4R^2 + r^2} - r)$. Because $R - r\sqrt{2} \geq 0$, from the inequality above we obtain $4r + \beta(R - r\sqrt{2}) \leq 4r + \frac{8\sqrt{2}}{3}(R - r\sqrt{2}) \leq 2(\sqrt{4R^2 + r^2} - r)$, if $\beta \leq \frac{8\sqrt{2}}{3}$. With this, the left member of the inequality (31) is proved.

Because g' is an increasing function, it results that $g'(x) \leq \lim_{x\to+\infty} g'(x) \leq 0$ for any $x \in [\sqrt{2}, +\infty)$, so $g'(x) \leq 0$, for any $x \in [\sqrt{2}, +\infty)$

x	$\sqrt{2}$ $\qquad$ $+\infty$
$g'(x)$	$-$
$g(x)$	$\searrow$

From the function variation table, it follows that $g(x) \leq 0$ for any $x \in [\sqrt{2}, +\infty)$. So, from (32), (34), (36) and (38) results that $2(\sqrt{4R^2 + r^2} - r) \leq (4 - \delta\sqrt{2})r + \delta R$ for any $\delta \geq 4$, equivalent to $2(\sqrt{4R^2 + r^2} - r) \leq 4r + \delta(R - r\sqrt{2})$. Because $R - r\sqrt{2} \geq 0$, from the inequality above we obtain $2(\sqrt{4R^2 + r^2} - r) \leq 4r + 4(R - r\sqrt{2}) \leq 4r + \delta(R - r\sqrt{2})$. With this, the right member of the inequality (31) is proved. □

Corollary 3.1. *In any bicentric quadrilateral $ABCD$, the following inequalities hold*

$$4r \leq 4r + \frac{8\sqrt{2}}{3}(R - r\sqrt{2}) \leq r_a + r_b + r_c + r_d \leq 4r + 4(R - r\sqrt{2}) \leq 3R\sqrt{2} - 2r. \tag{39}$$

Proof. The inequalities $4r + \frac{8\sqrt{2}}{3}(R - r\sqrt{2}) \leq r_a + r_b + r_c + r_d \leq 4r + 4(R - r\sqrt{2})$ follows from (31). The inequality $4r \leq 4r + \frac{8\sqrt{2}}{3}(R - r\sqrt{2})$ is evident because $R \geq r\sqrt{2}$. The inequality $4r + 4(R - r\sqrt{2}) \leq 3R\sqrt{2} - 2r$ is equivalent to $0 \leq (R - r\sqrt{2})(3\sqrt{2} - 4)$, which is a true inequality. From the inequalities above, inequalities from (39) are obtained. □

Remark 3.1. The inequalities in (39) are stronger than those in (25).

Using the idea from Theorem 3.2 we prove a new inequality in Theorem 3.3.

Theorem 3.3. *In any bicentric quadrilateral $ABCD$, the following inequalities hold*

$$\frac{8\sqrt{2}}{27}\cdot\frac{1}{R}+\frac{100}{27}\cdot\frac{1}{r}\le\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}+\frac{1}{r_d}\le-\frac{28}{27}\cdot\frac{1}{r}+\frac{64\sqrt{2}}{27}\cdot\frac{R}{r^2}+\frac{4}{27}\cdot\frac{R^2}{r^3}. \tag{40}$$

Proof. Taking (15) into account, the first inequality from (40) is equivalent to

$$\frac{8\sqrt{2}}{27}\cdot\frac{1}{R}+\frac{100}{27}\cdot\frac{1}{r}\le\frac{2(\sqrt{4R^2+r^2}-r)}{r^2}.$$

Multiplying by r, noting $x=\frac{R}{r}$, where $x\ge\sqrt{2}$, we have equivalent that

$$\frac{8\sqrt{2}}{27}\cdot\frac{1}{x}+\frac{100}{27}\le 2(\sqrt{4x^2+1}-1)$$

equivalent to

$$2\sqrt{4x^2+1}-2-\frac{8\sqrt{2}}{27}\cdot\frac{1}{x}-\frac{100}{27}\ge 0,\quad x\in[\sqrt{2},+\infty).$$

Let $f\colon[\sqrt{2},+\infty)\to\mathbb{R}$ be a function defined by $f(x)=2\sqrt{4x^2+1}-2-\frac{8\sqrt{2}}{27}\cdot\frac{1}{x}-\frac{100}{27}$.

We have that $f'(x)=\frac{8x}{\sqrt{4x^2+1}}+\frac{8\sqrt{2}}{27}\cdot\frac{1}{x^2}\ge 0$, for any $x\in[\sqrt{2},+\infty)$, so f is increasing function, from where it follows that $f(x)\ge f(\sqrt{2})=0$, what we had to prove.

The second inequality, similarly is equivalent to

$$2\sqrt{4x^2+1}-2+\frac{28}{27}-\frac{64\sqrt{2}}{27}x-\frac{4}{27}x^2\le 0,\quad x\in[\sqrt{2},+\infty).$$

Let $g\colon[\sqrt{2},+\infty)\to\mathbb{R}$ be a function defined by $g(x)=2\sqrt{4x^2+1}-2+\frac{28}{27}-\frac{64\sqrt{2}}{27}x-\frac{4}{27}x^2$.

We have $g'(x)=\frac{8x}{\sqrt{4x^2+1}}-\frac{64\sqrt{2}}{27}-\frac{8}{27}x$, and $g''(x)=\frac{8}{(4x^2+1)\sqrt{4x^2+1}}-\frac{8}{27}\le 0$, because $x\ge\sqrt{2}$.

Then g' is decreasing function and we have $g'(x)\le g'(\sqrt{2})=0$, $x\in[\sqrt{2},+\infty)$, so g is decreasing function. In the end we have that $g(x)\le g(\sqrt{2})=0$, $x\in[\sqrt{2},+\infty)$, what had to be demonstrated. □

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